

Equations, inequalities and graphs

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Read first: [Quadratic functions](#), [Factors of polynomials](#)

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What you need to know

By the end of this topic you should be able to:

1. **Solve modulus equations**, with algebra or with an accurate graph, of these kinds: $|ax + b| = c$ (with $c \geq 0$), $|ax + b| = cx + d$, $|ax + b| = |cx + d|$ and $|ax^2 + bx + c| = d$.
2. **Solve modulus inequalities**, with algebra or with an accurate graph, of these kinds: $k|ax + b| > c$ and $k|ax + b| \leq c$ (with $k > 0$), $k|ax + b| \leq |cx + d|$, $|ax + b| \leq cx + d$, and $|ax^2 + bx + c| > d$ or $\leq d$.
3. **Spot a "hidden quadratic"**: choose a substitution, such as $u = 2^x$, $u = \sqrt{x}$ or $u = \ln x$, that turns an equation into a quadratic in u , solve it, then get back to x .
4. **Sketch a cubic given as a product of three linear factors**, such as $y = (x + 1)(x - 3)(x - 6)$, and sketch its modulus $y = |f(x)|$, with the points where the graph meets the axes clearly labelled.
5. **Solve cubic inequalities from a graph**: $f(x) > d$, $f(x) \geq d$, $f(x) < d$ or $f(x) \leq d$, where $f(x)$ is a product of three linear factors and d is a constant.

Sketching $y = |f(x)|$ for linear and quadratic f also belongs here, because modulus equations and inequalities are often solved from those graphs.

Understand it

What the modulus means

The **modulus** $|x|$ is the size of x without its sign: $|5| = 5$ and $|-5| = 5$. Think of it as the **distance from 0** on the number line. So $|x - 4| = 3$ asks "which numbers are 3 away from 4?", and the answers are 1 and 7.

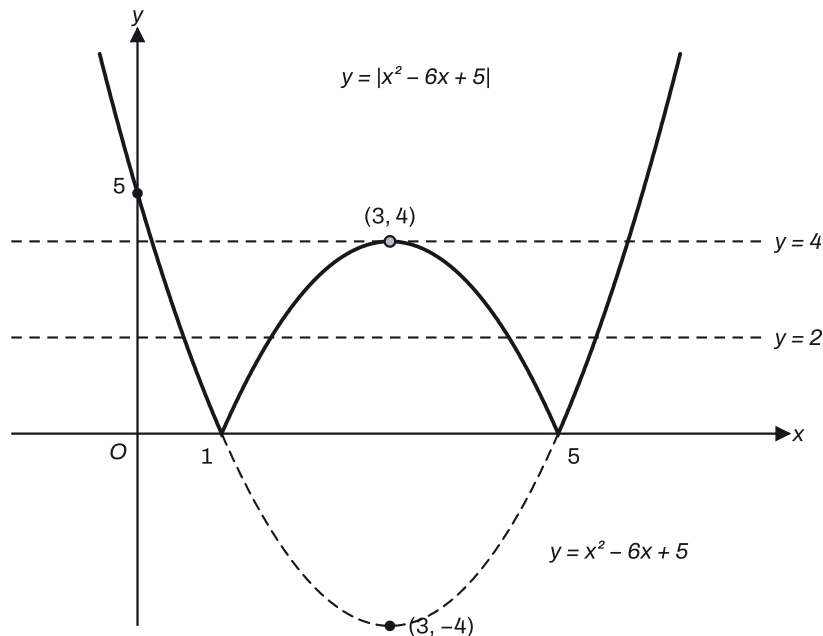
Two facts do most of the work:

- $|X| = c$ (with $c \geq 0$) means $X = c$ or $X = -c$.
- A modulus is never negative, so $|X| = -2$ has no solutions, and $|X| = |Y|$ is the same as $X^2 = Y^2$.

The graph of $y = |f(x)|$

Draw $y = f(x)$ first. Wherever the graph is **below** the x -axis, reflect that part in the x -axis; the parts above stay where they are. The graph of $y = |f(x)|$ is never below the x -axis.

- $y = |2x - 3|$ is a **V**: two straight pieces meeting at a vertex **on the x -axis** at $x = \frac{3}{2}$ (where $2x - 3 = 0$). It meets the y -axis at $|-3| = 3$. Draw both pieces with a ruler.
- For a quadratic or a cubic, each place the curve crosses the x -axis becomes a sharp corner, called a **cusp**. The reflected piece keeps its curve, so a dip below the axis becomes a hump above it.



The height of that reflected hump matters. Here the parabola's lowest point is $(3, -4)$ (complete the square: $x^2 - 6x + 5 = (x - 3)^2 - 4$), so the hump's peak is $(3, 4)$. A horizontal line $y = k$ therefore meets the graph:

- **4 times** if $0 < k < 4$ (twice on the outer arms, twice on the hump);
- **3 times** if $k = 4$ (it touches the peak);
- **2 times** if $k > 4$, or if $k = 0$ (at the cusps).

That is how you count the solutions of $|f(x)| = k$.

