

Series

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Read first: [Permutations and combinations](#), [Logarithmic and exponential functions](#)

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What you need to know

By the end of this topic you should be able to:

1. **Expand $(a + b)^n$ with the binomial theorem**, where n is a positive whole number, and simplify every coefficient to a single number (or a simple expression if the bracket holds a letter such as k).
2. **Use the general term $\binom{n}{r}a^{n-r}b^r$** to pick out one term without writing the whole expansion: a given power of x , the term independent of x , or an unknown constant hidden in a term.
3. **Tell arithmetic progressions from geometric ones.** In an arithmetic progression you add the same number each time; in a geometric progression you multiply by the same number each time.
4. **Use the n th-term and sum-to- n -terms formulas** for both kinds of progression, including problems set in a real situation.
5. **Decide whether a geometric progression has a sum to infinity**, explain why it does or doesn't, and use $S_\infty = \frac{a}{1-r}$ when it does.

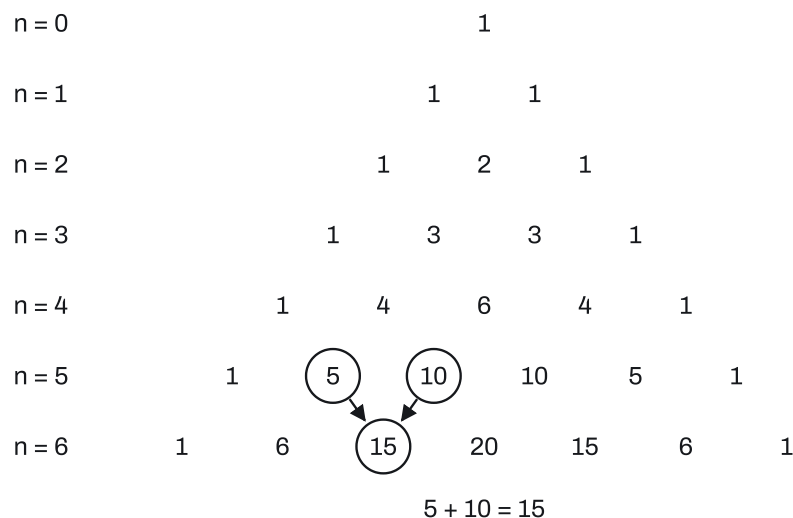
Two things are **not** needed: finding the greatest term of an expansion, and the special properties of binomial coefficients. Expansions with negative or fractional powers, such as $(1 + x)^{-1}$, are not in this syllabus either.

Understand it

Expanding a bracket to a power

Multiply out $(a + b)^n$ and every term has the form (number) $\times a^{\text{something}} b^{\text{something}}$, where the two powers always add up to n . The power of a starts at n and goes down by one each term; the power of b starts at 0 and goes up by one. So the only real work is finding the numbers in front, the **binomial coefficients**.

The coefficients for each n are one row of **Pascal's triangle**. Each row starts and ends with 1, and every other number is the sum of the two just above it.



For $(2 + x)^4$, row 4 gives 1, 4, 6, 4, 1:

$$(2+x)^4 = 2^4 + 4(2^3)x + 6(2^2)x^2 + 4(2)x^3 + x^4 = 16 + 32x + 24x^2 + 8x^3 + x^4.$$

For bigger n you don't draw the triangle: the r th number along row n (counting from $r = 0$) is $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$, the number of ways of choosing which r of the n brackets supply a b . Your calculator has an nC_r key. Useful ones to know by heart: $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{2} = \frac{n(n-1)}{2}$, $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$, and the row is symmetric, so $\binom{n}{r} = \binom{n}{n-r}$.

The general term

The term with b^r in it is

$$\binom{n}{r} a^{n-r} b^r.$$

This is the key to almost every binomial question. When b is something like $-\frac{3}{x^2}$, put the **whole** of it in a bracket: $(-\frac{3}{x^2})^r = (-3)^r x^{-2r}$. The number, its sign and the power of x are all raised to the power r .

"Ascending powers of x " means start with the constant term, then x , then x^2 . **"Descending powers"** means start with the highest power. For $(3x + 1)^5$ in descending powers, $a = 3x$ comes first; in ascending powers, write it as $(1 + 3x)^5$ so that the constant is a .

"The term independent of x " means the term with no x in it, where the powers of x cancel to x^0 .

Arithmetic progressions

A **sequence** is a list of numbers in order, $u_1, u_2, u_3, \dots$. A **series** is what you get when you add terms up.

