Cambridge IGCSE · Additional Mathematics 0606 · Notes

Logarithmic and exponential functions

Read online at **pastopic.com/cambridge/additional-mathematics-0606/notes/logarithmic-and-exponential-functions**

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Read first: **Functions**, **Quadratic functions**

What you need to know

By the end of this topic you should be able to:

1. **Use the simple properties of logarithms and exponentials**, in any base, including the natural ones: e^x and $\ln x$ (and $\lg x$ for base 10). Know that e^x and $\ln x$ are inverse functions: each undoes the other.
2. **Sketch and read their graphs**, of the forms $y = ke^{nx} + a$ and $y = k\ln(ax + b)$ where k, n, a and b are integers. Know that these graphs have **asymptotes** (lines the curve gets closer and closer to but never reaches) and state their equations, along with where the curve meets the axes.
3. **Use the laws of logarithms**: the product, quotient and power laws, and **change of base**, for example to write an expression as a single logarithm or to solve an equation.
4. **Solve equations of the form** $a^x = b$, and the harder equations built from them.

Series expansions of e^x or $\ln x$ are **not** needed. Differentiating and integrating e^x and $\ln x$ belong to the calculus topics; this note only covers the logarithm work those questions end with.

Understand it

What a logarithm is

A logarithm answers the question "what power?". $\log_2 8 = 3$ because $2^3 = 8$. In general

$$\log_a b = c \quad \text{means exactly the same as} \quad a^c = b.$$

The number a is the **base**; it must be positive and not 1. The number b (the **argument**) must be **positive**, because a positive base raised to any power is positive. The answer c can be anything, including negative: $\log_2 \frac{1}{4} = -2$, because $2^{-2} = \frac{1}{4}$.

Some values follow straight from the definition, in every base:

- $\log_a 1 = 0$ (since $a^0 = 1$) and $\log_a a = 1$ (since $a^1 = a$);
- $\log_a a^k = k$, for example $\log_5 125 = 3$ and $\log_9 3 = \frac{1}{2}$;
- $a^{\log_a x} = x$: raising the base to "the power that gives x" gives x.

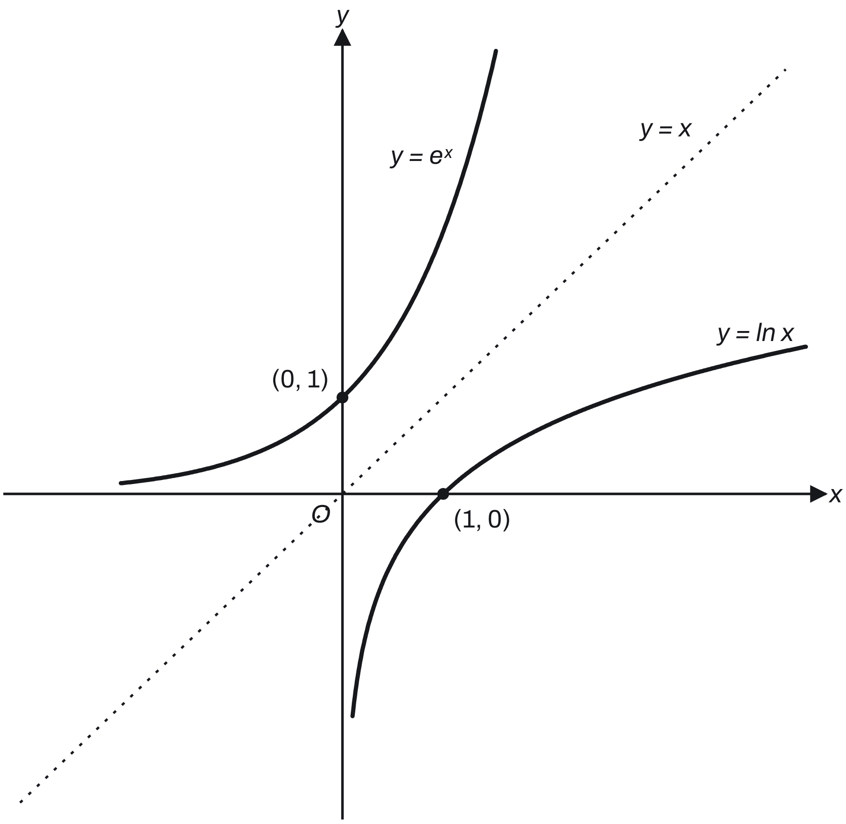
Two bases have their own names. $\lg x$ means $\log_{10} x$, and $\ln x$ means $\log_e x$, the **natural logarithm**, where $e = 2.718\ldots$ is a special constant (your calculator has keys for e^x and $\ln$).

e^x and $\ln x$ are inverses

Because $\ln$ asks "what power of e?", the two functions undo each other:

$$e^{\ln x} = x \quad (x > 0), \qquad \ln\left(e^x\right) = x \quad (\text{every } x).$$

So their graphs are reflections of each other in the line $y = x$. Everything about one graph swaps over to the other: $y = e^x$ crosses the y-axis at $(0, 1)$, so $y = \ln x$ crosses the x-axis at $(1, 0)$; $y = e^x$ has the x-axis ($y = 0$) as an asymptote, so $y = \ln x$ has the y-axis ($x = 0$) as an asymptote.