Solving modulus equations

$|ax + b| = c$. Write two equations, $ax + b = c$ and $ax + b = -c$, and solve both.

$|ax + b| = cx + d$. Again split into $ax + b = cx + d$ and $ax + b = -(cx + d)$. But the right-hand side could be negative, and a modulus cannot equal a negative number, so **check each answer in the original equation**. For $|2x + 1| = x + 5$: $2x + 1 = x + 5$ gives $x = 4$, and $2x + 1 = -(x + 5)$ gives $x = -2$. Both check ($9 = 9$ and $3 = 3$). But for $|x + 3| = x - 1$, the only candidate is $x = -1$, where $x - 1 = -2 < 0$, so there are **no** solutions.

$|ax + b| = |cx + d|$. Either $ax + b = cx + d$ or $ax + b = -(cx + d)$. Or square both sides (safe, because both sides are ≥ 0) and solve the quadratic. For $|x - 1| = |2x + 4|$: $x - 1 = 2x + 4$ gives $x = -5$, and $x - 1 = -(2x + 4)$ gives $x = -1$.

$|ax^2 + bx + c| = d$. Solve the two quadratics $ax^2 + bx + c = d$ and $ax^2 + bx + c = -d$. Each can give 0, 1 or 2 roots, so there can be up to **four** answers. If one quadratic has a negative discriminant, it simply gives none.

Solving modulus inequalities

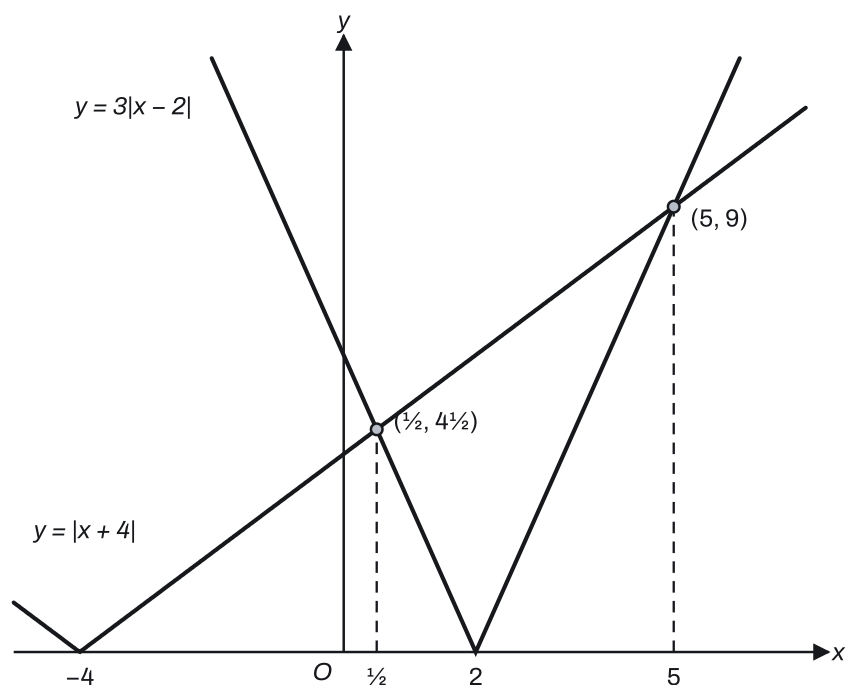
With $c > 0$:

$$|X| < c \iff -c < X < c, \quad |X| > c \iff X < -c \text{ or } X > c.$$

"Less than" gives one interval **between** two values; "greater than" gives two pieces **outside** them. So $|x - 3| < 5$ means $-5 < x - 3 < 5$, that is $-2 < x < 8$, and $|x - 3| \geq 5$ means $x \leq -2$ or $x \geq 8$. With a number in front, divide first: $2|x + 1| > 6$ means $|x + 1| > 3$, so $x < -4$ or $x > 2$.

For **harder inequalities** the safest method has three steps:

1. **Find the critical values**: solve the matching equation (with $=$ in place of the inequality sign).
2. **Decide which regions work**, using a sketch of both sides or by testing one number in each region.
3. **Write the answer** with the right signs: strict ($<$, $>$) stays strict, and "or equal to" ($\leq$, $\geq$) stays inclusive.



For $3|x - 2| \geq |x + 4|$, the diagram shows where $y = 3|x - 2|$ is on or above $y = |x + 4|$: to the left of the first crossing and to the right of the second. Squaring gives the crossings exactly (Example 2 below): $x = \frac{1}{2}$ and $x = 5$. So the answer is $x \leq \frac{1}{2}$ or $x \geq 5$. Testing agrees: at $x = 0$, $6 \geq 4$ is true; at $x = 2$, $0 \geq 6$ is false.

