**CAMBRIDGE INTERNATIONAL EXAMINATIONS**

**Cambridge International Advanced Level**

# MARK SCHEME for the October/November 2015 series

## 9231 FURTHER MATHEMATICS

**9231/23** Paper 2, maximum raw mark 100

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| Question Number | Mark Scheme Details | | Part Mark | Total |
|---|---|---|---|---|
| **1** | Find 3 independent equations for $T$, $R_A$, $R_B$: | | | |
| | Resolve horizontally: | $R_B = T\cos\alpha$ **M1 A1** | | |
| | Resolve vertically: | $R_A = W + T\sin\alpha$ **M1 A1** | | |
| | Take moments about $A$:<br>($a$ may be omitted from moment eqns) | $R_B\,3a\sin\theta = W(3a/2)\cos\theta$<br>$+ T\,a(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$<br>*or* $+ T\,a\sin(\alpha+\theta)$<br>*or* $+ T\,3a\cos\theta\sin\alpha$ **M1 A1** | | |
| | Take moments about $B$: | $R_A 3a\cos\theta = W(3a/2)\cos\theta$<br>$+ T\,2a(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$<br>*or* $+ T\,2a\sin(\alpha+\theta)$<br>*or* $+ T\,3a\sin\theta\cos\alpha$ **(M1 A1)** | | |
| | Take moments about $C$: | $R_A\,a\cos\theta + W(a/2)\cos\theta$<br>$= R_B\,2a\sin\theta$ **(M1 A1)** | | |
| | Take moments about $D$: | $R_A\,3a\cos\theta - W(3a/2)\cos\theta$<br>$= R_B\,3a\sin\theta$ **(M1 A1)** | | |
| | Solve for $T$, $R_A$, $R_B$ (AEF in $W$ and $\alpha$): | $T = W/2\sin\alpha$ *or* $\frac{1}{2}W\operatorname{cosec}\alpha$ **B1**<br>$R_A = 3W/2$ **B1**<br>$R_B = W/2\tan\alpha$ *or* $\frac{1}{2}W\cot\alpha$ **B1** | 9 | **9** |
| **2** | For $A$ & $B$ use conservation of momentum, | e.g.:$2mv_A + mv_B = 2mu$<br>(allow $2v_A + v_B = 2u$) **M1** | | |
| | Use Newton's law of restitution (consistent signs): | $v_B - v_A = eu$ **M1** | | |
| | Combine to find $v_A$ and $v_B$: | $v_A = (2-e)\,u/3$, $v_B = 2(1+e)\,u/3$ **A1, A1** | 4 | |
| | Find $e$ from $v_A = \lvert v_B' \rvert$ with $v_B' = [-]\,0{\cdot}4\,v_B$: | $(2-e) = 0{\cdot}8\,(1+e)$, $e = 2/3$ **M1 A1** | 2 | |
| | *EITHER:* Equate times in terms of reqd. distance $x$:<br>[speeds need not be found:<br>Use $v_A = v_B' = 0{\cdot}4\,v_B$ to solve for $x$: | $(d-x)/v_A = d/v_B + x/v_B'$ (AEF) **M1 A1**<br>$v_A = v_B' = 4u/9$, $v_B = 10u/9$]<br>$d - x = 0{\cdot}4\,d + x$, $x = 0{\cdot}3\,d$ **M1 A1** | | |
| | *OR:* Find dist. moved by $A$ when $B$ reaches wall:<br>Find reqd. distance $x$: | $d_A = (d/v_B)\,v_A = 0{\cdot}4\,d$ **(M1 A1)**<br>$x = \frac{1}{2}(d - d_A) = 0{\cdot}3\,d$ **(M1 A1)** | 4 | **10** |