Notice that e^x is **always positive**, so an equation such as $e^x = -3$ has no solution, and $\ln x$ only exists for $x > 0$.

The laws of logarithms

Logarithms are powers, so the laws of indices turn into laws of logarithms. Since $a^m \times a^n = a^{m+n}$, multiplying numbers adds their logs:

Law	In symbols	Example
Product	$\log_a(xy) = \log_a x + \log_a y$	$\lg 4 + \lg 25 = \lg 100 = 2$

Law	In symbols	Example
Quotient	$\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$	$\log_3 54 - \log_3 2 = \log_3 27 = 3$
Power	$\log_a\left(x^k\right) = k\log_a x$	$\ln\sqrt{x} = \frac{1}{2}\ln x$

There are **no** laws for $\log(x+y)$, for $\log x \times \log y$ or for $\dfrac{\log x}{\log y}$. In particular, $\dfrac{\log_a x}{\log_a y}$ is **not** $\log_a x - \log_a y$.

A number on its own can always be written as a logarithm, which lets you combine it with the others: $2 = \lg 100$, $3 = \log_2 8$, $1 = \ln e$, $-2 = \ln e^{-2}$.

Change of base

To switch a logarithm to a different base c, divide:

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

For example, $\log_8 x = \dfrac{\log_2 x}{\log_2 8} = \dfrac{\log_2 x}{3}$. Two special cases come up again and again:

- **Swapping base and argument** turns a log into its reciprocal: $\log_b a = \dfrac{1}{\log_a b}$. So if $u = \log_3 x$, then $\log_x 3 = \dfrac{1}{u}$ and $\log_x 9 = \dfrac{2}{u}$.
- **A base that is a power**: $\log_{a^n} x = \dfrac{1}{n}\log_a x$. So $\log_{16} x = \frac{1}{4}\log_2 x$, and $\log_{\sqrt{5}} x = 2\log_5 x$.

Pick the base already in the question (usually the smallest one). Changing everything to $\lg$ or $\ln$ also works, but it takes longer and gives more chances to slip.

Solving $a^x = b$

To bring x down from the power, take logarithms of both sides and use the power law. For $5^x = 40$:

$$x\lg 5 = \lg 40 \ \Rightarrow\ x = \frac{\lg 40}{\lg 5} = 2.29\ (3\text{ s.f.}).$$

$\ln$ works just as well. When the base is e, use $\ln$: $e^{3x} = 20$ gives $3x = \ln 20$, so $x = \frac{1}{3}\ln 20$ exactly.

Graphs of $y = ke^{nx} + a$

Start from $y = e^x$ and see what each number does:

- a moves the curve up or down, so the **asymptote is** $y = a$;
- the y**-intercept** is $ke^0 + a = k + a$;
- the sign of n decides the direction: for $n > 0$ the curve approaches the asymptote on the **left**; for $n < 0$ it approaches it on the **right**;
- the sign of k decides whether the curve lies **above** ($k > 0$) or **below** ($k < 0$) its asymptote;

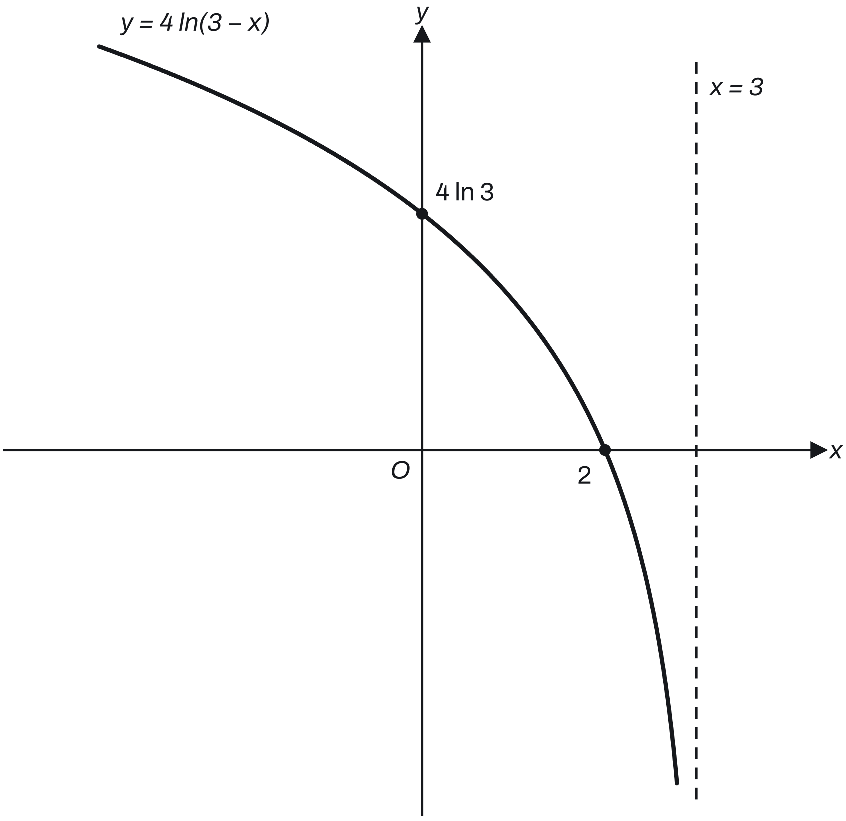
- it meets the x-axis only if $ke^{nx} = -a$ has a solution, that is when $-\frac{a}{k} > 0$.