Two ways to get the critical values when both sides have a modulus, $k|ax + b|$ against $|cx + d|$:

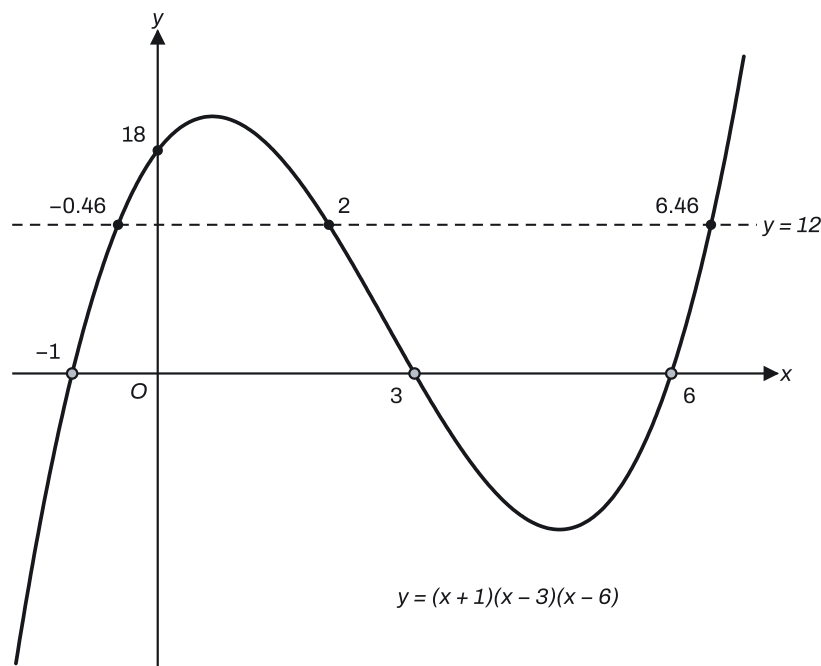
- **Square both sides.** Both sides are never negative, so squaring keeps the inequality true. You get a quadratic inequality: factorise it, then a sketch of the parabola picks the region (" ≤ 0 ": between the roots; " ≥ 0 ": outside them). Square the number in front too: $(3(x - 2))^2 = 9(x - 2)^2$.
- **Two linear equations:** $k(ax + b) = cx + d$ and $k(ax + b) = -(cx + d)$. Quicker to solve, but you must then decide the regions with a sketch or a test value.

When the right-hand side is $cx + d$ (no modulus), **don't square**: $cx + d$ can be negative. Use the two linear equations, keep only the critical values where $cx + d \geq 0$, and test the regions.

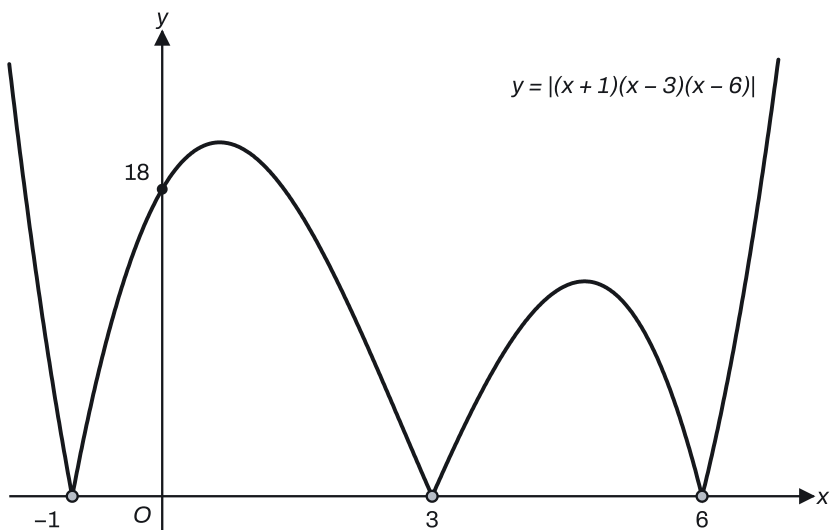
Cubic graphs from three linear factors

For $y = (x + 1)(x - 3)(x - 6)$:

- **Roots:** each bracket equal to zero gives a crossing, at $x = -1, 3$ and 6 .
- **y-intercept:** put $x = 0$: $(1)(-3)(-6) = 18$.
- **Which way round:** multiply the x -terms of the brackets: $x \times x \times x = x^3$. A **positive** x^3 term means the graph starts **low on the left** and ends **high on the right**. A negative one, such as from a bracket written $(6 - x)$ or a minus sign in front, means it starts high and ends low.
- Between neighbouring roots the curve has one turning point. You don't need its exact position for a sketch, but it must be a smooth curve with a clear maximum and minimum, not straight lines or a pointed tip.
- A **squared factor**, such as $(x + 2)^2$, means the curve **touches** the axis at $x = -2$ and turns back, instead of crossing.



For its modulus, reflect the parts below the axis (to the left of -1 , and between 3 and 6). The y -intercept stays 18 , because it is already above the axis, and each root becomes a cusp.



Cubic inequalities from a graph

" $f(x) > 0$ " asks: where is the curve **above the x -axis**? From the first cubic graph, $(x + 1)(x - 3)(x - 6) > 0$ when $-1 < x < 3$ or $x > 6$.

