

CAMBRIDGE INTERNATIONAL EXAMINATIONS

Cambridge International Advanced Level

MARK SCHEME for the October/November 2015 series

9231 FURTHER MATHEMATICS

9231/21

Paper 2, maximum raw mark 100

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Question Number	Mark Scheme Details	Part Mark	Total
1	Find 3 independent equations for T, R_A, R_B: Resolve horizontally: $R_B = T \cos \alpha$ M1 A1 Resolve vertically: $R_A = W + T \sin \alpha$ M1 A1 Take moments about A: (a may be omitted from moment eqns) $R_B 3a \sin \theta = W (3a/2) \cos \theta$ $+ T a (\sin \alpha \cos \theta + \cos \alpha \sin \theta)$ or $+ T a \sin (\alpha + \theta)$ or $+ T 3a \cos \theta \sin \alpha$ M1 A1 Take moments about B: $R_A 3a \cos \theta = W (3a/2) \cos \theta$ $+ T 2a (\sin \alpha \cos \theta + \cos \alpha \sin \theta)$ or $+ T 2a \sin (\alpha + \theta)$ or $+ T 3a \sin \theta \cos \alpha$ (M1 A1) Take moments about C: $R_A a \cos \theta + W (a/2) \cos \theta$ $= R_B 2a \sin \theta$ (M1 A1) Take moments about D: $R_A 3a \cos \theta - W (3a/2) \cos \theta$ $= R_B 3a \sin \theta$ (M1 A1) Solve for T, R_A, R_B (AEF in W and α): $T = W / 2 \sin \alpha$ or $\frac{1}{2}W \operatorname{cosec} \alpha$ B1 $R_A = 3W / 2$ B1 $R_B = W / 2 \tan \alpha$ or $\frac{1}{2}W \cot \alpha$ B1	9	9
2	For A & B use conservation of momentum, e.g.: $2mv_A + mv_B = 2mu$ (allow $2v_A + v_B = 2u$) M1 Use Newton's law of restitution (consistent signs): $v_B - v_A = eu$ M1 Combine to find v_A and v_B: $v_A = (2 - e)u/3, v_B = 2(1 + e)u/3$ A1, A1 Find e from $v_A = v_B'$ with $v_B' = [-] 0.4 v_B$: $(2 - e) = 0.8(1 + e), e = 2/3$ M1 A1 <i>EITHER:</i> Equate times in terms of reqd. distance x: $(d - x)/v_A = d/v_B + x/v_B'$ (AEF) M1 A1 [speeds need not be found: $v_A = v_B' = 4u/9, v_B = 10u/9$] Use $v_A = v_B' = 0.4 v_B$ to solve for x: $d - x = 0.4 d + x, x = 0.3 d$ M1 A1 <i>OR:</i> Find dist. moved by A when B reaches wall: $d_A = (d/v_B) v_A = 0.4 d$ (M1 A1) Find reqd. distance x: $x = \frac{1}{2}(d - d_A) = 0.3 d$ (M1 A1)	4 2 4	10

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Question Number	Mark Scheme Details	Part Mark	Total
3	Find k by equating equilibrium tensions: (vertical motion can earn M1 only)	$mg(a/2)/a = 2mg(3a/2 - ka)/ka$ $\frac{1}{2} = 3/k - 2, \quad k = 6/5 \text{ or } 1.2$ M1 A1	3
	Apply Newton's law at general point, e.g.: (lose A1 for each incorrect term)	$m \frac{d^2x}{dt^2} = -mg(a/2 + x)/a$ $+ 2mg(3a/2 - ka - x)/ka$ or $m \frac{d^2y}{dt^2} = +mg(a/2 - y)/a$ $- 2mg(3a/2 - ka + y)/ka$ M1 A2	
	Simplify to give standard SHM eqn, e.g.: S.R.: B1 if no derivation (max 2/5)	$\frac{d^2x}{dt^2} = - (1 + 2/k)gx/a$ $= - 8gx/3a$ A1	
	State or find period using $2\pi/\omega$ with $\omega = \sqrt{(8g/3a)}$: $T = 2\pi\sqrt{(3a/8g)}$ or $\pi\sqrt{(3a/2g)}$ ($\sqrt{\quad}$ on ω)	or $3.85\sqrt{(a/g)}$ or $1.22\sqrt{a}$ [s] B1	5
	Substitute values in $v^2 = \omega^2(x_0^2 - x^2)$:	$0.7^2 = (8g/3a)\{(0.2a)^2 - (0.05a)^2\}$ M1 A1	
	Solve to find numerical value of a :	$0.49 = (8g/3) \times 0.0375a, \quad a = 0.49$ A1	3
			11