An **arithmetic progression** (AP) goes up or down by the same amount each time, the **common difference** d . For 5, 9, 13, 17, ... the first term is $a = 5$ and $d = 4$. To reach the n th term you add d a total of $n - 1$ times (not n times: the first term has no d added):

$$u_n = a + (n - 1)d.$$

To add the first n terms, write the sum forwards and backwards. Each pair (first + last, second + second-last, ...) adds to the same total, $a + l$, where l is the last term, and there are n pairs, which counts every term twice:

$$S_n = \frac{1}{2}n(a + l) = \frac{1}{2}n\{2a + (n - 1)d\}.$$

For 5, 9, 13, ...: $u_{20} = 5 + 19 \times 4 = 81$ and $S_{20} = \frac{1}{2} \times 20 \times (5 + 81) = 860$.

Geometric progressions

A **geometric progression** (GP) is multiplied by the same number each time, the **common ratio** r . For 3, 6, 12, 24, ... the first term is $a = 3$ and $r = 2$. The ratio can be a fraction (81, 27, 9, ... has $r = \frac{1}{3}$) or negative (4, -8, 16, ... has $r = -2$, so the signs alternate). The n th term has been multiplied by r a total of $n - 1$ times:

$$u_n = ar^{n-1}.$$

To add the first n terms, write $S_n = a + ar + \dots + ar^{n-1}$, multiply everything by r and subtract. All the middle terms cancel, leaving $S_n - rS_n = a - ar^n$, so

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad (r \neq 1).$$

For 3, 6, 12, ...: $S_6 = \frac{3(1-2^6)}{1-2} = 189$. When $r > 1$ the same formula written as $\frac{a(r^n-1)}{r-1}$ avoids negative signs.

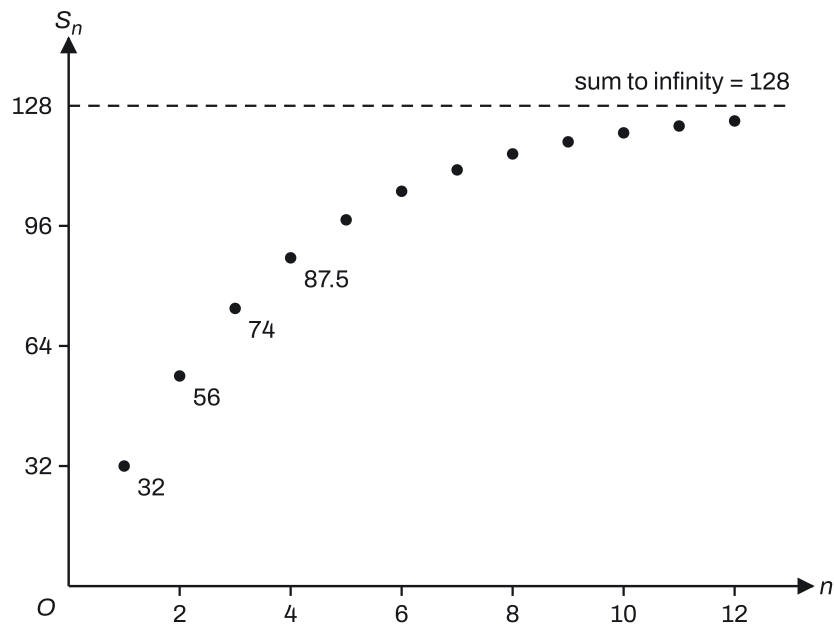
Which kind is it? Subtract neighbouring terms: if the differences are all the same, it is an AP. Divide neighbouring terms: if the ratios are all the same, it is a GP. 2, 6, 10 is an AP ($d = 4$); 2, 6, 18 is a GP ($r = 3$).

For three terms x, y, z in a row:

- AP: $y - x = z - y$, that is $2y = x + z$ (the middle term is the average of the others);
- GP: $\frac{y}{x} = \frac{z}{y}$, that is $y^2 = xz$.

Sum to infinity

Keep adding terms of 32, 24, 18, 13.5, ... (here $r = \frac{3}{4}$) and the total climbs, but by less each time, because each new term is only three-quarters of the last. The running totals get closer and closer to 128 without ever passing it.



