

Differentiation

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Read first: [Logarithmic and exponential functions](#), [Trigonometry](#)

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What you need to know

By the end of this topic you should be able to:

1. **Explain what a derived function is.** The gradient of a curve changes from point to point; the derived function (the derivative) gives that gradient for every x . You need the idea of a limit only informally: you will **not** be asked to differentiate from first principles.
2. **Read and write the notation:** $f'(x)$ and $f''(x)$; $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, where $\frac{d^2y}{dx^2}$ means $\frac{d}{dx} \left(\frac{dy}{dx} \right)$; and δx , δy for small changes, with $\frac{\delta y}{\delta x} \rightarrow \frac{dy}{dx}$ as $\delta x \rightarrow 0$.
3. **Differentiate the standard functions** x^n (for any rational n), $\sin x$, $\cos x$, $\tan x$, e^x and $\ln x$, together with constant multiples, sums, and functions of a function (the **chain rule**), such as $(3x^2 + 1)^5$, $\sin 4x$ or e^{2x-1} . Angles are always in radians.
4. **Differentiate products and quotients** of these functions with the **product rule** and the **quotient rule**.

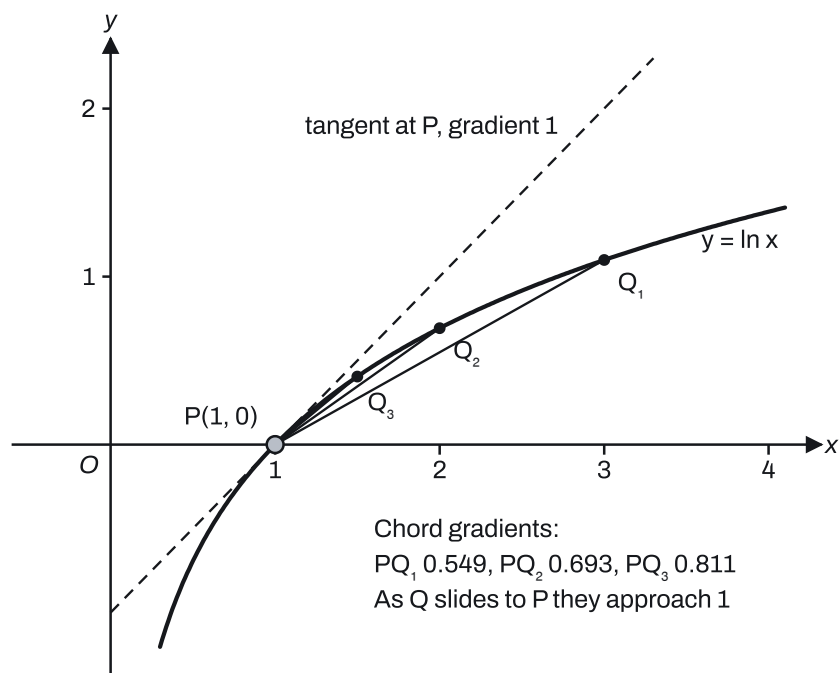
Exam questions on this topic often go on to use the derivative: to find a stationary point, a tangent or normal, a small change or a rate of change, or to integrate "hence". Those methods belong to Applications of differentiation and Integration; "How to do it" (type 1 and type 4) gives the short version you need to finish these questions.

Understand it

The gradient of a curve

A straight line has one gradient. A curve does not: it is steep in some places and flat in others. The **gradient of a curve at a point P** is the gradient of the **tangent** there, the line that touches the curve at P and runs in the same direction as it.