For example, $y = 3e^{-2x} + 1$ has asymptote $y = 1$, y-intercept 4, lies above $y = 1$ and never meets the x-axis. Its range is $y > 1$.

Graphs of $y = k\ln(ax + b)$

The log only exists where the bracket is positive, and it heads to $\pm\infty$ as the bracket shrinks to 0:

- the **asymptote is the vertical line where** $ax + b = 0$, that is $x = -\frac{b}{a}$;
- the curve meets the x**-axis where** $ax + b = 1$ (since $\ln 1 = 0$);
- the y**-intercept** is $k\ln b$, which exists only if $b > 0$;
- over its whole domain the curve takes every real value, so its **range is all real numbers**.



For $y = 4\ln(3 - x)$: the domain is $x < 3$ and the asymptote is $x = 3$; $3 - x = 1$ gives the x-intercept $x = 2$; the y-intercept is $4\ln 3$. Because $a = -1$ is negative, the curve goes **down** as x increases and approaches its asymptote on the right.

Key facts and formulas

"Given" means it is in the List of formulas printed in the exam paper. None of the logarithm facts are given: learn them all.

Formula	Status
$\log_a b = c \iff a^c = b$, with $a > 0, a \neq 1, b > 0$	Learn it
$\log_a 1 = 0,\ \log_a a = 1,\ \log_a a^k = k,\ a^{\log_a x} = x$	Learn it
$\lg x = \log_{10} x,\ \ln x = \log_e x;\ e^{\ln x} = x,\ \ln e^x = x$	Learn it
$\log_a(xy) = \log_a x + \log_a y$	Learn it
$\log_a \dfrac{x}{y} = \log_a x - \log_a y$	Learn it
$\log_a x^k = k \log_a x$	Learn it
$\log_a b = \dfrac{\log_c b}{\log_c a}$	Learn it
$\log_b a = \dfrac{1}{\log_a b},\ \log_{a^n} x = \dfrac{1}{n} \log_a x$	Learn it
$a^x = b \Rightarrow x = \dfrac{\lg b}{\lg a} = \dfrac{\ln b}{\ln a}$	Learn it
$y = ke^{nx} + a$: asymptote $y = a$, y-intercept $k + a$	Learn it
$y = k\ln(ax + b)$: asymptote $x = -\dfrac{b}{a}$, x-intercept where $ax + b = 1$	Learn it
$e^x > 0$ for every x; $\ln x$ needs $x > 0$	Learn it
Quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Given

How to do it

In the Paper 1 and Paper 2 questions we have tagged for this topic (72 questions, 2021–2025), the types came up this often. 35 of the 72 questions mix two types, so the counts add up to more than 72.

Question type	Questions
Solving logarithmic equations	32
Solving exponential equations	18
Log and exponential functions: domain, range, inverse, composites	15
Sketching log and exponential graphs	13
Using the laws: a single logarithm, simplifying, rearranging	12
Simultaneous equations with logs or exponentials	8
Progressions with logarithm terms	5
Logarithms in calculus answers	4

1. Solving logarithmic equations (32 questions)

1. **Note where each log exists**: every argument must be positive (and any base must be positive and not 1).
2. **Get every log into the same base** with change of base.
3. **Combine** the logs with the laws until there is one log on each side, or one log and a number. Write any number as a log in that base if it helps.
4. **Remove the logs**: $\log_a X = c$ gives $X = a^c$; $\log_a X = \log_a Y$ gives $X = Y$.
5. **Solve**, then **check each answer** in the original equation. Reject an answer only if it makes an argument zero or negative, or a base equal to 1 or not positive.

For example, $\log_2(x+6) = 3\log_8 x + 1$. Since $\log_8 x = \frac{1}{3}\log_2 x$, this is $\log_2(x+6) = \log_2 x + \log_2 2 = \log_2(2x)$, so $x + 6 = 2x$ and $x = 6$.