The reason is in the formula. If $-1 < r < 1$, then r^n gets closer and closer to 0 as n grows ($0.75^{10} \approx 0.056$, $0.75^{50} \approx 0.0000006$). So $S_n = \frac{a(1-r^n)}{1-r}$ gets closer to

$$S_\infty = \frac{a}{1-r}, \quad \text{valid only when } |r| < 1.$$

Here $S_\infty = \frac{32}{1-0.75} = 128$. Such a progression is called **convergent**. If $|r| \geq 1$ the terms don't shrink, so the total keeps growing (or keeps jumping about, as with $r = -1$ or $r = -2$): there is **no sum to infinity**. An AP never has one either, unless every term is 0.

Key facts and formulas

"Given" means it is in the list of formulas printed on page 2 of the exam paper. "Learn it" means you must remember it.

Formula	Status
$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n, n \text{ a positive integer}$	Given
$\binom{n}{r} = \frac{n!}{(n-r)!r!}$	Given
General term: $\binom{n}{r}a^{n-r}b^r$ (the term containing b^r)	Given (inside the binomial theorem)
$\binom{n}{1} = n, \binom{n}{2} = \frac{n(n-1)}{2}, \binom{n}{3} = \frac{n(n-1)(n-2)}{6}$	Learn it
AP: $u_n = a + (n-1)d$	Given
AP: $S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a + (n-1)d\}$	Given
GP: $u_n = ar^{n-1}$	Given

Formula	Status
GP: $S_n = \frac{a(1-r^n)}{1-r}, r \neq 1$	Given
GP: $S_\infty = \frac{a}{1-r}$, for $ r < 1$	Given
$d = u_2 - u_1; r = \frac{u_2}{u_1}$	Learn it
Three terms x, y, z in a row: AP if $2y = x + z$; GP if $y^2 = xz$	Learn it
$u_n = S_n - S_{n-1}$	Learn it
A sum to infinity exists only when $-1 < r < 1$	Learn it

How to do it

In the Paper 1 and Paper 2 questions we have tagged for this topic (78 questions, 2021–2025), the types came up this often. 46 of the 78 questions mix two or more types (a binomial part and a progression part, say), so the counts add up to more than 78.

Question type	Questions
Sum to infinity and convergence	27
Arithmetic progressions: terms and sums	21
Binomial: unknown constants from given terms	20
Binomial: expansions and coefficients	16
Geometric progressions: terms and sums	15
Binomial: the term independent of x	13
Progressions with logarithm or trigonometry terms	11
An arithmetic and a geometric progression linked	8
Progressions in a real situation	2

1. Sum to infinity and convergence (27 questions)

- Find r .** Divide the second term by the first (check with the third).
- Does a sum to infinity exist?** Write down the value of r , then say whether $-1 < r < 1$. " $r = -\frac{2}{5}$, so $|r| < 1$ and the sum to infinity exists." With two possible values of r , give the reason for **each** one.
- Find it** with $S_\infty = \frac{a}{1-r}$.

Variations you will meet:

- S_∞ is given and something else is unknown. $S_\infty = 45$ with $r = -0.8$ gives $a = 45 \times 1.8 = 81$.

- **Two facts, two unknowns.** Write one equation for each fact in a and r , for example a sum of terms ($a + ar = \dots$, or $S_n = \dots$) and $\frac{a}{1-r} = \dots$. Make a the subject of the simpler one and substitute. You often get a quadratic in r ; keep only roots with $|r| < 1$ (and $r > 0$ if you are told the ratio is positive). Factorising $1 - r^2 = (1 - r)(1 + r)$ or $1 - r^3 = (1 - r)(1 + r + r^2)$ often helps cancel.
- **"Find the sum of all the terms after the k th":** $S_\infty - S_k = \frac{ar^k}{1-r}$.
- **A ratio that contains θ ,** such as $r = 1 - 2\cos\theta$ or $r = \tan^2\theta$. Solve $-1 < r < 1$ for θ in the given interval: sketch the trig graph, find where it crosses each boundary, and give the answer as inequalities, such as $0^\circ < \theta < 90^\circ$. Remember that $\sin$ and $\cos$ always lie between -1 and 1 , which sometimes makes one side automatic.
- **"Show that the progression has a sum to infinity"** with $r = \cos\theta$ and a range of θ : find the values of $\cos\theta$ at the ends of the range and show they lie strictly between -1 and 1 , then conclude.
- **Two progressions built from one.** If you take every second term of a GP with ratio r , you get another GP with ratio r^2 . Write its first term and ratio, then use $\frac{a}{1-r}$ again.

2. Arithmetic progressions: terms and sums (21 questions)

1. **Turn every fact into an equation in a and d ,** using $u_n = a + (n - 1)d$ and $S_n = \frac{1}{2}n\{2a + (n - 1)d\}$.
2. **Solve** them together (subtracting two term equations removes a).
3. **Answer the question asked:** a term, a sum, or a number of terms.