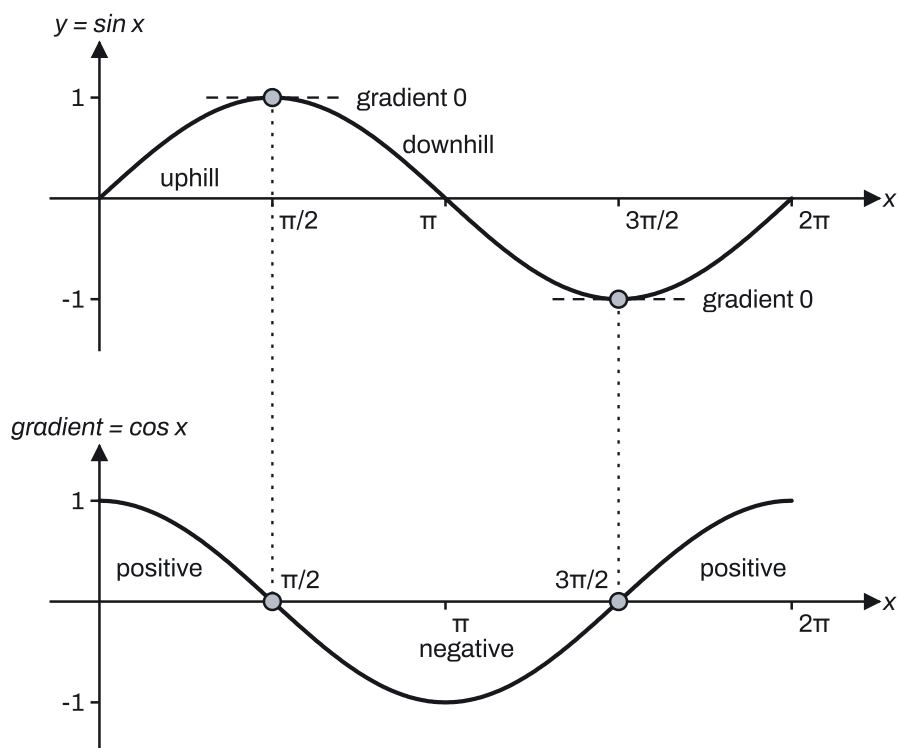
You can't find the tangent's gradient from two points, because you only know one point on it. So take a second point Q on the curve, a small step δx along from P , and find the gradient of the chord PQ : $\frac{\delta y}{\delta x}$. Now slide Q towards P . The chords turn towards the tangent, and their gradients settle on one number: that number is the gradient at P . In symbols, $\frac{\delta y}{\delta x} \rightarrow \frac{dy}{dx}$ as $\delta x \rightarrow 0$.



On $y = \ln x$ with $P(1, 0)$, the chord gradients are 0.549, 0.693, 0.811 and, with Q at $x = 1.1$, 0.953. They are heading for 1, and indeed the gradient of $\ln x$ at $x = 1$ is $\frac{1}{1} = 1$.

The derived function

Doing this at every point gives a new function, the **derived function** (or derivative), written $\frac{dy}{dx}$ or $f'(x)$. Put in a value of x and it gives the gradient of the curve there. $\frac{dy}{dx}$ is one symbol meaning "the rate at which y changes as x changes"; it is not a fraction you can cancel.



The picture shows why the derivative of $\sin x$ is $\cos x$. Where $\sin x$ climbs, the gradient is positive; at its peak ($x = \frac{\pi}{2}$) and trough ($x = \frac{3\pi}{2}$) the tangent is flat, so the gradient is 0; where it falls, the gradient is negative. The gradient is steepest at $x = 0, \pi$ and 2π , with values 1, -1 and 1. That is exactly the graph of $\cos x$. This only works when x is in **radians**, which is why calculus always uses radians.

Differentiating a second time gives the **second derivative**, $\frac{d^2y}{dx^2}$ or $f''(x)$: the rate at which the gradient itself changes. For example, $y = x^3 - 4e^{2x}$ gives $\frac{dy}{dx} = 3x^2 - 8e^{2x}$ and $\frac{d^2y}{dx^2} = 6x - 16e^{2x}$.

The standard derivatives

You must know these six results by heart; none of them is printed in the exam.

y	$\frac{dy}{dx}$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
e^x	e^x
$\ln x$	$\frac{1}{x}$

A constant multiple stays in front, a sum is differentiated term by term, and a constant on its own gives 0.

Rewrite roots and fractions as powers first: $5x^4 - \frac{6}{x} + 2\sqrt{x} = 5x^4 - 6x^{-1} + 2x^{1/2}$ differentiates to $20x^3 + 6x^{-2} + x^{-1/2}$.

The chain rule: a function of a function

$(x^2 + 5)^6$ is built in two layers: first work out the inside, $u = x^2 + 5$, then apply the outside, u^6 . Each layer changes at its own rate, and the rates multiply:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

In practice: **differentiate the outside as if the inside were a single letter, then multiply by the derivative of the inside.**

- $(x^2 + 5)^6 \rightarrow 6(x^2 + 5)^5 \times 2x = 12x(x^2 + 5)^5$
- $\sin 4x \rightarrow \cos 4x \times 4 = 4 \cos 4x$, and $\cos \frac{x}{2} \rightarrow -\frac{1}{2} \sin \frac{x}{2}$
- $\tan(3x - 1) \rightarrow 3 \sec^2(3x - 1)$
- $e^{5-2x} \rightarrow -2e^{5-2x}$, and $e^{x^2+1} \rightarrow 2xe^{x^2+1}$: e^{inside} **stays the same and is multiplied by the inside's derivative**
- $\ln(3x - 2) \rightarrow \frac{3}{3x - 2}$, and $\ln(x^2 + 4) \rightarrow \frac{2x}{x^2 + 4}$: $\ln(\text{inside})$ **gives** $\frac{\text{inside}'}{\text{inside}}$
- $\sin^3 x = (\sin x)^3 \rightarrow 3(\sin x)^2 \cos x = 3 \sin^2 x \cos x$