" $f(x) \geq d$ " asks: where is the curve **on or above the line $y = d$** ? Draw the line, read off the x -values where it meets the curve, and read the regions. For $f(x) < 12$ above, the line $y = 12$ meets the curve at about -0.46 , 2 and 6.46 ; the curve is below it for $x < -0.46$ and for $2 < x < 6.46$ (Example 3 finds these values exactly). A cubic inequality usually has **two pieces** in its answer, often one of them an interval between two values.

Hidden quadratics

Some equations are quadratics in disguise. In $4^x - 6(2^x) + 8 = 0$, notice $4^x = (2^x)^2$. Let $u = 2^x$:

$$u^2 - 6u + 8 = 0 \Rightarrow (u - 2)(u - 4) = 0 \Rightarrow 2^x = 2 \text{ or } 2^x = 4 \Rightarrow x = 1 \text{ or } x = 2.$$

The pattern: find a quantity u such that the equation has a u^2 term, a u term and a constant. Common choices:

The equation has	Let	Because
a^{2x} and a^x (or a^{x+1})	$u = a^x$	$a^{2x} = u^2$, $a^{x+1} = a \cdot u$
e^x and e^{-x}	$u = e^x$	$e^{-x} = \frac{1}{u}$; multiply through by u
x and $\sqrt{x}$	$u = \sqrt{x}$	$x = u^2$
$x^{2/3}$ and $x^{1/3}$	$u = x^{1/3}$	$x^{2/3} = u^2$
$(\ln x)^2$ and $\ln x$	$u = \ln x$	the square is already there

After solving for u , **go back to x** and check each value makes sense: 2^x , e^x and $\sqrt{x}$ can never be negative, so a negative u is rejected there. A cube root **can** be negative, so $x^{1/3} = -2$ is fine and gives $x = -8$.

Key facts and formulas

"Given" means it is in the List of formulas on page 2 of the exam paper. "Learn it" means you must remember it.

Formula	Status
Quadratic formula: $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Given
$ X = c (c \geq 0) \iff X = c \text{ or } X = -c$	Learn it
$ X = Y \iff X = \pm Y \iff X^2 = Y^2$	Learn it
$ X < c \iff -c < X < c$	Learn it
$ X > c \iff X < -c \text{ or } X > c$	Learn it
$ X \leq Y \iff X^2 \leq Y^2 \text{ (both sides } \geq 0)$	Learn it
$y = f(x) $: reflect the parts of $y = f(x)$ below the x -axis in the x -axis	Learn it
$y = ax + b $: V shape, vertex $(-\frac{b}{a}, 0)$, y -intercept $ b $	Learn it
Cubic $k(x - p)(x - q)(x - r)$: roots p, q, r ; y -intercept $-kpqr$; $k > 0$ starts low and ends high, $k < 0$ the other way	Learn it
$a^{2x} = (a^x)^2$, $a^{x+n} = a^n \cdot a^x$, $e^{-x} = \frac{1}{e^x}$, $x = (\sqrt{x})^2$	Learn it

How to do it

In the Paper 1 and Paper 2 questions we have tagged for this topic (76 questions, 2021–2025), the types came up this often. 36 of the 76 questions mix types (a graph, then an inequality from it), so the counts add up to more than 76.

Question type	Questions
Hidden quadratics: substitution	24
Cubic graphs from three linear factors	21
Graphs of modulus functions	21
Modulus inequalities	18
Number of solutions of $f(x) = k$	11

Question type	Questions
Modulus equations	10
Cubic inequalities from a graph	9
Curves meeting in a cubic equation	2

1. Hidden quadratics: substitution (24 questions)

- Rewrite the powers** so that one quantity appears squared and on its own: $9^x = (3^x)^2$, $3^{x+1} = 3 \times 3^x$, $x^{2/3} = (x^{1/3})^2$, $\lg(x^3) = 3 \lg x$.
- State the substitution** ("let $u = 3^x$ ") and write the quadratic in u , with every term on one side. If there is a $\frac{1}{u}$ term, multiply through by u .
- Solve the quadratic** by factorising, or with the formula if it won't factorise.
- Go back to x** for each u : $a^x = u$ gives $x = \frac{\ln u}{\ln a}$ (or $\log_a u$); $\sqrt{x} = u$ gives $x = u^2$; $x^{1/3} = u$ gives $x = u^3$; $\ln x = u$ gives $x = e^u$.
- Reject impossible values** with a reason ($3^x > 0$ for all x), and give the accuracy asked for: "exact" means $\ln 3$ or $\log_5 7$, not 1.10.