Variations you will meet:

- **A log and its reciprocal**, such as $\log_3 x$ and $\log_x 9$. Let $u = \log_3 x$; then $\log_x 9 = \frac{2}{u}$. Multiply through by u to get a quadratic in u, solve it, then turn each u back into $x = 3^u$. This is the most common single question in the topic (see Example 4).
- **Bases that are powers of each other**, such as $\log_5$ and $\log_{25}$: use $\log_{25} x = \frac{1}{2}\log_5 x$, and write a number like $\frac{1}{2}$ as $\log_5 \sqrt{5}$ if you want a single log.
- **A negative logarithm is fine.** $\log_2(x+1) = -1$ gives $x + 1 = \frac{1}{2}$, a perfectly good answer. Only the **argument** has to be positive, not the value of the log.
- **A quadratic in $\ln x$.** $(\ln x)^2$ means $\ln x$ squared; $\ln x^2 = 2\ln x$ is something else. Substitute $u = \ln x$, solve, and finish with $x = e^u$. The same happens when you find where a curve like $y = \ln x$ meets another curve built from $\ln x$: set the two equal, multiply out the fractions and solve the quadratic in $\ln x$.
- **"Give y in terms of p"**: the answer is a power of p, such as $y = p^2$ or $y = p^{-1/2}$, found the same way with $u = \log_p y$.
- **"Find the value of p"** from an equation like $2\log_a p - \log_a 3 = \log_a 12$, where $p > 0$: combine into $\log_a \frac{p^2}{3} = \log_a 12$, so $p^2 = 36$ and $p = 6$ (not -6, since $\log_a p$ needs $p > 0$). The base a never needs a value.
- **"Write down the set of values of x for which $\log(\ldots)$ exists"**: solve (argument) > 0.

2. Solving exponential equations (18 questions)

1. **One power on its own**: make it the subject, then take $\ln$ (or $\lg$) of both sides. $4e^{2x-1} = 28$ gives $e^{2x-1} = 7$, so $2x - 1 = \ln 7$ and $x = \frac{1+\ln 7}{2}$.
2. **Two different bases**, such as $4^x = 3^{2-x}$: take $\lg$ of both sides, bring the powers down, collect the x terms: $x\lg 4 = (2-x)\lg 3 \Rightarrow x(\lg 4 + \lg 3) = 2\lg 3 \Rightarrow x = \dfrac{\lg 9}{\lg 12} = 0.884$ (3 s.f.).
3. **The same base on both sides**, such as $e^A = ke^B$: divide to get $e^{A-B} = k$, so $A - B = \ln k$. Numbers that are powers of each other can be matched too: $27^x = 3^{3x}$, so $27^x = 3^{x+4}$ gives $3x = x + 4$.

4. **A hidden quadratic**: look for one power that is the square of another, such as $e^{2x} = (e^x)^2$ or $9^x = (3^x)^2$. Substitute $u = e^x$ (or $u = 3^x$), solve the quadratic, then solve $e^x = u$ for each root. **A negative or zero u gives no solution**, because e^x and 3^x are always positive; say so.

Variations you will meet:

- **Negative powers or powers in the denominator.** Multiply every term by the power that clears them. For $\dfrac{e^x+8}{e^x-1} = e^x$ multiply by $e^x - 1$: $e^x + 8 = e^{2x} - e^x$, so $u^2 - 2u - 8 = 0$ with $u = e^x$, giving $u = 4$ (reject $u = -2$) and $x = \ln 4$. With messier powers, such as $\frac{a}{e^{2t+1}}$, multiply by that exact power and use $e^m \times e^n = e^{m+n}$ on every term; then look for a power that is the square of another, like $e^{4t-2} = (e^{2t-1})^2$.
- **A cubic in e^x**: after factorising the cubic (factor theorem), each factor gives $e^x =$ a number; keep the positive ones.
- **"Give your answer correct to 3 decimal places"** (or 3 significant figures): work exactly, round only at the end.
- **Paper 1 has no calculator**: leave answers such as $\ln 5$ or $\frac{1}{2}\ln 3$ exact.

3. Log and exponential functions: domain, range, inverse, composites (15 questions)

These are Functions questions with e and $\ln$ inside; the Functions note covers the general method.

1. **Least value of the domain** for $\ln(ax+b)$: the bracket must be positive, so the boundary is where $ax + b = 0$.
2. **Range.** Over its largest domain, $k\ln(ax+b)$ takes every real value. For $ke^{nx} + a$ the range stops at the asymptote: $k > 0$ gives $f(x) > a$; $k < 0$ gives $f(x) < a$. With a restricted domain, find the value at the end point too.
3. **Inverse**: write $y = f(x)$, rearrange for x (an exponential becomes a log and a log becomes an exponential), then swap to $f^{-1}(x)$. Its **domain is the range of f** and its **range is the domain of f**.
4. **Composites**: $e^{\ln X} = X$ and $\ln e^X = X$ often simplify $fg(x)$ a lot. Then solve as in types 1 and 2.

For example, $f(x) = 2 + 3\ln(x-1)$ for $x > 1$ has range $f(x) \in \mathbb{R}$; $y - 2 = 3\ln(x-1)$ gives $x = 1 + e^{(y-2)/3}$, so $f^{-1}(x) = 1 + e^{(x-2)/3}$ for $x \in \mathbb{R}$, with range $f^{-1}(x) > 1$. And $g(x) = 4e^{-2x} + 1$ for $x \geqslant 0$ has range $1 < g(x) \leqslant 5$ (it starts at $g(0) = 5$ and falls towards 1); $g^{-1}(x) = -\frac{1}{2}\ln\left(\frac{x-1}{4}\right)$ for $1 < x \leqslant 5$.