Two shortcuts with logarithms. $\ln(kx) = \ln k + \ln x$, so its derivative is just $\frac{1}{x}$ (for example, $\ln 7x \rightarrow \frac{1}{x}$). And use the laws of logarithms **before** differentiating a log of a product, quotient or power: $\ln((2x-1)^3\sqrt{x}) = 3\ln(2x-1) + \frac{1}{2}\ln x$, which differentiates to $\frac{6}{2x-1} + \frac{1}{2x}$. Never split $\ln(a+b)$: it is **not** $\ln a + \ln b$.

The product rule

When $y = uv$, with u and v both functions of x , both parts change, and each contributes:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

"First times derivative of second, plus second times derivative of first." For $y = x^2 \cos x$: $u = x^2$, $v = \cos x$, so $\frac{dy}{dx} = x^2(-\sin x) + \cos x(2x) = 2x \cos x - x^2 \sin x$.

For $y = e^{3x} \cos x$: $\frac{dy}{dx} = e^{3x}(-\sin x) + \cos x(3e^{3x}) = e^{3x}(3 \cos x - \sin x)$. Taking out the common factor e^{3x} makes the answer easier to use.

The quotient rule

When $y = \frac{u}{v}$:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

The order matters because of the minus sign: **bottom times derivative of top, minus top times derivative of bottom, all over bottom squared**. For $y = \frac{x+3}{x^2+1}$:

$$\frac{dy}{dx} = \frac{(x^2+1)(1) - (x+3)(2x)}{(x^2+1)^2} = \frac{1-6x-x^2}{(x^2+1)^2}.$$

You may instead write a quotient as a product, $u \times v^{-1}$, and use the product and chain rules; both earn full marks. But if the top is just a number, don't use either rule: $\frac{3}{(4x-1)^2} = 3(4x-1)^{-2}$ needs only the chain rule.

Key facts and formulas

The syllabus says no calculus formulas are given in the List of formulas, so every result here is "Learn it".

Formula	Status
$\frac{d}{dx}(x^n) = nx^{n-1}$, for any rational n	Learn it
$\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(\tan x) = \sec^2 x$ (x in radians)	Learn it

Formula	Status
$\frac{d}{dx}(e^x) = e^x, \frac{d}{dx}(\ln x) = \frac{1}{x}$	Learn it
Constant multiples and sums: $\frac{d}{dx}(kf(x) + g(x)) = kf'(x) + g'(x)$; a constant gives 0	Learn it
Chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	Learn it
$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}, \frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$	Learn it
$\frac{d}{dx}\sin(ax + b) = a\cos(ax + b), \frac{d}{dx}\cos(ax + b) = -a\sin(ax + b),$ $\frac{d}{dx}\tan(ax + b) = a\sec^2(ax + b)$	Learn it
Product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	Learn it
Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	Learn it
$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$	Learn it
Small changes: $\delta y \approx \frac{dy}{dx}\delta x$; rates: $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$	Learn it
Rewrite before differentiating: $\sqrt{x} = x^{1/2}, \frac{1}{x^n} = x^{-n}, \frac{k}{\sqrt{ax+b}} = k(ax+b)^{-1/2}, \sec^2 x = \frac{1}{\cos^2 x}$ and $\sec^2 x = 1 + \tan^2 x$ (from Trigonometry).	

How to do it

In the Paper 1 and Paper 2 questions we have tagged for this topic (53 questions, 2021–2025), the types came up this often. Most questions ask you to differentiate with one rule and then do something with the result, so 50 of the 53 mix two or more types and the counts add up to more than 53.

Question type	Questions
Using the derivative in a later part (stationary points, tangents, small changes, rates)	29
Differentiating a quotient	24
Differentiating a product	19
Differentiate, then "hence" integrate	19
Showing that $\frac{dy}{dx}$ can be written in a given form	18
Chain rule and standard derivatives on their own	13

Question type	Questions
Exact value of $\frac{dy}{dx}$ at a point	6
Equations involving the derivatives	4

1. Using the derivative in a later part (29 questions)

The first part asks for $\frac{dy}{dx}$ (often "show that"); the later parts use it. Always use the given form from a "show that" part, even if you didn't manage to reach it.