Variations you will meet:

- **A formula for S_n is given,** such as $S_n = 3n^2 + 2n$. Then $u_1 = S_1 = 5$ and $u_n = S_n - S_{n-1} = 6n - 1$, so $d = 6$.
- **Least number of terms for the sum to pass a value.** With $a = 5$, $d = 6$ and "greater than 1000": $\frac{1}{2}n(10 + 6(n - 1)) > 1000$ becomes $3n^2 + 2n - 1000 > 0$. The positive root of $3n^2 + 2n - 1000 = 0$ is $n = 17.9\dots$, so the least whole number is $n = 18$. Check: $S_{17} = 901$, $S_{18} = 1008$.
- **Which term first passes a value:** solve $a + (n - 1)d > k$ and round **up**.
- **Sums of blocks of terms,** such as the sum of the 11th to the 25th terms ($S_{25} - S_{10}$) or the sum of the last few terms: write each block with the formulas.
- **A given number of terms.** If you know the first term, the last term and the total, $S_n = \frac{1}{2}n(a + l)$ gives n straight away.
- **Terms written in terms of one letter.** If $u_4 = 7$, then $u_1 = 7 - 3d$ and $u_2 = 7 - 2d$; substitute into the extra fact (a sum of squares, a product) to get a quadratic in d .
- **"Show that" an expression in two sums,** such as $S_{2k} - S_k$: write each sum with the formula in terms of k and simplify.

3. Binomial: unknown constants from given terms (20 questions)

1. **Write the first few terms of the expansion with the letters left in,** using the general term. For $(p + qx)^n$: p^n , then $\binom{n}{1}p^{n-1}qx$, then $\binom{n}{2}p^{n-2}q^2x^2$.
2. **Match coefficients** with the terms you are given, one equation per term.

3. **Solve**, usually by dividing one equation by another so that most letters cancel.

4. **Check signs and conditions**. A square such as $a^2 = 9$ gives $a = \pm 3$; use the other facts (a positive coefficient, " $a > 0$ ", "integers") to keep the right one.

For example, $(1 + ax)^n = 1 + 24x + 252x^2 + \dots$ gives $na = 24$ and $\frac{n(n-1)}{2}a^2 = 252$. Divide the second by the square of the first: $\frac{n(n-1)a^2}{n^2a^2} = \frac{504}{576}$, so $\frac{n-1}{n} = \frac{7}{8}$, giving $n = 8$ and $a = 3$.

Variations you will meet:

- **The first term is a power**: $(2 + \dots)^n$ starting with 256 means $2^n = 256$, so $n = 8$. Look out for a number in front of the bracket: $3(2 + \dots)^n$ starting with 96 means $3 \times 2^n = 96$, so $n = 5$.
- **An unknown squared in a given term**, such as b^2x : if $b^2 = 180$, then $b = \pm\sqrt{180} = \pm 6\sqrt{5}$, and both values stand unless you are told otherwise. And since b^2 can't be negative, the matching coefficient must be positive, which can decide the sign of another letter.
- **A condition on two coefficients**, such as "the coefficient of x^3 is 4 times the coefficient of x^2 " or "the coefficients of x^r and x^{r+1} are equal". Write both with the general term and cancel factorials: $\frac{n!}{(n-r)!r!}$ and $\frac{n!}{(n-r-1)!(r+1)!}$ differ by the factor $\frac{n-r}{r+1}$.
- **Two brackets multiplied together**: collect every pair of terms that makes the power you want (see type 4) and then match.
- **Terms that are equal**, such as the 3rd and 6th terms: the k th term uses $r = k - 1$.

4. Binomial: expansions and coefficients (16 questions)

1. **Identify a , b and n** , with the sign kept inside b : in $(2 - x)^6$, $a = 2$ and $b = -x$.
2. **Write each term** as $\binom{n}{r}a^{n-r}b^r$ with brackets, then work out each coefficient as a single number.
3. **Give the terms in the order asked** (ascending or descending) and as many as asked.