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|---|---|---|---|---|---|
| **3** | Find $k$ by equating equilibrium tensions:<br>(vertical motion can earn M1 only) | $mg\,(a/2)/a = 2mg\,(3a/2 - ka) / ka$<br>$½ = 3/k - 2, \quad k = 6/5$ *or* 1·2 | **M1 A1**<br>**A1** | 3 | |
| | Apply Newton's law at general point, e.g.:<br>(lose A1 for each incorrect term) | $m\,\mathrm{d}^2x/\mathrm{d}t^2 = -mg\,(a/2 + x)/a$<br>$+ 2mg\,(3a/2 - ka - x) / ka$<br>*or* $m\,\mathrm{d}^2y/\mathrm{d}t^2 = +mg\,(a/2 - y)/a$<br>$- 2mg\,(3a/2 - ka + y) / ka$ | **M1 A2** | | |
| | Simplify to give standard SHM eqn, e.g.:<br>**S.R.**: B1 if no derivation (max 2/5) | $\mathrm{d}^2x/\mathrm{d}t^2 = -(1 + 2/k)gx / a$<br>$= -8gx/3a$ | **A1** | | |
| | State or find period using $2\pi/\omega$ with $\omega = \sqrt{(8g/3a)}$:<br>($\sqrt{}$ on $\omega$) | $T = 2\pi\sqrt{(3a/8g)}$ *or* $\pi\sqrt{(3a/2g)}$<br>*or* $3{\cdot}85\sqrt{(a/g)}$ *or* $1{\cdot}22\sqrt{a}$ [s] | **B1**√ | 5 | |
| | Substitute values in $v^2 = \omega^2(x_0^2 - x^2)$: | $0{\cdot}7^2 = (8g/3a)\{(0{\cdot}2a)^2 - (0{\cdot}05a)^2\}$ | **M1 A1** | | |
| | Solve to find numerical value of $a$: | $0{\cdot}49 = (8g/3) \times 0{\cdot}0375a, \; a = 0{\cdot}49$ | **A1** | 3 | **11** |

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| **4** | *EITHER:* Find tension at top from $F = ma$ vertically: $T = mu^2/a - mg$ **B1** | | |
| | *OR:* Use energy at e.g. $\theta$ to upward vertical: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mga(1 - \cos\theta)$ | | |
| | Find tension $T$ by using $F = ma$ radially: $T' = mv^2/a - mg\cos\theta$ | | |
| | Eliminate $v^2$: $= mu^2/a + mg(2 - 3\cos\theta)$ | | |
| | Find $T$ at top by taking $\theta = 0$: $T = mu^2/a - mg$ **(B1)** | | |
| | Find $u_{min}$ by requiring $T \geqslant 0$ at top [or $T > 0$]: $u^2/a - g \geqslant 0$ so $u_{min} = \sqrt{ag}$ **A.G.** **B1** | 2 | |
| | Find $v$ at bottom from conservation of energy: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mg \times 2a$ **M1** | | |
| | $v^2 = ag + 4ag$, $v = \sqrt{(5ag)}$ **A1** | | |
| | Find new speed $V$ from conservation of momentum: $m'V = mv$ with $m' = m + \frac{1}{4}m$ **M1** | | |
| | $V = 4v/5 = 4\sqrt{(ag/5)}$ *or* $(4/5)\sqrt{(5ag)}$ AEF **A1** | 4 | |
| | Find $w^2$ at angle $\theta$ from conservation of energy: $\frac{1}{2}m'w^2 = \frac{1}{2}m'V^2$ (condone $m$ instead of $m'$ here since cancels out) | | |
| | $- m'ga(1 + \cos\theta)$ **M1 A1** | | |
| | $[w^2 = ag(6/5 - 2\cos\theta)]$ | | |
| | **S.R.** Invalid energy method (max 2/5): $\frac{1}{2}m'w^2 = \frac{1}{2}mu^2$ | | |
| | [gives $T' = (5mg/4)(2 - 3\cos\theta)$ ] $+ mga(1 - \cos\theta)$ | | |
| | $- \frac{1}{4}mga(1 + \cos\theta)$ **(B1)** | | |
| | Find tension $T'$ there by using $F = ma$ radially: $T' = m'w^2/a - m'g\cos\theta$ **B1** | | |
| | Eliminate $w^2$: $= m'V^2/a - m'g(2 + 3\cos\theta)$ **A1** | | |
| | Substitute for $m'$ and $V$: $= (5mg/4)(6/5 - 3\cos\theta)$ | 5 | |
| | AEF *or* $3mg/2 - (15/4)mg\cos\theta$ **A1** | | |
| | Find $\cos\theta$ when string becomes slack from $T' = 0$: $\cos\theta = \frac{1}{3} \times 6/5 = 2/5$ *or* 0·4 **M1 A1** | 2 | **13** |
| | **S.R.** Allow if found from $T' = mg(6/5 - 3\cos\theta)$ | | |
| **5** | Find or use sample mean $\bar{x} = 222{\cdot}8 / 10 = 22{\cdot}28$ | | |
| | and estimate population variance: $s^2 = 4{\cdot}12 / 9$ | | |
| | (allow biased here: 0·412 *or* $0{\cdot}642^2$) $= 0{\cdot}458$ *or* $103/225$ *or* $0{\cdot}677^2$ **M1** | | |
| | Find confidence interval (allow $z$ in place of $t$) e.g.: $22{\cdot}28 \pm t\sqrt{(0{\cdot}458 / 10)}$ **M1 A1** | | |
| | Use of correct tabular value: $t_{9,\,0.975} = 2{\cdot}26$[2] **A1** | | |
| | Evaluate C.I. correct to 3 s.f.: $22{\cdot}3 \pm 0{\cdot}48$[4] *or* [21·8, 22·8] **A1** | 5 | **5** |