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4	<i>EITHER:</i> Find tension at top from $F = ma$ vertically: $T = mu^2/a - mg$ B1		
	<i>OR:</i> Use energy at e.g. θ to upward vertical: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mga(1 - \cos \theta)$ Find tension T by using $F = ma$ radially: $T' = mv^2/a - mg \cos \theta$ Eliminate v^2 : $= mu^2/a + mg(2 - 3 \cos \theta)$ Find T at top by taking $\theta = 0$: $T = mu^2/a - mg$ (B1)		
	Find $u_{\min}$ by requiring $T \geq 0$ at top [$T > 0$]: $u^2/a - g \geq 0$ so $u_{\min} = \sqrt{ag}$ A.G. B1	2	
	Find v at bottom from conservation of energy: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mg \times 2a$ $v^2 = ag + 4ag$, $v = \sqrt{5ag}$ M1 A1		
	Find new speed V from conservation of momentum: $m'V = mv$ with $m' = m + \frac{1}{4}m$ M1 $V = 4v/5 = 4\sqrt{ag/5}$ or $(4/5)\sqrt{5ag}$ AEF A1	4	
	Find w^2 at angle θ from conservation of energy: $\frac{1}{2}m'w^2 = \frac{1}{2}m'V^2$ (condone m instead of m' here since cancels out) $-m'ga(1 + \cos \theta)$ M1 $[w^2 = ag(6/5 - 2 \cos \theta)]$ A1		
	S.R. Invalid energy method (max 2/5): [gives $T' = (5mg/4)(2 - 3 \cos \theta)$] $\frac{1}{2}m'w^2 = \frac{1}{2}mu^2 + mga(1 - \cos \theta) - \frac{1}{4}mga(1 + \cos \theta)$ (B1)		
	Find tension T' there by using $F = ma$ radially: $T' = m'w^2/a - m'g \cos \theta$ B1		
	Eliminate w^2 : $= m'V^2/a - m'g(2 + 3 \cos \theta)$ A1		
	Substitute for m' and V : $= (5mg/4)(6/5 - 3 \cos \theta)$ AEF or $3mg/2 - (15/4)mg \cos \theta$ A1	5	
5	Find $\cos \theta$ when string becomes slack from $T' = 0$: $\cos \theta = \frac{1}{3} \times 6/5 = 2/5$ or 0.4 M1 S.R. Allow if found from $T' = mg(6/5 - 3 \cos \theta)$ A1	2	13
	Find or use sample mean $\bar{x} = 222.8 / 10 = 22.28$ and estimate population variance: $s^2 = 4.12 / 9$ (allow biased here: 0.412 or 0.642 ²) $= 0.458$ or 103/225 or 0.677 ² M1		
	Find confidence interval (allow z in place of t) e.g.: $22.28 \pm t\sqrt{(0.458 / 10)}$ M1 A1		
	Use of correct tabular value: $t_{9, 0.975} = 2.26[2]$ A1		
	Evaluate C.I. correct to 3 s.f.: $22.3 \pm 0.48[4]$ or [21.8, 22.8] A1	5	5

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6	Find prob. p of head from mean = $2 \times$ variance:	$1/p = 2 \times (1 - p)/p^2, \quad p = 2/3 \quad \mathbf{A.G.} \quad \mathbf{M1 \ A1}$		2	8	
	(i) Find $P(X = 4)$ (denoting $1 - p$ by q [= $1/3$]):	$P(X = 4) = q^3 \times p$ $= 2/81 \text{ or } 0.0247 \quad \mathbf{B1}$		1		
	(ii) Find or state $P(X > 4)$:	$P(X > 4) [= 1 - (1 + q + q^2 + q^3) \times p$ $= 1 - (1 - q^4)] = q^4$ $= 1/81 \text{ or } 0.0123 \quad \mathbf{M1 \ A1}$		2		
	(iii) Formulate condition for N : Take logs (any base) to give bound for N : Find $N_{\min}$: ($N < 6.29$ or $N = 6.29$ earns M2 A0)	$1 - q^N > 0.999, \quad [(1/3)^N < 0.001]$ $N > \log 0.001 / \log 1/3$ $N > 6.29, \quad N_{\min} = 7 \quad \mathbf{M1}$ $\mathbf{M1}$ $\mathbf{A1}$		3		
7	Find $F(x)$ for $1 \leq x \leq 4$:	$F(x) = (x^3 - 1)/63 \quad \mathbf{B1}$		5		
	Find $G(y)$ from $Y = X^2$ for $1 \leq x \leq 4$: (result may be stated)	$G(y) = P(Y < y) = P(X^2 < y)$ $= P(X < y^{1/2}) = F(y^{1/2})$ $= (y^{3/2} - 1)/63 \quad \mathbf{M1 \ A1}$				
	Find $g(y)$ for corresponding range of y :	$g(y) = y^{1/2}/42 \quad \mathbf{A.G.} \quad \mathbf{A1}$				
	Find or state corresponding range of y :	$1 \leq y \leq 16 \quad \mathbf{A.G.} \quad \mathbf{B1}$				
	(i) Find median value m of Y :	$(m^{3/2} - 1)/63 = 1/2$ $m = 32.5^{2/3} = 10.2 \quad \mathbf{M1 \ A1}$				2
	(ii) Find $E(Y)$ [or equivalently $E(X^2)$]:	$E(Y) = \int y g(y) dy = \int y^{3/2} dy / 42$ $= [y^{5/2}]_1^{16} / 105 = 1023/105$ $= 341/35 \text{ or } 9.74 \quad \mathbf{M1 \ A1}$				2