Variations you will meet:

- Even powers**, such as $x^4 - 5x^2 + 4 = 0$: let $u = x^2$, so $u^2 - 5u + 4 = 0$ and $u = 1$ or 4 . Then $x^2 = 1$ or $x^2 = 4$, so $x = \pm 1$ or $x = \pm 2$: four answers, because a square has two square roots.
- A cubic in u that an earlier part has factorised**. If part (b) showed $p(x) = (x - 3)(x + 2)(2x - 1)$, then $p(e^{2y}) = 0$ needs no new work: $u = e^{2y}$ is a root of p , so $e^{2y} = 3, -2$ or $\frac{1}{2}$. Reject $e^{2y} = -2$ (an exponential is always positive) and solve the others: $y = \frac{1}{2} \ln 3$ or $y = \frac{1}{2} \ln \frac{1}{2} = -\frac{1}{2} \ln 2$.
- Powers with different bases that are all powers of 2** (or of 3): $64^x = (2^{3x})^2 = (8^x)^2$. Rewrite everything in one base first.
- Two curves meet**, such as $y = a \ln x$ and $y = b - \frac{c}{\ln x}$: set them equal, multiply through by $\ln x$, and you get a quadratic in $\ln x$.
- Logarithms in two bases**, such as $\log_p y$ and $\log_y p$: use $\log_y p = \frac{1}{\log_p y}$, then it is a quadratic in $u = \log_p y$.
- A completed square used later**: once a quadratic in x is written as $a(x + p)^2 + q$, the same quadratic with x replaced by $y^{1/3}$ (or $\sqrt{y}$) is solved the same way for x , then you cube (or square) at the end.

2. Cubic graphs from three linear factors (21 questions)

Sketching $y = f(x)$:

- Roots**: set each bracket to zero, including fractions: $(2x - 5) = 0$ gives $x = 2.5$.
- y -intercept**: substitute $x = 0$ into the whole product.
- End behaviour**: multiply the x -terms of the brackets (and any number in front). Positive x^3 : low on the left, high on the right. Negative: the opposite.

4. **Draw** a smooth curve through the intercepts, with one turning point between each pair of neighbouring roots, and **label every intercept** on the axes (the numbers, not just marks).

Finding $f(x)$ from a graph:

1. Read the roots from the graph and write $f(x) = k(x - p)(x - q)(x - r)$. For a root at $-\frac{1}{2}$ use $(2x + 1)$, to keep integer coefficients.
2. Use the y -intercept to find k . For roots $-3, 2, 4$ and y -intercept -48 : $k(3)(-2)(-4) = 24k = -48$, so $f(x) = -2(x + 3)(x - 2)(x - 4)$.
3. If the diagram shows $y = |f(x)|$, you cannot tell whether the original was above or below the axis, so there are **two possible expressions**: $f(x)$ and $-f(x)$. Give both.

Variations you will meet:

- **A repeated factor**, such as $(2 - x)(x + 1)^2$: the curve touches the x -axis at -1 . Earlier parts often ask for the stationary points first; use them, and make sure the turning points on your sketch match (a maximum found at a non-zero x is not on the y -axis).
- **Sketch $y = |f(x)|$ after $y = f(x)$** : reflect the parts below the axis, keep cusps on the axis, and keep the curvature.

3. Graphs of modulus functions (21 questions)

1. **Linear**, $y = |ax + b|$: mark the vertex at $x = -\frac{b}{a}$ on the x -axis and the y -intercept $|b|$, then rule two straight arms. If the question gives a domain (such as $-2 \leq x \leq 8$), stop at its ends.
2. **With a number added or subtracted**, such as $y = 2 + |x - 3|$ or $y = 7 - |2x + 1|$: the vertex moves up to $(3, 2)$, or the V turns upside down with its top at $(-\frac{1}{2}, 7)$.
3. **Quadratic or cubic**: sketch $y = f(x)$, then reflect the parts below the axis. Label the intercepts and show **cusps** (sharp points) on the x -axis, not rounded minima.
4. When the question says **draw** on a grid, plot accurately, because you will read values off it.

Variations you will meet:

- **"By drawing a suitable graph on the same diagram, solve ..."**: draw the other side of the inequality (usually a straight line or another V), find where they cross, and read off the region.
- **Find a line $y = ax + b$ that is on or above a V-shaped graph only for $p \leq x \leq q$** : the line must pass through the graph's points at $x = p$ and $x = q$. Find those two points, then the line through them.

4. Modulus inequalities (18 questions)

1. **One modulus and a number**, $k|ax + b| > c$: divide by k , then use $|X| > c \iff X < -c \text{ or } X > c$ (or $|X| < c \iff -c < X < c$).
2. **Modulus against modulus**, $k|ax + b| \leq |cx + d|$: square both sides, collect into $Ax^2 + Bx + C \leq 0$, factorise to get the critical values, and use the parabola: between the roots for ≤ 0 , outside them for ≥ 0 .

- Modulus against a line**, $|ax + b| \leq cx + d$: solve $ax + b = cx + d$ and $ax + b = -(cx + d)$, keep critical values where $cx + d \geq 0$, and test a value in each region. For $|x - 5| \leq 2x - 1$: $x - 5 = 2x - 1$ gives $x = -4$, but there $2x - 1 < 0$, so reject it; $-(x - 5) = 2x - 1$ gives $x = 2$. The answer is $x \geq 2$.
- Modulus of a quadratic**, $|ax^2 + bx + c| < d$: write $-d < ax^2 + bx + c < d$, solve both quadratic inequalities, and take the x -values that satisfy **both**. A sketch with the lines $y = d$ is the clearest check.
- Graphical**: from accurate graphs, read the x -values of the crossings (to the grid's accuracy) and state where one graph is above the other.