4. Sketching log and exponential graphs (13 questions)

1. **Asymptote first**: $y = a$ for $ke^{nx} + a$; $x = -\frac{b}{a}$ for $k\ln(ax+b)$. Draw it as a dashed line and write its equation.
2. **Intercepts**, as exact values: put $x = 0$, and put $y = 0$ (for a log: the bracket equals 1). Some curves miss an axis; don't invent an intercept.
3. **Shape**: decide which side of the asymptote the curve lies on and which way it goes (see "Understand it"), and show it clearly getting closer to the asymptote without touching or crossing it.
4. **Only the domain given**: a log curve exists only on one side of its asymptote; a function with $x \geqslant 0$ starts at the y-axis.

Variations you will meet:

- **f and f^{-1} on the same axes**: draw f, then reflect it in $y = x$. Every intercept and asymptote swaps: $(0, c)$ becomes $(c, 0)$, and $y = a$ becomes $x = a$. If the two curves meet, they meet **on** the line $y = x$. Label each curve.
- **Reading a graph back**: if you are told an asymptote or an intercept, the facts above give equations for unknown constants (Example 3).

5. Using the laws: a single logarithm, simplifying, rearranging (12 questions)

1. **Numbers become logs** in the right base: $3 = \lg 1000$, $2 = \log_2 4$, $1 = \log_a a$.
2. **Power law first**: move every coefficient inside as a power ($2\ln x = \ln x^2$, $\frac{1}{2}\ln x = \ln\sqrt{x}$).
3. **Then combine**: plus terms multiply, minus terms divide. Simplify the fraction.

For example, $\frac{1}{2}\ln x + \ln 4 - 2 = \ln\sqrt{x} + \ln 4 - \ln e^2 = \ln\dfrac{4\sqrt{x}}{e^2}$.

Variations you will meet:

- **"Find the value of rs"** from an equation with two different bases: change to one base, combine, remove the log.
- **"Express y in terms of x"** from an equation such as $\log_y x = \ldots$: collect the logs into one, then undo it.
- **"Write in the form $c + d\log_a 9$"**: change every other base to a and every argument into powers of 3 or 9.

6. Simultaneous equations with logs or exponentials (8 questions)

1. **Turn each equation into ordinary algebra.** A log equation: combine and remove the logs ($\lg x + \lg y = 2$ gives $xy = 100$). An exponential equation: write both sides with the same base and match the powers ($2^x \times 4^y = 2^{x+2y}$), or combine powers of e ($e^a \div e^b = e^{a-b}$, and $e^c = 1$ means $c = 0$).
2. **Solve** the pair by substitution or elimination.
3. **Check** that every x and y you keep makes each log's argument positive.

Variations you will meet:

- **Linear in the logs** (each equation is a number times $\log x$ plus a number times $\log y$): treat $X = \log x$ and $Y = \log y$ as the unknowns, solve the linear pair for X and Y, then undo the logs, for example $x = 3^X$ for base 3. Change any other base first, for example $\log_9 y = \frac{1}{2}\log_3 y$.
- **e^x and e^y as the unknowns**, or $\ln y$ with x: solve for them first, then take $\ln$.

7. Progressions with logarithm terms (5 questions)

1. **Write each term as a multiple of one log**: $\ln x^5 = 5\ln x$, $\lg\sqrt{z} = \frac{1}{2}\lg z$.
2. **Arithmetic**: d = second term minus first; **geometric**: r = second term divided by first. Then use the series formulas (given).

3. **Set up the equation** the question gives; the common log (such as $\ln x$) usually cancels, leaving an equation in n. A second fact, such as "this sum equals -60", then gives $\ln x$ and so x.

For example, $\ln 3, \ln 12, \ln 48$ form an arithmetic progression, because $\ln 12 - \ln 3 = \ln 4 = \ln 48 - \ln 12$. Its nth term is $\ln 3 + (n-1)\ln 4 = \ln\left(3 \times 4^{n-1}\right)$.

8. Logarithms in calculus answers (4 questions)

Integrating gives terms like $k\ln(ax+b)$ (see the calculus notes). The logarithm work comes at the end:

1. After substituting limits, **combine** the $\ln$ terms into one, using the power and quotient laws.
2. If the answer equals a given log, **match the arguments**: $\ln\left(a^2-4\right) - \ln a = \ln 3$, with a a positive constant, gives $\ln \frac{a^2-4}{a} = \ln 3$, so $a^2 - 3a - 4 = 0$ and $a = 4$ (reject $a = -1$).
3. **Limits from intersections** of exponential curves often need a hidden quadratic in e^x (type 2), with the negative root rejected.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

It is given that $\lg 2 = p$ and $\lg 3 = q$.