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| **6** | Find prob. $p$ of head from mean = 2 × variance: | $1/p = 2 \times (1-p)/p^2$, $p = ⅔$ **A.G.** | **M1 A1** | 2 | |
| **(i)** | Find P($X = 4$) (denoting $1 - p$ by $q$ [= ⅓]): | P($X = 4$) = $q^3 \times p$<br>= 2/81 *or* 0·0247 | **B1** | 1 | |
| **(ii)** | Find or state P($X > 4$): | P($X > 4$) [= $1 - (1 + q + q^2 + q^3) \times p$<br>= $1 - (1 - q^4)$ ] = $q^4$<br>= 1/81 *or* 0·0123 | **M1 A1** | 2 | |
| **(iii)** | Formulate condition for $N$:<br>Take logs (any base) to give bound for $N$:<br>Find $N_{min}$:<br>($N < 6{\cdot}29$ *or* $N = 6{\cdot}29$ earns M2 A0) | $1 - q^N > 0{\cdot}999$, [ $(⅓)^N < 0{\cdot}001$ ]<br>$N >$ log 0·001 / log ⅓<br>$N > 6{\cdot}29$, $N_{min} = 7$ | **M1**<br>**M1**<br>**A1** | 3 | **8** |
| **7** | Find F($x$) for $1 \leqslant x \leqslant 4$: | F($x$) = $(x^3 - 1)/63$ | **B1** | | |
| | Find G($y$) from $Y = X^2$ for $1 \leqslant x \leqslant 4$:<br>(result may be stated) | G($y$) = P($Y < y$) = P($X^2 < y$)<br>= P($X < y^{1/2}$) = F($y^{1/2}$)<br>= $(y^{3/2} - 1)/63$ | **M1 A1** | | |
| | Find g($y$) for corresponding range of y: | g($y$) = $y^{1/2}/42$ **A.G.** | **A1** | | |
| | Find or state corresponding range of $y$ : | $1 \leqslant y \leqslant 16$ **A.G.** | **B1** | 5 | |
| **(i)** | Find median value $m$ of $Y$: | $(m^{3/2} - 1)/63 = ½$<br>$m = 32{\cdot}5^{2/3} = 10{\cdot}2$ | **M1 A1** | 2 | |
| **(ii)** | Find E($Y$) [or equivalently E($X^2$)]: | E($Y$) = $\int y\, g(y)\, dy = \int y^{3/2}\, dy /42$<br>= $[y^{5/2}]_1^{16}/105 = 1023/105$<br>= 341/35 *or* 9·74 | **M1 A1** | 2 | **9** |