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8	Find mean of sample data [for use in Poisson distn.]: $\lambda = 220/100 = 2.2$ B1 State (at least) null hypothesis (AEF): H_0: Poisson distn. fits data or $\lambda = 2.2$ B1 Find expected values $100\lambda^r e^{-\lambda}/r!$ (to 1 d.p.): (ignore incorrect final value here for M1) 11.080 24.377 26.814 19.664 10.8151 4.759 2.491 M1 A1 Combine last two cells so that exp. value ≥ 5: O_i: 3 E_i: 7.25 M1* Calculate value of χ^2 (to 2 d.p.; A1 dep M1*): $\chi^2 = 0.076 + 2.879 + 0.653 + 1.448$ $+ 0.441 + 2.491$ (allow 7.95 if 1 d.p. exp.values used) $= 7.99$ M1 A1 State or use consistent tabular value (to 3 s.f.): 5 cells: $\chi_{3,0.95}^2 = 7.815$ 6 cells: $\chi_{4,0.95}^2 = 9.488$ (correct) 7 cells: $\chi_{5,0.95}^2 = 11.07$ B1 State or imply valid method for conclusion e.g.: Accept H_0 if $\chi^2 < \text{tabular value}$ M1 Conclusion (AEF, requires both values correct): Distn fits or $\lambda = 2.2$ A1 Not combining cells [so $\chi^2 = 8.64$] can earn B1 B1 M1 A1 M0 M1 B1 M1 (max 7)	10	10
9	Calculate gradient b_1 in $y - \bar{y} = b_1(x - \bar{x})$: $S_{xy} = 24\,879 - 472 \times 400/8$ $= 1\,279$ $S_{xx} = 29\,950 - 472^2/8 = 2\,102$ $b_1 = S_{xy} / S_{xx} = 0.6085$ (3 s.f.) M1 A1 Find regression line of y on x: $y = 400/8 + b_1(x - 472/8)$ M1 A1 $= 50 + 0.6085(x - 59)$ $= 0.6085x + 14.1$ Find y when $x = 72$: $= 57.9$ or 58 Allow use of regression line of x on y (since neither variable clearly independent): $S_{yy} = 21\,226 - 400^2/8 = 1\,226$ $b_2 = S_{xy} / S_{yy} = 1.043$ (M1 A1) $x = 472/8 + b_2(y - 400/8)$ (M1 A1) $= 1.043y + 6.85$ $Y = 62.5$ or 62 (A1)	5	