Variations you will meet:

- **A coefficient in a product**, such as the x^2 term of $(1 + ax)(2 - x)^6$: the 1 pairs with the x^2 term of the second bracket and the ax pairs with its x term. Add these products. With two binomial brackets, expand each as far as needed first ("hence" usually means use your earlier expansion).
- **A quadratic raised to a power**: first spot that it is a perfect square, for example $4x^2 - 4x + 1 = (2x - 1)^2$, so $(4x^2 - 4x + 1)^3 = (2x - 1)^6$.
- **Surds**: in $(p + \sqrt{q})^n - (p - \sqrt{q})^n$ the terms with even powers of $\sqrt{q}$ cancel and the terms with odd powers double. Simplify powers of the surd, such as $(\sqrt{3})^3 = 3\sqrt{3}$. Work without a calculator when told to.
- **Approximations**: $(1 + 2x)^6 \approx 1 + 12x + 60x^2$, so $x = 0.01$ gives $1.02^6 \approx 1 + 0.12 + 0.006 = 1.126$. Show the substitution.
- **Other letters**: $(2 + e^x)^3 = 8 + 12e^x + 6e^{2x} + e^{3x}$, because $(e^x)^2 = e^{2x}$.

5. Geometric progressions: terms and sums (15 questions)

1. **Turn each fact into an equation in a and r** with $u_n = ar^{n-1}$ and $S_n = \frac{a(1-r^n)}{1-r}$.

- Divide** one equation by the other to remove a : $u_2 = -6$ and $u_5 = 162$ give $\frac{ar^4}{ar} = r^3 = -27$, so $r = -3$ and $a = 2$.
- Find what is asked.** An even power such as $r^2 = \frac{9}{4}$ gives $r = \pm\frac{3}{2}$; use the conditions (positive terms, positive ratio) to choose.

Variations you will meet:

- Least n for a sum or term to pass a value:** use logarithms. For $a = 5$, $r = 2$ and $S_n > 10\,000$: $5(2^n - 1) > 10\,000$, so $2^n > 2001$ and $n > \frac{\lg 2001}{\lg 2} = 10.97\dots$, giving $n = 11$. If $r < 1$, $\lg r$ is negative, so dividing by it turns $<$ into $>$.
- Sum of a block of terms**, such as the 11th to the 20th: $S_{20} - S_{10}$, or treat the block as a new GP starting at u_{11} .
- A block of terms against an earlier block:** each term of the second block is the matching term of the first multiplied by the same power of r . For example, $u_4 + u_5 + u_6 = ar^3 + ar^4 + ar^5 = r^3(a + ar + ar^2)$, so if the sum of the 4th, 5th and 6th terms is 64 times the sum of the first three, then $r^3 = 64$ and $r = 4$.
- The terms are expressions in x** (such as $x + 1$, $x + 5$, $3x + 7$): use $y^2 = xz$ for three terms in a row, solve for x , then find r .

6. Binomial: the term independent of x (13 questions)

- Write the general term** with every part bracketed: for $(2x^2 - \frac{1}{x})^9$ it is $\binom{9}{r}(2x^2)^{9-r}(-\frac{1}{x})^r$.
- Collect the powers of x :** $x^{18-2r} \times x^{-r} = x^{18-3r}$.
- Set the power equal to 0** (or to the power you want) and solve for r . It must be a whole number from 0 to n ; if it isn't, there is no such term.
- Evaluate** the term with that r , sign included.

Variations you will meet:

- A product**, such as the term independent of x in $(1 + x^2)(x - \frac{2}{x})^6$. The general term of the second bracket is $\binom{6}{r}(-2)^r x^{6-2r}$. The 1 needs the x^0 term ($r = 3$: $20 \times (-8) = -160$) and the x^2 needs the x^{-2} term ($r = 4$: $15 \times 16 = 240$). The answer is $-160 + 240 = 80$.
- "Exact" or "rational":** give a fraction, not a decimal.
- The variable is another letter** (such as y): nothing changes.

7. Progressions with logarithm or trigonometry terms (11 questions)

- Rewrite every term as a multiple of one simple expression** using the log laws ($\ln x^k = k \ln x$) or the trig identities. Then d or r is easy to read off.
- Use the usual formulas** with that a and d (or r), keeping the log or trig factor as a common factor.
- Finish** by solving for n , for x , or for the angle.

For example, $\ln 12$, $\ln 36$, $\ln 108$: each term is the one before plus $\ln 3$ (since $36 = 12 \times 3$), so it is an AP with $a = \ln 12$ and $d = \ln 3$. Then $S_n = \frac{1}{2}n\{2 \ln 12 + (n-1) \ln 3\} = n \ln 12 + \frac{n(n-1)}{2} \ln 3$.