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| **8** | Find mean of sample data [for use in Poisson distn.]: | $\lambda = 220/100 = 2{\cdot}2$ **B1** | | |
| | State (at least) null hypothesis (AEF): | $H_0$: Poisson distn. fits data<br>*or* $\lambda = 2{\cdot}2$ **B1** | | |
| | Find expected values $100\lambda^r e^{-\lambda}/r!$ (to 1 d.p.):<br>(ignore incorrect final value here for M1) | 11·080 24·377 26·814 19·664<br>10·8151 4·759 2·491 **M1 A1** | | |
| | Combine last two cells so that exp. value $\geqslant 5$: | $O_i$: 3<br>$E_i$: 7·25 **M1*** | | |
| | Calculate value of $\chi^2$ (to 2 d.p.; A1 dep M1*):<br>(allow 7·95 if 1 d.p. exp.values used) | $\chi^2 = 0{\cdot}076 + 2{\cdot}879 + 0{\cdot}653 + 1{\cdot}448 + 0{\cdot}441 + 2{\cdot}491$<br>$= 7{\cdot}99$ **M1 A1** | | |
| | State or use consistent tabular value (to 3 s.f.): | 5 cells: $\chi^2_{3,0.95} = 7{\cdot}815$<br>6 cells: $\chi^2_{4,0.95} = 9{\cdot}488$ (correct)<br>7 cells: $\chi^2_{5,0.95} = 11{\cdot}07$ **B1** | | |
| | State or imply valid method for conclusion e.g.: | Accept $H_0$ if $\chi^2 <$ tabular value **M1** | | |
| | Conclusion (AEF, requires both values correct):<br>Not combining cells [so $\chi^2 = 8{\cdot}64$] can earn B1 B1 M1 A1 M0 M1 B1 M1 (max 7) | Distn fits *or* $\lambda = 2{\cdot}2$ **A1** | 10 | **10** |
| **9** | Calculate gradient $b_1$ in $y - \bar{y} = b_1(x - \bar{x})$: | $S_{xy} = 24\,879 - 472 \times 400/8$<br>$= 1\,279$<br>$S_{xx} = 29\,950 - 472^2/8 = 2\,102$<br>$b_1 = S_{xy} / S_{xx} = 0{\cdot}6085$ (3 s.f.) **M1 A1** | | |
| | Find regression line of $y$ on $x$: | $y = 400/8 + b_1 (x - 472/8)$ **M1 A1**<br>$= 50 + 0{\cdot}6085 (x - 59)$<br>$= 0{\cdot}6085x + 14{\cdot}1$ | | |
| | Find $y$ when $x = 72$:<br>Allow use of regression line of $x$ on $y$<br>(since neither variable clearly independent): | $= 57{\cdot}9$ *or* 58<br>$S_{yy} = 21\,226 - 400^2/8 = 1\,226$<br>$b_2 = S_{xy} / S_{yy} = 1{\cdot}043$ **(M1 A1)**<br>$x = 472/8 + b_2 (y - 400/8)$ **(M1 A1)**<br>$= 1{\cdot}043\, y + 6{\cdot}85$<br>$Y = 62{\cdot}5$ *or* 62 **(A1)** | 5 | |

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| | Find product moment correlation coefficient $r$: | $r = 1\,279 / \sqrt{(2\,102 \times 1\,226)}$ *or* $\sqrt{(0{\cdot}6085 \times 1{\cdot}043)} = 0{\cdot}797$ **M1 A1*** | | |
| | State both hypotheses (B0 for $r$ …): | $H_0$: $\rho = 0$, $H_1$: $\rho \neq 0$ **B1** | | |
| | State or use correct tabular two-tail $r$-value: | $r_{8,\,5\%} = 0{\cdot}707$ **B1*** | | |
| | State or imply valid method for conclusion e.g.: | Reject $H_0$ if $\lvert r\rvert >$ tab. value (AEF) **M1** | | |
| | Correct conclusion (AEF, dep A1*, B1*): | There is non-zero correlation **A1** | 6 | **11** |
| **10A** | Find MI of lamina about $Q$: | $I_{\text{lamina}} = \frac{1}{3}m\{(3a)^2 + (3a/2)^2\} + m(9a/2)^2$ **M1 A1** $[= (15/4 + 81/4)\,ma^2 = 24\,ma^2]$ | | |
| | State or find MI of rod about $Q$: | $I_{\text{rod}} = (\frac{1}{3} + 1)\,M(3a/2)^2$ $[= 3Ma^2]$ **B1** | | |
| | Sum to find MI of object about $Q$: | $I_1 = 24\,ma^2 + 3\,Ma^2$ $= 3\,(8m + M)\,a^2$ **A.G.** **A1** | 4 | |
| | Find MI of object about mid-point of $PQ$: | $I_2 = (15/4 + 3^2)\,ma^2 + \frac{1}{3}\,M(3a/2)^2$ $= (51/4)\,ma^2 + \frac{3}{4}\,Ma^2$ $= \frac{3}{4}\,(17m + M)\,a^2$ **A.G.** **M1 A1** | 2 | |
| | Use eqn of circular motion to find $d^2\theta/dt^2$ for axis $l_1$: | $[-]I_1\,d^2\theta/dt^2 = mg \times (9a/2)\sin\theta + Mg \times (3a/2)\sin\theta$ $[= (9m/2 + 3M/2)\,ga\sin\theta]$ **M1 A1** | | |
| | [Approximate sin $\theta$ by $\theta$ and] find $\omega_1^2$ in SHM eqn: | $\omega_1^2 = (3m + M)g / 2(8m + M)\,a$ **M1** | | |
| | Find period $T_1$ for axis $l_1$ from $2\pi/\omega_1$: (AEF) | $T_1 = 2\pi\sqrt{\{2(8m + M)\,a / (3m + M)g\}}$ **A1** | | |
| | Use eqn of circular motion to find $d^2\theta/dt^2$ for axis $l_2$: | $[-]I_2\,d^2\theta/dt^2 = mg \times 3a\sin\theta$ **M1** | | |
| | [Approximate sin $\theta$ by $\theta$ and] find $\omega_2^2$ in SHM eqn: | $\omega_2^2 = 4mg / (17m + M)\,a$ **M1** | | |
| | Find period $T_2$ for axis $l_2$ from $2\pi/\omega_2$: (AEF) | $T_2 = 2\pi\sqrt{\{(17m + M)\,a / 4mg\}}$ **A1** | | |
| | Verify that $T_1 = T_2$ when $m = M$: (AEF) [Taking $m = M$ throughout 2nd part can earn: M1 A1 M1 A0 M1 M1 A0 B1 (max 6/8)] | $T_1 = 2\pi\sqrt{(18\,a / 4g)} = T_2$ **B1** | 8 | **14** |