(a) Express $\lg 72$ in terms of p and q. **[2]**

(b) Express $\lg 7.5$ in terms of p and q. **[2]**

(c) Express $\log_6 12$ in terms of p and q. **[2]**

(a) $72 = 2^3 \times 3^2$, so $\lg 72 = 3\lg 2 + 2\lg 3$ **M1** (product and power laws) $= 3p + 2q$. **A1**

(b) $7.5 = \dfrac{3 \times 10}{4} = \dfrac{3 \times 10}{2^2}$, so $\lg 7.5 = \lg 3 + \lg 10 - 2\lg 2$ **M1** $= q + 1 - 2p$. **A1** Remember $\lg 10 = 1$.

(c) Change to base 10: $\log_6 12 = \dfrac{\lg 12}{\lg 6}$ **M1**, and $\lg 12 = 2p + q$, $\lg 6 = p + q$, so

$$\log_6 12 = \frac{2p+q}{p+q}.$$

A1. You cannot cancel further: this is a fraction of two logs, not a log of a fraction.

Example 2

(a) Solve the equation $x^{\lg x} = 100x$. **[4]**

(b) Solve the equation $\ln\left(e^{2x} - 3\right) = x + \ln 6$, giving your answer in exact form. **[5]**

(a) Take $\lg$ of both sides and use the power and product laws: **B1**

$$\lg x \times \lg x = \lg 100 + \lg x \;\Rightarrow\; (\lg x)^2 = 2 + \lg x.$$

Let $u = \lg x$: $u^2 - u - 2 = 0$ **M1**, so $(u-2)(u+1) = 0$ and $u = 2$ or $u = -1$. **M1**

$\lg x = 2$ gives $x = 100$; $\lg x = -1$ gives $x = \frac{1}{10}$. **A1** Both are positive, so both are valid.

(b) Write x as $\ln e^x$, so the right side is $\ln e^x + \ln 6 = \ln\left(6e^x\right)$. **B1** Removing the logs:

$$e^{2x} - 3 = 6e^x.$$

M1. With $u = e^x$: $u^2 - 6u - 3 = 0$, so $u = \dfrac{6 \pm \sqrt{48}}{2} = 3 \pm 2\sqrt{3}$. **M1 A1**

$3 - 2\sqrt{3} = -0.46\ldots$ is negative, and e^x is never negative, so reject it. Then $e^x = 3 + 2\sqrt{3}$ and

$$x = \ln\left(3 + 2\sqrt{3}\right) \;\; (= 1.87 \text{ to } 3 \text{ s.f.}).$$

A1, with the rejection shown. (Check: $e^{2x} - 3 = 6e^x > 0$, so the log on the left exists.)

Example 3

The curve $y = ke^{2x} + a$, where k and a are constants, has the line $y = 4$ as an asymptote and crosses the x-axis where $x = \frac{1}{2}\ln 2$.

(a) Find the values of a and k, and the y-intercept of the curve. **[3]**

The function f is defined by $f(x) = ke^{2x} + a$ for $x \in \mathbb{R}$, with these values.

(b) Find an expression for $f^{-1}(x)$ and state its domain. **[3]**

(c) Sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, showing where each meets the axes and the equations of the asymptotes. **[4]**

(a) The asymptote of $ke^{2x} + a$ is $y = a$, so $a = 4$. **B1**

At $x = \frac{1}{2}\ln 2$: $e^{2x} = e^{\ln 2} = 2$, so $2k + 4 = 0$ and $k = -2$. **M1**

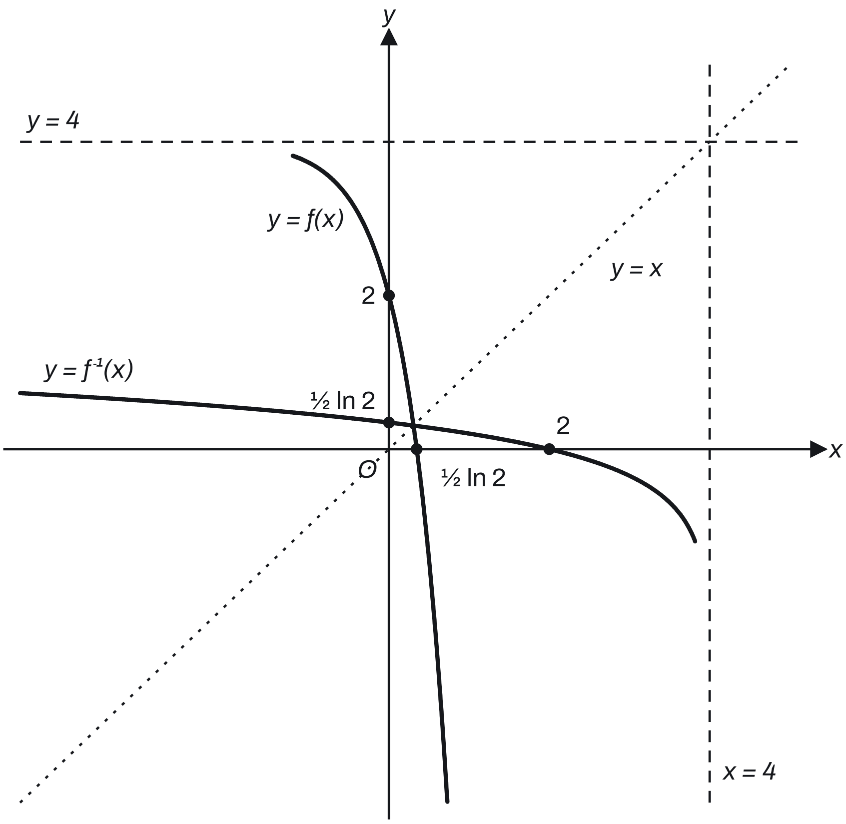
y-intercept: $-2e^0 + 4 = 2$. **A1** So $f(x) = 4 - 2e^{2x}$. Because $k < 0$, the curve lies **below** its asymptote, and because $n = 2 > 0$ it approaches the asymptote on the **left** and falls steeply on the right.