- **Stationary points:** set $\frac{dy}{dx} = 0$. A factor $e^{\dots}$ is never 0, and a denominator can't make a fraction 0, so only the numerator's other factors matter. For $\frac{dy}{dx} = (4 - x)e^{-x/3}$, the only solution is $x = 4$. Find y by substituting into the **curve's equation**, exactly if asked. Reject any x outside the domain (for example one that makes a logarithm's inside negative), and say why.
- **Tangent or normal at a point:** gradient m from $\frac{dy}{dx}$, then $y - y_1 = m(x - x_1)$; for the normal use $-\frac{1}{m}$.
- **Small changes:** " x increases from a to $a + h$, where h is small" means $\delta y \approx \frac{dy}{dx} \times h$, with $\frac{dy}{dx}$ worked out at $x = a$. For $y = x^2 + \frac{4}{x}$ at $x = 2$: $\frac{dy}{dx} = 2x - \frac{4}{x^2} = 3$, so $\delta y \approx 3h$. If you are given δy and asked for the change in x , use $\delta x \approx \delta y \div \frac{dy}{dx}$.
- **Rates of change:** $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$. If y is increasing at 6 units per second at a point where $\frac{dy}{dx} = 3$, then $\frac{dx}{dt} = 6 \div 3 = 2$ units per second. "Decreasing" means a negative rate. A negative answer for $\frac{dx}{dt}$ means x is decreasing.
- **Velocity:** if s is a displacement in terms of t , the velocity is $\frac{ds}{dt}$; "never at rest" means showing $\frac{ds}{dt}$ can never be 0.
- **Nature of a stationary point:** check the sign of $\frac{dy}{dx}$ just either side (for a maximum it goes $+$, 0 , $-$; for a minimum $-$, 0 , $+$), or find $\frac{d^2y}{dx^2}$ there (negative means maximum, positive means minimum).

2. Differentiating a quotient (24 questions)

1. Name the parts: u = top, v = bottom. Write down $\frac{du}{dx}$ and $\frac{dv}{dx}$ separately first, using the chain rule where needed.
2. Substitute into $\frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$ with **brackets round every part**.
3. Simplify only as far as the question needs. "Find $\frac{dy}{dx}$ " accepts an unsimplified correct answer; a later part may need it tidier.

Variations you will meet:

- **A logarithm on top**, such as $\frac{\ln(1 + x^3)}{\sqrt{x}}$: the top's derivative is $\frac{3x^2}{1+x^3}$, so the first term of the numerator is $\sqrt{x} \times \frac{3x^2}{1+x^3}$. Leaving a fraction inside the big fraction is fine.
- **An exponential on top**, such as $\frac{e^{4x}}{\cos x}$: $\frac{du}{dx} = 4e^{4x}$; take out e^{4x} as a common factor at the end.
- **A trigonometric quotient**, such as $\frac{\sin x}{2 + \cos x}$ (Example 3): look for $\sin^2 x + \cos^2 x = 1$ in the numerator.

- **Products inside a quotient**, such as $\frac{x^2 \sin x}{e^x + 1}$: the top needs the product rule first ($u' = 2x \sin x + x^2 \cos x$), then the quotient rule.
- **A power of a quotient**, such as $\left(\frac{x+2}{x-3}\right)^3$: chain rule outside ($3(\dots)^2 \times$), quotient rule inside.

3. Differentiating a product (19 questions)

1. Name the parts: u and v . Write $\frac{du}{dx}$ and $\frac{dv}{dx}$ first.
2. $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$: **add**, never multiply the two derivatives.
3. Take out any common factor, especially if a later part solves $\frac{dy}{dx} = 0$.

Watch for the product that hides: $e^{2x} \ln x$, $x \ln(x+4)$, $\sqrt{x} \cos 3x$, $(2x-1) \tan x$ and $\tan x \cos 2x$ are all products. A number in front, like $5e^{2x}$, is **not** a product: it is a constant multiple. A product of two brackets like $x^2(x-1)^4$ can be done with the product rule; factorise the answer, $x(x-1)^3(2(x-1)+4x) = x(x-1)^3(6x-2)$.

4. Differentiate, then "hence" integrate (19 questions)

"Hence" means the integral comes from reversing your derivative, not from a new method. Integration is the reverse of differentiation: if $\frac{d}{dx} F(x) = f(x)$, then $\int f(x) dx = F(x) + c$.

1. Write the result of the first part as an integral: $\int (\text{your } \frac{dy}{dx}) dx = y + c$.
2. Split it so that the integral you want stands alone. Move the other terms across; they are usually easy to integrate.
3. Adjust for any constant multiple, add $+c$ for an indefinite integral, and substitute limits if there are any.