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| **10B** | State hypotheses (B0 for $\bar{x}$ …), e.g.: | $H_0$: $\mu_X = \mu_Y$, $H_1$: $\mu_X \neq \mu_Y$ **B1** | | |
| | State assumption (AEF): | Distributions have equal variances **B1** | | |
| | Find sample means and estimate popln. variances: | $\bar{x} = 4{\cdot}2$, $\bar{y} = 4{\cdot}8$ | | |
| | (allow biased here: $0{\cdot}36$ *or* $0{\cdot}6^2$) | $s_X^2 = (180 - 42^2/10) / 9$ = $0{\cdot}4$ *or* $0{\cdot}6325^2$ | | |
| | (allow biased here: $0{\cdot}4183$ *or* $0{\cdot}6468^2$) | $s_Y^2 = (281{\cdot}5 - 57{\cdot}6^2/12) / 11$ = $0{\cdot}4564$ *or* $251/550$ *or* $0{\cdot}6755^2$ **M1** | | |
| | Estimate (pooled) common variance: (note $s_X^2$ and $s_Y^2$ not needed explicitly) | $s^2 = (9\,s_X^2 + 11\,s_Y^2) / 20$ (AEF) *or* $(180 - 42^2/10 + 281{\cdot}5 - 57{\cdot}6^2/12) / 20$ = $0{\cdot}431$ *or* $0{\cdot}6565^2$ **M1 A1** | | |
| | Calculate value of $t$ (to 3 s.f.): | [-] $t = (\bar{y} - \bar{x}) / s\sqrt{(1/10 + 1/12)}$ = $2{\cdot}13$ **M1 A1** | | |
| | State or use correct tabular $t$ value: (or can compare $\bar{y} - \bar{x} = 0{\cdot}6$ with $0{\cdot}586$) | $t_{20,\,0.975} = 2{\cdot}086$ [allow $2{\cdot}09$] **B1*** | | |
| | Correct conclusion (AEF, √ on $t$, dep *B1): | $t > 2{\cdot}09$ so mean masses not same **B1√** | 9 | |
| | **S.R.** Implicitly taking $s_X^2$, $s_Y^2$ as popln. variances: (may also earn first B1 B1 M1) | $z = (\bar{y} - \bar{x}) / \sqrt{(s_X^2/10 + s_Y^2/12)}$ = $0{\cdot}6 / \sqrt{(0{\cdot}078} = 2{\cdot}15$ | | |
| | Comparison with $z_{0.975}$ and conclusion: (can earn at most 5/9) | $2{\cdot}15 > 1{\cdot}96$ so mean masses not same **(B1)** | | |
| | State hypotheses (B0 for $\bar{x}$ …), e.g.: | $H_0$: $\mu_X = 3{\cdot}8$, $H_1$: $\mu_X > 3{\cdot}8$ *or* $H_0$: $\mu_X = \mu_Z$, $H_1$: $\mu_X > \mu_Z$ **B1** | | |
| | Calculate value of $t$ using $s_X$ from above: | $t = (4{\cdot}2 - 3{\cdot}8) / (s_X/\sqrt{10}) = 2{\cdot}0$ **M1 A1** | | |
| | State or use correct tabular $t$ value: (or can compare $0{\cdot}4$ with $0{\cdot}367$) | $t_{9,\,0.95} = 1{\cdot}833$ [allow $1{\cdot}83$] **B1*** | | |
| | Correct conclusion (A.E.F., √ on $t$, dep *B1): | $t > 1{\cdot}833$, so claim is justified *or* mean mass of Royals > mean mass of Crowns **B1√** | 5 | **14** |