Variations you will meet:

- **A GP of logs:** $\lg x^6, \lg x^4, \lg x^{8/3}$ are $6 \lg x, 4 \lg x, \frac{8}{3} \lg x$, so $r = \frac{2}{3}$ and $S_\infty = \frac{6 \lg x}{1 - \frac{2}{3}} = 18 \lg x$.
- **" S_n is equal to $k \ln x$ ":** cancel the $\ln x$ (it isn't 0) and solve the quadratic in n . For "greater than", **check the sign of $\ln x$ first**: if $x > 1$ then $\ln x > 0$ and dividing keeps the inequality the same way.
- **The sum also equals a number**, such as 150: then $k \ln x = 150$, so $\ln x = \frac{150}{k}$ and $x = e^{150/k}$ exactly.
- **Terms like $\sin^2 x, \tan^2 x$** as a common factor: find S_n in terms of that factor, set it equal to the given value, then solve the trig equation in the given interval.
- **Powers:** terms like q^{2x}, q^{5x}, q^{8x} form a GP with $r = q^{3x}$; the n th term is $q^{2x} (q^{3x})^{n-1} = q^{(3n-1)x}$.

8. An arithmetic and a geometric progression linked (8 questions)

1. **Use different letters** for the two progressions: a and d for the AP, and write the GP's terms in terms of those, or give the GP its own first term, such as A .
2. **Write the shared terms** with $u_n = a + (n-1)d$ or ar^{n-1} .
3. **Link them.** If three AP terms are three GP terms in a row, the ratios are equal: $\frac{2\text{nd}}{1\text{st}} = \frac{3\text{rd}}{2\text{nd}}$, or (middle)² = first $\times$ third.
4. **Expand and simplify.** Usually the squared terms cancel and you are left with a simple relation between a and d ; $d \neq 0$ lets you divide by d .

For example, if the 1st, 4th and 13th terms of an AP with $d \neq 0$ are three terms of a GP in a row, then $(a + 3d)^2 = a(a + 12d)$. Expanding, $a^2 + 6ad + 9d^2 = a^2 + 12ad$, so $9d^2 = 6ad$. Dividing by $3d$ (allowed, as $d \neq 0$) gives $3d = 2a$, so $a = 1.5d$. The terms are $1.5d, 4.5d, 13.5d$, so $r = 3$.

Variations you will meet:

- **"The 2nd term of the GP equals the 4th term of the AP"** and similar: write each equality as an equation in the unknowns, then eliminate.
- **After finding r** , questions often ask whether the GP converges, or for a sum of either progression.

9. Progressions in a real situation (2 questions)

1. **Decide which kind it is.** "Increases by the same amount each week" is an AP. "Increases by 10% each week" is a GP with $r = 1.1$; "decreases by 10%" has $r = 0.9$.
2. **Match the words to n .** The amount in week n is u_n ; the total by the end of week n is S_n .
3. **Answer in context**, as a whole number of weeks where needed, rounding **up** for "first week in which ...".

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

(a) Find the term independent of x in the expansion of $\left(2x^2 - \frac{1}{x}\right)^9$. **[3]**

(b) Find the coefficient of $\frac{1}{x^3}$ in this expansion. **[2]**

(c) Explain why every power of x in this expansion is a multiple of 3. **[1]**

(a) The general term is

$$\binom{9}{r} (2x^2)^{9-r} \left(-\frac{1}{x}\right)^r = \binom{9}{r} 2^{9-r} (-1)^r x^{18-3r}.$$

B1 for a correct general term (or the correct term written down).

$18 - 3r = 0$ gives $r = 6$. **M1**

The term is $\binom{9}{6} \times 2^3 \times (-1)^6 = 84 \times 8 = 672$. **A1**

(b) $\frac{1}{x^3} = x^{-3}$, so $18 - 3r = -3$, giving $r = 7$. **M1**

Coefficient = $\binom{9}{7} \times 2^2 \times (-1)^7 = 36 \times 4 \times (-1) = -144$. **A1**

(c) The power of x is $18 - 3r = 3(6 - r)$, and $6 - r$ is a whole number, so every power is a multiple of 3 (from x^{18} down to x^{-9}). So, for example, there is no term in x^2 . **B1**

Example 2

(a) Find the first four terms in the expansion of $(2 - x)^6$ in ascending powers of x . Simplify each coefficient. **[3]**

(b) The coefficient of x^2 in the expansion of $(1 + ax)(2 - x)^6$ is zero. Find the value of the constant a . **[2]**

(c) Explain why, with this value of a , the expansion of $(4 + 5x)(2 - x)^6$ also has no term in x^2 . **[1]**

(a) With $a = 2$, $b = -x$ and row 6 of Pascal's triangle (1, 6, 15, 20, ...):

$$2^6 + 6(2^5)(-x) + 15(2^4)(-x)^2 + 20(2^3)(-x)^3 = 64 - 192x + 240x^2 - 160x^3.$$

B1 for each correct term after the first $(-192x, 240x^2, -160x^3)$.