Writing the answer: two separate pieces are written with "or", or as a list with a comma, such as $x < -4$, $x > 2$. A single interval is written as one statement, $-2 < x < 8$; don't split it into " $x > -2$, $x < 8$ " unless you join them with **and**.

5. Number of solutions of $f(x) = k$ (11 questions)

- Sketch** $y = f(x)$ (often $y = |\text{quadratic}|$ or a cubic), with its turning points' y -values.
- Imagine a horizontal line** $y = k$ moving up the page, and count where it meets the graph.
- The number of solutions changes at the heights of the turning points (and at $y = 0$ for a modulus graph). Write the answer as a value (" $k = 9$ ") or a range (" $0 < k < 9$ ").

For $|x^2 - 4x - 5| = k$ with $k > 0$: the parabola $y = (x + 1)(x - 5)$ has its lowest point at $x = 2$, $y = (3)(-3) = -9$, so the hump peaks at 9. There are exactly **2** solutions when $k > 9$, **3** when $k = 9$, and **4** when $0 < k < 9$.

For a cubic with a maximum at height M and a minimum on the x -axis (a repeated root), $f(x) = k$ has three distinct roots when $0 < k < M$. The maximum usually comes from an earlier "find the stationary points" part.

6. Modulus equations (10 questions)

- $|ax + b| = c$ (or with the modulus taken away from a number, such as $10 - |3x + 1| = 3$): first get the modulus on its own ($|3x + 1| = 7$), then solve $ax + b = \pm c$ (here $x = 2$ or $x = -\frac{8}{3}$).
- $|ax + b| = cx + d$: solve $ax + b = cx + d$ and $ax + b = -(cx + d)$, then check each answer in the original; reject any that make $cx + d$ negative.
- $|ax + b| = |cx + d|$: solve $ax + b = \pm(cx + d)$, or square both sides.
- $|ax^2 + bx + c| = d$: solve $ax^2 + bx + c = d$ and $ax^2 + bx + c = -d$. Give **every** root, including negative ones and surd forms such as $-1 \pm \sqrt{2}$.
- After sketching two graphs**: the number of crossings tells you how many solutions to expect.

7. Cubic inequalities from a graph (9 questions)

- Sketch or use the given graph** of $y = f(x)$, and draw the line $y = d$ (for $d = 0$, it is the x -axis).
- Find the critical values**: the roots (for $d = 0$), or where the line meets the curve, read from an accurate graph. If the line crosses at $x = 0$, the y -intercept tells you so exactly.
- Read the regions**: " $> d$ " means the curve is above the line. Include the critical values for $\geq$ and $\leq$.

4. Write both pieces, such as $x < -2, 1 < x < 4$.

8. Curves meeting in a cubic equation (2 questions)

1. Set the two expressions equal (or set $y = 0$ for where a curve cuts the x -axis) and rearrange to a cubic equal to zero.
2. Factorise it, using a factor from an earlier part or the factor theorem, into a linear factor times a quadratic.
3. Solve each factor equal to zero, and give exact answers if asked. The examiners want to see the factorisation, not just calculator roots.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
-

Example 1

(a) Solve the equation $|3x - 7| = x + 1$. **[3]**

(b) Solve the equation $|x^2 - 6x + 5| = 4$, giving your answers in exact form. **[5]**

(a) $3x - 7 = x + 1 \Rightarrow 2x = 8 \Rightarrow x = 4$. **M1** for one equation solved.

$3x - 7 = -(x + 1) \Rightarrow 4x = 6 \Rightarrow x = \frac{3}{2}$. **M1** for the equation with the sign changed.

Check: $|12 - 7| = 5 = 4 + 1$ and $|\frac{9}{2} - 7| = \frac{5}{2} = \frac{3}{2} + 1$, so $x = 4$ or $x = \frac{3}{2}$. **A1** for both.

(b) $x^2 - 6x + 5 = 4 \Rightarrow x^2 - 6x + 1 = 0$. This does not factorise, so use the formula: **M1**

$$x = \frac{6 \pm \sqrt{36 - 4}}{2} = \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}.$$

A1

$x^2 - 6x + 5 = -4 \Rightarrow x^2 - 6x + 9 = 0 \Rightarrow (x - 3)^2 = 0 \Rightarrow x = 3$. **M1 A1**. A repeated root counts once.

There are three solutions: $3 - 2\sqrt{2}$, 3 and $3 + 2\sqrt{2}$. **A1** for all three and no others. The first diagram in "Understand it" shows why: the line $y = 4$ crosses the two outer arms and just touches the peak (3, 4).