(b) $y = 4 - 2e^{2x} \Rightarrow e^{2x} = \dfrac{4-y}{2} \Rightarrow 2x = \ln\left(\dfrac{4-y}{2}\right)$ **M1**, so

$$f^{-1}(x) = \frac{1}{2}\ln\left(\frac{4-x}{2}\right).$$

A1. The range of f is $f(x) < 4$, so the domain of f^{-1} is $x < 4$. **B1**

(c) **B1** for $y = f(x)$: close to $y = 4$ on the far left, then falling ever more steeply. **B1** for its intercepts $(0, 2)$ and $\left(\frac{1}{2}\ln 2, 0\right)$ and the asymptote $y = 4$. **B1** for $y = f^{-1}(x)$ as the reflection in $y = x$, falling towards the asymptote $x = 4$. **B1** for its intercepts $(2, 0)$ and $\left(0, \frac{1}{2}\ln 2\right)$ and the asymptote $x = 4$.



Example 4

Let $u = \log_4 x$, where $x > 0$ and $x \neq 1$.

(a) Show that $\log_x 64 = \dfrac{3}{u}$ and $\log_4(16x) = 2 + u$. **[3]**

(b) Hence solve the equation $\log_4(16x) = \log_x 64$. **[4]**

(c) Hence write down all the solutions of $\log_4\left(16y^2\right) = \log_{y^2} 64$. **[2]**

(a) Change of base to 4: $\log_x 64 = \dfrac{\log_4 64}{\log_4 x}$ **B1** $= \dfrac{3}{u}$, since $4^3 = 64$. **B1**

Product law: $\log_4(16x) = \log_4 16 + \log_4 x = 2 + u$. **B1**

(b) $2 + u = \dfrac{3}{u} \Rightarrow u^2 + 2u - 3 = 0$ **M1** $\Rightarrow (u+3)(u-1) = 0$ **M1**, so $u = 1$ or $u = -3$. **A1**

$\log_4 x = 1$ gives $x = 4$; $\log_4 x = -3$ gives $x = 4^{-3} = \frac{1}{64}$. **A1** Both are positive and not 1, so both stand. (Don't reject $u = -3$: a log can be negative.)

(c) This is the equation in (b) with $x = y^2$, so $y^2 = 4$ or $y^2 = \frac{1}{64}$. **M1**

$y = \pm 2$ or $y = \pm\frac{1}{8}$. **A1** The negative values are valid here, because only y^2 appears inside the logs, and $y^2 > 0$.

Common mistakes

- **Inventing log laws.** $\log(x+3)$ is not $\log x + \log 3$, and $\dfrac{\log_a x}{\log_a y}$ is not $\log_a x - \log_a y$. Examiners see the second one every year, often leading to a "correct" answer that earns nothing.
- **Mixing up $(\ln x)^2$ and $\ln x^2$.** In a quadratic in a logarithm, the square is on the whole log. Writing $u = \log_2(x+1)$ before rearranging avoids the confusion.
- **Leaving a number out of the combining.** "Write $3 + \ldots$ as a single logarithm" needs $3 = \lg 1000$ (or $\log_2 8$); forgetting to convert it leaves two terms.
- **Rejecting a negative logarithm.** $\log_2(x+1) = -1$ is fine: it gives $x + 1 = \frac{1}{2}$. Only arguments must be positive. Examiners report candidates throwing away correct answers this way.
- **Keeping an impossible root.** $e^x = -2$ has no solution; writing $x = \ln(-2)$ or $-\ln 2$ loses the mark. Equally, check that each x makes every log's argument positive and reject any that don't, with a reason.
- **Decimals when "exact" is asked,** or on Paper 1, where no calculator is allowed: $\ln 5$ must stay $\ln 5$, not 1.61. Sketches need exact intercepts too, such as $5\ln 3$.
- **Wrong range of a log function.** Over its largest domain, $\ln(ax+b)$ takes every real value; write $f(x) \in \mathbb{R}$, using f (or y), not x.
- **Weak sketches.** Show the asymptote as a line with its equation, draw the curve clearly approaching it on the correct side, stay inside the domain (a log curve on one side of its asymptote only), and draw f^{-1} as a true reflection in $y = x$.
- **Missing brackets.** $\ln 2x + 1$ means $(\ln 2x) + 1$. Write $\ln(2x+1)$, especially after integrating.
- **Losing roots.** $x^2 = \ldots$ inside a composite gives $\pm$; keep both unless the domain rules one out.
- **Changing to $\lg$ or $\ln$ when you don't need to.** It works, but makes the algebra longer and slips more likely; use the base in the question.