For example, $\frac{d}{dx}(\cos^4 x) = 4 \cos^3 x \times (-\sin x) = -4 \cos^3 x \sin x$. Reversing, and dividing by -4 :

$$\int \cos^3 x \sin x dx = -\frac{1}{4} \cos^4 x + c.$$

When your derivative has two terms (as a product-rule answer does), the integral you want is one of them: write the other term on the other side and integrate it separately.

If the derivative is a multiple of the integrand, divide: when $\frac{d}{dx}(\ln(\dots)) = 2 \times (\text{integrand})$, the integral is $\frac{1}{2} \ln(\dots) + c$. An identity can help: since $\frac{d}{dx}(\tan x) = \sec^2 x = 1 + \tan^2 x$, a derivative in terms of $\tan x$ can be reversed to integrate $\tan^2$ expressions.

5. Showing that $\frac{dy}{dx}$ can be written in a given form (18 questions)

The printed answer tells you where you are going: work towards it and show every step.

1. Differentiate carefully (types 2, 3 and 6).
2. **Roots and negative powers**: multiply the top and bottom of the whole fraction by the same root, or take out the **lowest** power of each bracket as a common factor. For $y = \frac{\sqrt{x^2+9}}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{x}{\sqrt{x^2+9}} - \sqrt{x^2+9}}{x^2} = \frac{x^2 - (x^2+9)}{x^2\sqrt{x^2+9}} = \frac{-9}{x^2\sqrt{x^2+9}},$$

multiplying top and bottom by $\sqrt{x^2+9}$ in the second step.

3. **Common factors:** take out $e^{\dots}$, a power of a bracket, or a trig function, and collect what is left into a single bracket such as $(Ax + B)$.

4. Match your final line **exactly** to the given form, then state the constants: " $A = -3, B = -4$ ".

Sometimes it is easier to simplify y first. $\frac{x^2 - 6\sqrt{x}}{x}$ splits into $x - 6x^{-1/2}$, which needs only the power rule; and a log of a product or quotient splits by the log laws before you differentiate.

6. Chain rule and standard derivatives on their own (13 questions)

1. Rewrite as powers (roots, reciprocals) or split logs with the log laws.
2. Spot the inside and the outside. Differentiate the outside, keep the inside unchanged, multiply by the inside's derivative.
3. For more than two layers, keep multiplying: $\sin^2(3x) = (\sin 3x)^2 \rightarrow 2 \sin 3x \times \cos 3x \times 3 = 6 \sin 3x \cos 3x$.

Variations: "in terms of $\tan x$ " means using $\sec^2 x = 1 + \tan^2 x$ on your answer; "simplify" a log derivative such as $\frac{4x^3+2x}{x^4+x^2}$ means cancelling the common factor x in top and bottom, giving $\frac{4x^2+2}{x^3+x}$; a trig reciprocal such as $\frac{1}{\cos^3 x} = (\cos x)^{-3}$ is a chain rule job, not a quotient.

7. Exact value of $\frac{dy}{dx}$ at a point (6 questions)

1. Differentiate and **write the derivative down** before substituting; the value alone earns little.
2. Substitute the value of x and simplify **exactly**: use exact trig values at $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ (radians), $e^0 = 1, \ln 1 = 0$.
3. Put the answer in the requested form. For "in the form $\ln p$ ", use the log laws: $-\frac{2}{3} \ln 8 = \ln 8^{-2/3} = \ln \frac{1}{4}$.

On the non-calculator paper (Paper 1) show each simplification step.

8. Equations involving the derivatives (4 questions)

- **Solve an equation made from $\frac{dy}{dx}$** , usually a trigonometric one. For $y = 4 \sin x - \cos 2x$: $\frac{dy}{dx} = 4 \cos x + 2 \sin 2x = 4 \cos x + 4 \sin x \cos x = 4 \cos x(1 + \sin x)$. Setting it to 0 gives $\cos x = 0$ or $\sin x = -1$, so for $0 < x < 2\pi$, $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Use identities to reach one function, and give answers in the range asked for, in radians unless the question uses degrees.
- **Find constants from a relation that holds for all x** . Work out y , $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, substitute them into the relation, and match the coefficients on both sides. For $y = e^{kx}$ and $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$: $k^2 e^{kx} - 5k e^{kx} + 6e^{kx} = 0$, and $e^{kx} \neq 0$, so $k^2 - 5k + 6 = 0$ and $k = 2$ or 3 .
- **Show that a combination of y and its derivatives simplifies** to a given expression: substitute, put everything over a common denominator, and simplify towards the printed answer.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