(b) The x^2 terms come from $1 \times 240x^2$ and $ax \times (-192x)$:

$$240 - 192a = 0 \Rightarrow a = \frac{240}{192} = \frac{5}{4}.$$

M1 for adding the two products, **A1**.

(c) $4 + 5x = 4\left(1 + \frac{5}{4}x\right) = 4(1 + ax)$, so $(4 + 5x)(2 - x)^6$ is 4 times the product in part (b). Every coefficient is multiplied by 4, and $4 \times 0 = 0$. **B1**

(Check directly: $4 \times 240 + 5 \times (-192) = 960 - 960 = 0$.)

Example 3

The positive numbers p , 10 and q , where $p < q$, are consecutive terms of an arithmetic progression. The numbers p , 8 and q are consecutive terms of a geometric progression.

(a) Find the values of p and q , and the common ratio of the geometric progression. **[4]**

(b) An arithmetic progression has first term p and common difference 6. Find the least number of terms for which its sum is greater than 500. **[4]**

(a) AP: 10 is the average of p and q , so $p + q = 20$. GP: $8^2 = pq$, so $pq = 64$. **B1 B1**

Substituting $q = 20 - p$: $p(20 - p) = 64$, so $p^2 - 20p + 64 = 0$, which is $(p - 4)(p - 16) = 0$. **M1** for forming and solving a quadratic in one letter.

Since $p < q$: $p = 4$, $q = 16$, and the common ratio is $\frac{8}{4} = 2$. **A1**

(The AP here is 4, 10, 16, with $d = 6$, which is the progression used in part (b).)

(b) With $a = 4$ and $d = 6$:

$$S_n = \frac{1}{2}n\{8 + 6(n - 1)\} = n(3n + 1).$$

M1 for using the sum formula. Set $3n^2 + n > 500$, that is $3n^2 + n - 500 > 0$. **A1**

The positive root of $3n^2 + n - 500 = 0$ is $n = \frac{-1 + \sqrt{6001}}{6} = 12.74 \dots$ **M1**

So the least number of terms is $n = 13$. **A1** Check: $S_{12} = 12 \times 37 = 444$ and $S_{13} = 13 \times 40 = 520$.

Example 4

A geometric progression has a sum to infinity that is 4 times its first term. Its third term is 18.

(a) Find the common ratio and the first term. **[3]**

(b) Find the sum to infinity. **[1]**

(c) Find the sum of all the terms after the 6th term. Give your answer as an exact fraction. **[2]**

(d) Find the smallest value of n for which the n th term is less than 1. **[3]**

(a) $\frac{a}{1-r} = 4a$, and $a \neq 0$, so $1-r = \frac{1}{4}$, giving $r = \frac{3}{4}$. **M1 A1**

$ar^2 = 18$, so $a \times \frac{9}{16} = 18$ and $a = 32$. **A1**

(b) $S_{\infty} = \frac{32}{1-\frac{3}{4}} = 128$. **B1** (This is the progression in the second diagram.)

(c) Sum after the 6th term $= S_{\infty} - S_6 = \frac{ar^6}{1-r} = 128 \times \left(\frac{3}{4}\right)^6$. **M1**

$$128 \times \frac{729}{4096} = \frac{729}{32}.$$

A1

(d) $32 \left(\frac{3}{4}\right)^{n-1} < 1$, so $\left(\frac{3}{4}\right)^{n-1} < \frac{1}{32}$. **B1**

Take logarithms: $(n-1) \ln 0.75 < \ln \frac{1}{32}$. Since $\ln 0.75 < 0$, dividing by it reverses the inequality:

$$n-1 > \frac{\ln(1/32)}{\ln 0.75} = 12.04 \dots$$

M1. So $n > 13.04 \dots$ and the smallest value is $n = 14$. **A1** Check: $u_{13} = 1.01 \dots$ and $u_{14} = 0.76 \dots$