Example 2

Solve the inequality $3|x - 2| \geq |x + 4|$. **[4]**

Both sides are never negative, so square:

$$9(x - 2)^2 \geq (x + 4)^2 \Rightarrow 9x^2 - 36x + 36 \geq x^2 + 8x + 16 \Rightarrow 8x^2 - 44x + 20 \geq 0.$$

M1 for squaring both sides, including the 3 as 9.

Divide by 4: $2x^2 - 11x + 5 \geq 0$, so $(2x - 1)(x - 5) \geq 0$. Critical values $x = \frac{1}{2}$ and $x = 5$. **A1 A1**

The parabola $y = (2x - 1)(x - 5)$ is on or above the axis outside its roots, so

$$x \leq \frac{1}{2} \text{ or } x \geq 5.$$

A1. This matches the diagram of the two V-shaped graphs above.

Example 3

The function f is defined by $f(x) = (x + 1)(x - 3)(x - 6)$.

(a) Sketch the graph of $y = f(x)$, stating the intercepts with the coordinate axes. **[3]**

(b) Show that $f(x) - 12 = (x - 2)(x^2 - 6x - 3)$. **[3]**

(c) Hence solve the inequality $f(x) < 12$, giving your answer in exact form. **[4]**

(a) Roots $x = -1, 3, 6$; y -intercept $(1)(-3)(-6) = 18$. The x -terms multiply to $+x^3$, so the curve starts low on the left and ends high on the right, with a maximum between -1 and 3 and a minimum between 3 and 6 . **B1** for the shape, **B1** for $-1, 3$ and 6 , **B1** for 18 . The sketch is the cubic graph in "Understand it".

(b) Expand: $(x + 1)(x - 3) = x^2 - 2x - 3$, and $(x^2 - 2x - 3)(x - 6) = x^3 - 8x^2 + 9x + 18$. **B1**

So $f(x) - 12 = x^3 - 8x^2 + 9x + 6$. When $x = 2$: $8 - 32 + 18 + 6 = 0$, so $(x - 2)$ is a factor. **M1**

Dividing (or comparing coefficients) gives $f(x) - 12 = (x - 2)(x^2 - 6x - 3)$. **A1**

(c) $f(x) = 12$ when $x = 2$ or $x^2 - 6x - 3 = 0$:

$$x = \frac{6 \pm \sqrt{36 + 12}}{2} = \frac{6 \pm 4\sqrt{3}}{2} = 3 \pm 2\sqrt{3} \quad (-0.46 \text{ and } 6.46).$$

M1 A1

The line $y = 12$ meets the curve three times. The curve is below it to the left of the first crossing (it starts low) and between the other two (the minimum is there), so

$$x < 3 - 2\sqrt{3} \text{ or } 2 < x < 3 + 2\sqrt{3}.$$

B1 B1, one for each piece. The inequality is strict, so the critical values are not included. From an accurate graph alone you would read $x < -0.5$ or $2 < x < 6.5$.

Example 4

(a) Solve the equation $9^x - 4(3^{x+1}) + 27 = 0$. **[4]**

(b) Solve the equation $(\lg x)^2 - \lg(x^3) + 2 = 0$. **[4]**

(a) $9^x = (3^x)^2$ and $3^{x+1} = 3 \times 3^x$. **B1**

Let $u = 3^x$: $u^2 - 12u + 27 = 0$. **M1** for a three-term quadratic in u .

$(u - 3)(u - 9) = 0$, so $3^x = 3$ or $3^x = 9$. **M1 A1**: $x = 1$ or $x = 2$.

(b) $\lg(x^3) = 3 \lg x$. **B1**

Let $u = \lg x$: $u^2 - 3u + 2 = 0 \Rightarrow (u - 1)(u - 2) = 0$. **M1**

$\lg x = 1$ or $\lg x = 2$, so $x = 10$ or $x = 100$. **M1 A1**

Common mistakes

- **Keeping only one case.** $|2x - 1| = 3$ has two equations, $2x - 1 = 3$ and $2x - 1 = -3$. Examiners regularly see just one, or one critical value such as 3 when ± 3 were both needed.
- **Rejecting good answers.** A negative x is fine in a modulus equation, and a negative cube root is fine. Only reject a value for a real reason: $cx + d < 0$ on the right of $|ax + b| = cx + d$, or $3^x = -2$.
- **Changing only some signs.** In $3(x - 2) = -(2x + 5)$ both terms on the right change sign, giving $-2x - 5$; writing $-2x + 5$ is a common slip.
- **Forgetting to square the number in front.** $(3(x - 2))^2 = 9(x - 2)^2$, not $3(x - 2)^2$. Write $(3x - 6)^2 \geq (x + 4)^2$ before expanding.
- **Squaring when the right-hand side has no modulus.** $|ax + b| < cx + d$ cannot be squared safely, because $cx + d$ can be negative.
- **Wrong final inequality after two linear equations.** Finding the critical values is only half the work; use a sketch or a test value to choose the region, then write it once, correctly joined.
- **Wrong inequality signs.** A strict question needs a strict answer; " $\geq$ " needs " $\geq$ " in the answer. Examiners penalise mixing them.
- **Modulus graphs with curves or minima.** $y = |ax + b|$ is two **ruled** straight lines; freehand lines lose marks. For a quadratic or cubic, the reflected graph has **cusps** on the x -axis, not smooth minima, and the reflected piece keeps its curvature.
- **Missing or wrong intercepts.** State every intercept on the axes. For $|2x + 5|$ the vertex is at $x = -\frac{5}{2}$, not $-\frac{2}{5}$.
- **A cubic sketched the wrong way round.** Multiply the x -terms of the brackets (and any minus sign in front) before drawing, and don't put a turning point on the y -axis unless it is really there.
- **Reading the wrong region** in a cubic inequality, such as solving $f(x) \geq d$ when asked for $f(x) \leq d$. Write the critical values first, then combine them.
- **Forgetting to go back to x** after a substitution, or leaving u values as the answer.