Differentiate with respect to x :

(a) $\frac{6}{\sqrt[3]{1-4x}}$ **[2]**

(b) $e^{\sin 2x}$ **[2]**

(c) $\ln(\cos x)$, simplifying your answer. **[2]**

(a) Rewrite as a power: $6(1-4x)^{-1/3}$.

$$\frac{dy}{dx} = 6 \times \left(-\frac{1}{3}\right) (1-4x)^{-4/3} \times (-4) = 8(1-4x)^{-4/3}.$$

B1 for $k(1-4x)^{-4/3}$, **B1** for $k = 8$ (the $\times(-4)$ from the chain rule makes it positive).

(b) The inside is $\sin 2x$, with derivative $2 \cos 2x$:

$$\frac{dy}{dx} = 2 \cos 2x e^{\sin 2x}.$$

B1 for $e^{\sin 2x}$ multiplied by something, **B1** for $2 \cos 2x$.

(c) $\ln(\text{inside})$ gives $\frac{\text{inside}'}{\text{inside}}$:

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x.$$

B1 for $\frac{-\sin x}{\cos x}$, **B1** for $-\tan x$.

Example 2

A curve has equation $y = e^{-x} \sin 2x$.

(a) Show that the gradient of the curve at the point where $x = \frac{\pi}{4}$ is $-e^{-\pi/4}$. **[4]**

(b) Find the x -coordinate of the point with $0 < x < \frac{\pi}{2}$ where the gradient of the curve is 0. Give your answer correct to 3 significant figures. **[3]**

(a) Product rule with $u = e^{-x}$, $v = \sin 2x$: $\frac{du}{dx} = -e^{-x}$ and $\frac{dv}{dx} = 2 \cos 2x$. **B1** for both.

$$\frac{dy}{dx} = e^{-x}(2 \cos 2x) + \sin 2x(-e^{-x}) = e^{-x}(2 \cos 2x - \sin 2x).$$

M1 for the product rule with the right structure, **A1** for a correct derivative.

At $x = \frac{\pi}{4}$: $\cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$, so

$$\frac{dy}{dx} = e^{-\pi/4}(0 - 1) = -e^{-\pi/4}.$$

A1, with the exact values shown (a "show that" needs the working).

(b) $e^{-x} > 0$ for every x , so the gradient is 0 only when $2 \cos 2x - \sin 2x = 0$. **M1** for setting the bracket to 0 and dismissing e^{-x} .

$$\sin 2x = 2 \cos 2x \Rightarrow \tan 2x = 2.$$

M1 for reaching $\tan 2x = 2$ and taking $\tan^{-1}$ in radians.

$0 < x < \frac{\pi}{2}$ means $0 < 2x < \pi$, and there $\tan 2x = 2$ only at $2x = 1.1071$, so $x = 0.554$. **A1**

Example 3

A curve has equation $y = \frac{\sin x}{2 + \cos x}$.

(a) Show that $\frac{dy}{dx} = \frac{A \cos x + B}{(2 + \cos x)^2}$, where A and B are integers to be found. **[4]**

(b) Find the exact values of x , for $0 \leq x \leq 2\pi$, at which the gradient of the curve is 0. **[2]**

(c) A point moves along the curve so that its x -coordinate increases at a constant rate of 0.3 units per second. Find the rate of change of its y -coordinate when $x = \frac{\pi}{3}$. **[2]**

(a) $u = \sin x$, $v = 2 + \cos x$, so $\frac{du}{dx} = \cos x$ and $\frac{dv}{dx} = -\sin x$. **B1**

$$\frac{dy}{dx} = \frac{(2 + \cos x) \cos x - \sin x(-\sin x)}{(2 + \cos x)^2}$$

M1 for the quotient rule with the right structure.

$$= \frac{2 \cos x + \cos^2 x + \sin^2 x}{(2 + \cos x)^2} = \frac{2 \cos x + 1}{(2 + \cos x)^2},$$

using $\cos^2 x + \sin^2 x = 1$. **A1** for the correct derivative, **A1** for $A = 2$, $B = 1$ with the identity shown.

(b) The denominator is never 0 (it is at least 1), so $2 \cos x + 1 = 0$, $\cos x = -\frac{1}{2}$. **M1**

$$x = \frac{2\pi}{3} \text{ or } x = \frac{4\pi}{3}. \text{ **A1**}$$

(c) At $x = \frac{\pi}{3}$: $\cos \frac{\pi}{3} = \frac{1}{2}$, so $\frac{dy}{dx} = \frac{2 \times \frac{1}{2} + 1}{\left(2\frac{1}{2}\right)^2} = \frac{2}{6.25} = 0.32$.