How to get the marks

- **Show the change of base** as a line of working, such as $\log_x 2 = \dfrac{1}{\log_2 x}$. It usually carries a mark of its own, and the method marks that follow depend on it.
- **Form the quadratic, then show how you solve it** (factorise or use the formula). The mark for solving needs a correct method seen.
- **"Solve"** means find **every** solution and give only the valid ones. If you reject one, say why ("$e^x > 0$", "the argument would be negative").

- **"Exact"** means answers like $\ln(3+2\sqrt{3})$, $e^{3/2}$ or $\frac{e^4-7}{2}$, with no decimals. Otherwise give 3 significant figures unless told otherwise (for example "correct to 3 decimal places").
- **"Write as a single logarithm"** means one log and nothing else; "to base 10" means $\lg$.
- **"Hence"** means use the part before, as in Example 4(c).
- **"Show that"**: the answer is printed, so every step must be visible, ending with exactly the given result.
- **"Sketch"**: the right shape and its key features (intercepts, asymptotes) labelled; it does not need to be to scale. "State" or "write down" needs just the answer.
- **Use the right notation** for domain and range: $x > -6$ for a domain, $f(x) \in \mathbb{R}$ or $f^{-1}(x) > 1$ for a range.
- **Method marks follow through**, so if an earlier value is wrong, keep going with it correctly.

Quick check

1. Without a calculator, find the exact values of (a) $\log_8 32$, (b) $e^{3\ln 2}$, (c) $\ln \dfrac{1}{\sqrt{e}}$.
2. The graph of $y = k\ln(ax+b)$, where a, b and k are integers and $a > 0$, has the asymptote $x = -\frac{5}{2}$, crosses the x-axis at $(-2, 0)$ and crosses the y-axis at $(0, 3\ln 5)$. Find a, b and k.
3. Solve the simultaneous equations $\log_2 x + \log_4 y = 4$ and $\log_{\sqrt{2}} x = \log_2 y$.
4. The first three terms of a geometric progression are $\log_2 x$, $\log_4 x$ and $\log_{16} x$, where $x > 1$. Find the common ratio, and find x given that the sum to infinity is 12.
5. The functions f and g are defined by $f(x) = \ln(x-2)$ for $x > 2$ and $g(x) = e^{2x} + 3$ for $x \in \mathbb{R}$. (a) Solve $fg(x) = \ln 10$, giving your answer in exact form. (b) Show that $gf(x) = (x-2)^2 + 3$.

Answers

1. (a) $32 = 8^{5/3}$ (since $8 = 2^3$ and $32 = 2^5$), so $\log_8 32 = \frac{5}{3}$. (b) $e^{3\ln 2} = e^{\ln 8} = 8$. (c) $\ln e^{-1/2} = -\frac{1}{2}$.
2. The asymptote is where $ax + b = 0$: $-\frac{b}{a} = -\frac{5}{2}$, so $b = \frac{5}{2}a$. The x-intercept is where $ax + b = 1$: $-2a + b = 1$. So $-2a + \frac{5}{2}a = 1$, giving $a = 2$, $b = 5$. Then $k \ln 5 = 3 \ln 5$, so $k = 3$: $y = 3\ln(2x + 5)$.
3. Let $X = \log_2 x$ and $Y = \log_2 y$. Then $\log_4 y = \frac{1}{2}Y$ and $\log_{\sqrt{2}} x = 2X$ (since $\sqrt{2} = 2^{1/2}$), so $X + \frac{1}{2}Y = 4$ and $2X = Y$. Substituting, $2X = 4$, so $X = 2$ and $Y = 4$: $x = 2^2 = 4$ and $y = 2^4 = 16$.
4. $\log_4 x = \frac{1}{2}\log_2 x$ and $\log_{16} x = \frac{1}{4}\log_2 x$, so the ratio is $r = \frac{1}{2}$. $S_\infty = \dfrac{\log_2 x}{1 - \frac{1}{2}} = 2\log_2 x = 12$, so $\log_2 x = 6$ and $x = 64$.
5. (a) $fg(x) = \ln\left(e^{2x} + 3 - 2\right) = \ln\left(e^{2x} + 1\right) = \ln 10$, so $e^{2x} = 9$ and $x = \frac{1}{2}\ln 9 = \ln 3$. (b) $gf(x) = e^{2\ln(x-2)} + 3 = e^{\ln(x-2)^2} + 3 = (x-2)^2 + 3$.

Past questions to try

- **0606/12/F/M/25 Q9**: change of base giving a quadratic in a logarithm.
- **0606/13/M/J/25 Q6**: where two curves built from $\ln x$ meet, with exact answers.
- **0606/12/M/J/25 Q9**: a log function, its inverse and both graphs with intercepts and asymptotes.
- **0606/22/O/N/24 Q5**: a log equation with two bases, then a hidden quadratic in an exponential.