How to get the marks

- **Show both cases** of a modulus equation or inequality. Each case is usually a method mark, even if the arithmetic slips later.
- **Critical values first.** Many schemes give marks for the correct critical values on their own, then a final mark for the correct inequality. So write the critical values clearly before you combine them.
- **"Solve the inequality"**: your answer must be a set of x -values with inequality signs, not the critical values. Write it once, neatly, at the end; that is the line that is marked.
- **"Sketch"**: the right shape, in the right place, with every intercept shown on the axes or stated. It does not need to be to scale, but modulus-of-linear graphs must be ruled.
- **"Draw"** (on a grid): plot accurately; your reading of the crossings is then accepted within the graph's accuracy.
- **"Use a graphical method" or "by drawing a suitable graph"**: you must draw the extra graph; algebra alone will not get the marks. "Solutions by accurate drawing will not be accepted" means the opposite: algebra only.
- **"Hence"**: use the previous part, such as the graph you just sketched or the factor you just found.
- **"Exact"**: surds, fractions, $\ln$ or $\log$ forms, such as $-1 + \sqrt{2}$ or $\log_5 7$. Otherwise give 3 significant figures, or the number of decimal places asked for.
- **Substitution questions**: show the quadratic in u (or in 3^x) and its factorisation or formula. On a paper without a calculator, every step must be visible.
- **Reject with a reason**: " $3^x > 0$, so $3^x = -2$ has no solution". An extra, wrong answer left in costs the accuracy mark.

Quick check

1. Solve the equation $|5 - 2x| = 9$.
2. Solve the inequality $|x + 3| \leq 2x$.
3. Solve the inequality $|x^2 - 2x - 3| < 5$.
4. Solve the equation $5^{2x} - 5^{x+1} - 14 = 0$, giving your answer correct to 2 decimal places.
5. Find the values of k for which the equation $|x^2 + 3x| = k$ has exactly two solutions.

Answers

1. $5 - 2x = 9$ gives $x = -2$; $5 - 2x = -9$ gives $x = 7$. So $x = -2$ or $x = 7$.
2. $x + 3 = 2x$ gives $x = 3$; $-(x + 3) = 2x$ gives $x = -1$, but there $2x = -2 < 0$, so reject it. Testing: at $x = 4$, $7 \leq 8$ is true; at $x = 0$, $3 \leq 0$ is false. The answer is $x \geq 3$.
3. $x^2 - 2x - 3 = 5$ gives $(x - 4)(x + 2) = 0$, so $x = 4$ or $x = -2$. $x^2 - 2x - 3 = -5$ gives $x^2 - 2x + 2 = 0$, with discriminant $4 - 8 = -4 < 0$: no roots. (The lowest point of $x^2 - 2x - 3 = (x - 1)^2 - 4$ is -4 , which is above -5 .) So the answer is $-2 < x < 4$.
4. Let $u = 5^x$: $u^2 - 5u - 14 = 0$, so $(u - 7)(u + 2) = 0$. $5^x = -2$ is impossible, so $5^x = 7$ and $x = \frac{\ln 7}{\ln 5} = 1.21$ (2 d.p.).
5. $x^2 + 3x = (x + \frac{3}{2})^2 - \frac{9}{4}$, so the lowest point $(-\frac{3}{2}, -\frac{9}{4})$ becomes a peak $(-\frac{3}{2}, \frac{9}{4})$ on $y = |x^2 + 3x|$, with cusps at $x = -3$ and $x = 0$. A line $y = k$ above the peak meets only the two outer arms, so $k > \frac{9}{4}$ gives exactly two solutions. So does $k = 0$: the line is the x -axis and meets the graph only at the cusps, $x = -3$ and $x = 0$. The answer is $k = 0$ or $k > \frac{9}{4}$.

Past questions to try

- **0606/21/O/N/25 Q2**: solving a modulus inequality by drawing a second graph.
- **0606/13/O/N/24 Q1**: sketching the modulus of a quadratic and counting the solutions of $f(x) = k$.
- **0606/11/M/J/24 Q1**: sketching a cubic from its factors, then a cubic inequality.
- **0606/21/M/J/23 Q4**: a hidden quadratic in e^{2x} .