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = 0.32 \times 0.3 = 0.096 \text{ units per second.}$$

M1 for the chain rule with $\frac{dx}{dt} = 0.3$ and their gradient, **A1**

Example 4

It is given that $y = \ln \left(\frac{x^2}{\sqrt{4x+1}} \right)$, for $x > 0$.

(a) Show that $\frac{dy}{dx} = \frac{k(3x+1)}{x(4x+1)}$, where k is an integer to be found. **[4]**

(b) Explain why the curve has no stationary points. **[2]**

(a) Split with the log laws first:

$$y = \ln x^2 - \ln(4x+1)^{1/2} = 2 \ln x - \frac{1}{2} \ln(4x+1).$$

B1 for a correct split.

$$\frac{dy}{dx} = \frac{2}{x} - \frac{1}{2} \times \frac{4}{4x+1} = \frac{2}{x} - \frac{2}{4x+1}.$$

M1 for differentiating $\ln(4x+1)$ as $\frac{4}{4x+1}$ (or any multiple of $\frac{1}{4x+1}$), **A1** for the whole derivative.

$$= \frac{2(4x+1) - 2x}{x(4x+1)} = \frac{6x+2}{x(4x+1)} = \frac{2(3x+1)}{x(4x+1)}, \quad k = 2.$$

A1

(You could differentiate $\ln \left(\frac{x^2}{\sqrt{4x+1}} \right)$ with the chain and quotient rules instead, but the log laws make it much shorter.)

(b) A stationary point needs $\frac{dy}{dx} = 0$, so $3x+1 = 0$, giving $x = -\frac{1}{3}$. **M1** for setting the numerator to 0.

But the curve is defined only for $x > 0$, and there $3x+1 > 0$, $x > 0$ and $4x+1 > 0$, so $\frac{dy}{dx} > 0$ for every x in the domain. The gradient is never 0, so there are no stationary points. **A1** for rejecting $x = -\frac{1}{3}$ with the reason.

Common mistakes

- **Leaving out the chain rule.** $\sin 3x$ becomes $3 \cos 3x$, not $\cos 3x$; e^{4x} becomes $4e^{4x}$, not e^{4x} or $4xe^{3x}$ (the power of e never drops by one). Examiners see these every year, and the error then spoils every later part.
- **Getting the quotient rule backwards.** The numerator is $v \frac{du}{dx} - u \frac{dv}{dx}$; swap them and every sign changes. Also don't forget to square the bottom, and never differentiate top and bottom separately.

- **Missing a product.** $\sqrt{x} e^{2x}$ or $x^2 \sin 3x$ needs the product rule; differentiating each factor and multiplying is wrong, and so is subtracting the two terms instead of adding. A product inside the numerator of a quotient needs its own product rule.
- **Missing brackets.** For $\frac{\ln(3x+1)}{x^2+4}$, write $(x^2 + 4) \left(\frac{3}{3x+1} \right) - \ln(3x + 1)(2x)$ with every bracket. Missing brackets are the main source of sign errors in quotient-rule answers.
- **Using the quotient rule when the top is a number.** $\frac{5}{(2x-3)^4}$ is $5(2x - 3)^{-4}$; the chain rule is quicker and safer. When the power is negative, it goes **down**: $-4 - 1 = -5$.
- **Breaking the log laws.** $\ln(x^2 + 5x)$ is **not** $\ln x^2 + \ln 5x$. The laws split products, quotients and powers only.
- **Degrees in calculus.** Every derivative of $\sin$, $\cos$ and $\tan$ assumes radians. Set your calculator to radians and give angles in radians.
- **Not showing the derivative.** Finding a gradient with a calculator's numerical derivative, or from a chord between two points, earns nothing. Write $\frac{dy}{dx}$ out in full, then substitute, and show the substitution.
- **Simplifying badly.** Many candidates reach a correct derivative, then make an error "tidying" it, which then spoils later parts. If the question only says "find", you may leave it unsimplified. In a "show that", take out common factors instead of expanding everything.
- **Ignoring "hence".** Integrating a product "term by term" or restarting with a new method gets no credit. Reverse your derivative, and remember $+c$.
- **Writing ambiguous products.** Write $e^{4x} \sec^2 x$, not $\sec^2 x e^{4x}$ run together so it looks like $\sec^2(xe^{4x})$.