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	Find product moment correlation coefficient r: $r = 1\,279 / \sqrt{(2\,102 \times 1\,226)}$ or $\sqrt{(0.6085 \times 1.043)} = 0.797$ M1 A1* State both hypotheses (B0 for $r \dots$): $H_0: \rho = 0, H_1: \rho \neq 0$ B1 State or use correct tabular two-tail r-value: $r_{8, 5\%} = 0.707$ B1* State or imply valid method for conclusion e.g.: Reject H_0 if $r > \text{tab. value}$ (AEF) M1 Correct conclusion (AEF, dep A1*, B1*): There is non-zero correlation A1	6	11
10A	Find MI of lamina about Q: $I_{\text{lamina}} = \frac{1}{3}m\{(3a)^2 + (3a/2)^2\}$ $+ m(9a/2)^2$ M1 A1 $[= (15/4 + 81/4) ma^2 = 24 ma^2]$ State or find MI of rod about Q: $I_{\text{rod}} = (\frac{1}{3} + 1) M(3a/2)^2$ $[= 3Ma^2]$ B1 Sum to find MI of object about Q: $I_1 = 24 ma^2 + 3 Ma^2$ $= 3(8m + M)a^2$ A.G. A1 Find MI of object about mid-point of PQ: $I_2 = (15/4 + 3^2) ma^2 + \frac{1}{3} M(3a/2)^2$ $= (51/4) ma^2 + \frac{3}{4} Ma^2$ $= \frac{3}{4}(17m + M)a^2$ A.G. M1 A1 Use eqn of circular motion to find $d^2\theta/dt^2$ for axis l_1: $[-]I_1 d^2\theta/dt^2 = mg \times (9a/2) \sin \theta$ $+ Mg \times (3a/2) \sin \theta$ M1 A1 $[= (9m/2 + 3M/2) ga \sin \theta]$ [Approximate $\sin \theta$ by θ and] find ω_1^2 in SHM eqn: $\omega_1^2 = (3m + M)g / 2(8m + M)a$ M1 Find period T_1 for axis l_1 from $2\pi/\omega_1$: (AEF) $T_1 = 2\pi\sqrt{\{2(8m + M)a / (3m + M)g\}}$ A1 Use eqn of circular motion to find $d^2\theta/dt^2$ for axis l_2: $[-]I_2 d^2\theta/dt^2 = mg \times 3a \sin \theta$ M1 [Approximate $\sin \theta$ by θ and] find ω_2^2 in SHM eqn: $\omega_2^2 = 4mg / (17m + M)a$ M1 Find period T_2 for axis l_2 from $2\pi/\omega_2$: (AEF) $T_2 = 2\pi\sqrt{\{(17m + M)a / 4mg\}}$ A1 Verify that $T_1 = T_2$ when $m = M$: (AEF) $T_1 = 2\pi\sqrt{(18a / 4g)} = T_2$ B1 [Taking $m = M$ throughout 2nd part can earn: M1 A1 M1 A0 M1 M1 A0 B1 (max 6/8)]	4 2 8	14

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10B	State hypotheses (B0 for $\bar{x}$...), e.g.: $H_0: \mu_X = \mu_Y, H_1: \mu_X \neq \mu_Y$ B1	9	
	State assumption (AEF): Distributions have equal variances B1		
	Find sample means <u>and</u> estimate popln. variances: $\bar{x} = 4.2, \bar{y} = 4.8$ $s_X^2 = (180 - 42^2/10) / 9$ (allow biased here: 0.36 or 0.6 ²) $= 0.4$ or 0.6325 ² $s_Y^2 = (281.5 - 57.6^2/12) / 11$ (allow biased here: 0.4183 or 0.6468 ²) $= 0.4564$ or 251/550 or 0.6755 ² M1		
	Estimate (pooled) common variance: (note s_X^2 and s_Y^2 not needed explicitly) $s^2 = (9 s_X^2 + 11 s_Y^2) / 20$ (AEF) or $(180 - 42^2/10 + 281.5 - 57.6^2/12) / 20$ $= 0.431$ or 0.6565 ² M1 A1		
	Calculate value of t (to 3 s.f.): $[-] t = (\bar{y} - \bar{x}) / s \sqrt{(1/10 + 1/12)}$ $= 2.13$ M1 A1		
	State or use correct tabular t value: (or can compare $\bar{y} - \bar{x} = 0.6$ with 0.586) $t_{20, 0.975} = 2.086$ [allow 2.09] B1*		
	Correct conclusion (AEF, $\sqrt$ on t , dep *B1): $t > 2.09$ so mean masses not same B1$\sqrt$		
	S.R. Implicitly taking s_X^2, s_Y^2 as popln. variances: $z = (\bar{y} - \bar{x}) / \sqrt{(s_X^2/10 + s_Y^2/12)}$ (may also earn first B1 B1 M1) $= 0.6 / \sqrt{(0.078)} = 2.15$ Comparison with $z_{0.975}$ and conclusion: $2.15 > 1.96$ (can earn at most 5/9) so mean masses not same (B1)		
	State hypotheses (B0 for $\bar{x}$...), e.g.: $H_0: \mu_X = 3.8, H_1: \mu_X > 3.8$ or $H_0: \mu_X = \mu_Z, H_1: \mu_X > \mu_Z$ B1		
	Calculate value of t using s_X from above: $t = (4.2 - 3.8) / (s_X / \sqrt{10}) = 2.0$ M1 A1		
	State or use correct tabular t value: (or can compare 0.4 with 0.367) $t_{9, 0.95} = 1.833$ [allow 1.83] B1*		
	Correct conclusion (A.E.F., $\sqrt$ on t , dep *B1): $t > 1.833$, so claim is justified or mean mass of Royals > mean mass of Crowns B1$\sqrt$	5	14