How to get the marks

- **Each part of a derivative earns its own mark.** Mark schemes typically give **B1** for differentiating an inner function correctly (such as $\frac{d}{dx} \ln(5x^2 + 1) = \frac{10x}{5x^2+1}$), **M1** for a product or quotient rule with the right structure, and **A1** for everything else right. Write down $\frac{du}{dx}$ and $\frac{dv}{dx}$ so the examiner can award them even if a later slip happens.
- **Method marks follow through.** A wrong inner derivative put correctly into the quotient rule still earns the **M1**, so keep going.
- **"Show that"** means the answer is printed. Every step must be visible, the last line must match the printed form exactly, and you must state constants such as A and B . Otherwise the final **A1** is lost.
- **"Hence"** means you must use the result just found. For "hence find $\int$ ", start from your derivative, not a new method.
- **"Find the exact value"** means surds, fractions, π , e and $\ln$: no decimals. If a form such as $\ln p$ is asked for, use the log laws to reach it.
- **Paper 1 is non-calculator.** Show exact trig values, powers of e and log simplifications step by step. On Paper 2, give non-exact answers to 3 significant figures unless told otherwise, and keep more figures in the working.
- **"Using differentiation" or "use calculus"** means a solution without a derivative (for example, working out y at two values of x for a small change) scores nothing.

- **Use x in radians** throughout calculus, and read the domain: reject values of x that lie outside it, and say why.

Quick check

1. Differentiate with respect to x : (a) $\sin \sqrt{x}$; (b) $\sqrt[4]{(2x^2 - 3)^3}$.
2. Find the exact gradient of the curve $y = \sin x \ln x$ at the point where $x = \frac{\pi}{2}$.
3. Given that $y = \frac{\tan x}{x}$, find the exact value of $\frac{dy}{dx}$ when $x = \frac{\pi}{4}$.
4. Given that $y = \sin 3x + \cos 3x$, show that $\frac{d^2y}{dx^2} = -9y$.
5. Show that the derivative of $\frac{x}{\sqrt{x^2 + 5}}$ is $\frac{5}{(x^2 + 5)^{3/2}}$.

Answers

- (a) $\sin(x^{1/2}) \rightarrow \cos(x^{1/2}) \times \frac{1}{2}x^{-1/2} = \frac{\cos \sqrt{x}}{2\sqrt{x}}$. (b) $(2x^2 - 3)^{3/4} \rightarrow \frac{3}{4}(2x^2 - 3)^{-1/4} \times 4x = 3x(2x^2 - 3)^{-1/4} = \frac{3x}{\sqrt[4]{2x^2 - 3}}$.
- Product rule: $\frac{dy}{dx} = \sin x \times \frac{1}{x} + \ln x \cos x$. At $x = \frac{\pi}{2}$: $\sin \frac{\pi}{2} = 1$ and $\cos \frac{\pi}{2} = 0$, so the gradient is $\frac{1}{\pi/2} + 0 = \frac{2}{\pi}$.
- Quotient rule: $\frac{dy}{dx} = \frac{x \sec^2 x - \tan x}{x^2}$. At $x = \frac{\pi}{4}$: $\sec^2 \frac{\pi}{4} = 2$ and $\tan \frac{\pi}{4} = 1$, so $\frac{dy}{dx} = \frac{\frac{\pi}{2} - 1}{\frac{\pi^2}{16}} = \frac{16(\frac{\pi}{2} - 1)}{\pi^2} = \frac{8\pi - 16}{\pi^2}$.
- $\frac{dy}{dx} = 3 \cos 3x - 3 \sin 3x$ and $\frac{d^2y}{dx^2} = -9 \sin 3x - 9 \cos 3x = -9(\sin 3x + \cos 3x) = -9y$.
- $u = x, v = (x^2 + 5)^{1/2}$, so $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{2}(x^2 + 5)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 5}}$. Then $\frac{dy}{dx} = \frac{\sqrt{x^2 + 5} - \frac{x^2}{\sqrt{x^2 + 5}}}{x^2 + 5}$. Multiply top and bottom by $\sqrt{x^2 + 5}$: $\frac{(x^2 + 5) - x^2}{(x^2 + 5)\sqrt{x^2 + 5}} = \frac{5}{(x^2 + 5)^{3/2}}$.

Past questions to try

- 0606/22/M/J/22 Q7**: a quotient whose top needs the chain rule, with $e^{\tan x}$ over $\ln x$.
- 0606/11/M/J/23 Q10**: a product with a logarithm, then "hence" integrate.
- 0606/11/O/N/23 Q10**: showing a quotient with a square root has a given form, then a small change.
- 0606/21/O/N/21 Q5**: a trigonometric derivative in a given form, then a trig equation from it.