Common mistakes

- **Losing the sign or the number inside b .** In $\left(2 - \frac{x}{3}\right)^5$ the second term is $5 \times 2^4 \times \left(-\frac{x}{3}\right)$, not $5 \times 2^4 \times x$. Bracket the whole of b , sign included, and raise all of it to the power.
- **Not understanding "independent of x ".** It means the power of x is 0. Find r from the powers first, instead of guessing which term it is.
- **Leaving coefficients unfinished.** $\binom{7}{3}(2^4)$ is not simplified; 560 is. "Simplify each term" also means including the power of x .
- **Ascending and descending mixed up.** Descending powers of x start with the highest power of x . Check your first term.
- **In a product of brackets, missing a pair.** List every pair of powers that adds up to the power you want before multiplying.
- **Mixing up the two kinds of progression.** Using $a + (n-1)d$ for a GP, or ar^{n-1} for an AP, is a common slip, especially in linked questions. Write the formula you are using before substituting. Also watch the $n-1$: the 12th term has $11d$, not $12d$.
- **Using the same letter for two progressions.** When an AP and a GP appear together, call their first terms different letters.
- **Using S_n when the question gives S_{∞}** (or the other way round). Read which sum it is.
- **Weak explanations of convergence.** "It gets smaller" or " r is a fraction" earns nothing. State the value of r and that $-1 < r < 1$ (or $|r| < 1$); for two possible ratios, deal with each.

- **Keeping a ratio that can't work.** If S_∞ is given, any r with $|r| \geq 1$ must be rejected; if the ratio or terms are positive, reject negative roots.
- **Getting the inequality the wrong way round** when dividing by the logarithm of a number less than 1, or rounding the answer for n down. The least n is always the next whole number above the root.
- **For log progressions, not simplifying the terms first.** Rewrite $\ln x^5$ as $5 \ln x$ and so on before finding d ; otherwise the common difference is easily wrong.
- **Rounding too early.** Keep full calculator values in working, and give exact answers (fractions, surds, e^k) when asked.

How to get the marks

- **Show the binomial terms you are using**, such as $\binom{8}{2}(3)^6(-2x)^2$, before the simplified number. A correct unsimplified term often earns a mark even if the arithmetic then slips.
- **"Simplify your coefficients"** means single numbers (or single fractions). "Exact" means fractions, surds or e^k , not decimals.
- **Write the equations for progressions in full**, such as $a + 11d = 64$ or $\frac{a}{1-r} = 25$. Method marks go to correct equations, even if the solving goes wrong later.
- **"Show that"** gives you the answer: every step must be visible, and the last line must match the printed result exactly.
- **"Explain" or "determine whether" a sum to infinity exists** needs the value of r and the condition $|r| < 1$ (or its failure), then a clear conclusion.
- **"Hence"** means use the part before: an expansion you have just written, or an a and d you have just found.
- **Least or smallest n :** give a whole number, and it helps to show the values either side (such as $S_{12} < 500 < S_{13}$).
- **Non-exact answers** should be given to 3 significant figures unless the question says otherwise; angles in degrees to 1 decimal place.
- **Some questions say "do not use a calculator":** then every step, such as expanding a surd bracket, must be shown.
- **Answer every part of the question.** "Find the values of a , b and n " needs all three, and "possible values" often means more than one.

Quick check

1. Find the coefficient of x^4 in the expansion of $(3 + 2x)^7$.
2. Find the term independent of x in the expansion of $\left(2x + \frac{1}{4x^3}\right)^8$.
3. A hall has 18 seats in the front row, and each row behind has 3 more seats than the row in front of it. There are 25 rows. Find the total number of seats, and find the first row with more than 60 seats.

4. The sum of the first two terms of a geometric progression is 20, and its second term is 12. Find the first term and the common ratio, explain why the progression has no sum to infinity, and find the sum of its first 6 terms.
5. The first three terms of a geometric progression are 3, $3(1 - 2 \cos \theta)$ and $3(1 - 2 \cos \theta)^2$. Find the values of θ , where $0^\circ < \theta < 360^\circ$, for which the progression has a sum to infinity.

Answers

1. The term is $\binom{7}{4} 3^3 (2x)^4 = 35 \times 27 \times 16x^4$, so the coefficient is 15 120.
2. The general term is $\binom{8}{r} (2x)^{8-r} \left(\frac{1}{4x^3}\right)^r$, with power of x equal to $8 - r - 3r = 8 - 4r$. So $r = 2$, and the term is $28 \times 2^6 \times \frac{1}{4^2} = 28 \times 64 \times \frac{1}{16} = 112$.
3. An AP with $a = 18$ and $d = 3$. Total $= \frac{1}{2} \times 25 \times \{36 + 24 \times 3\} = \frac{1}{2} \times 25 \times 108 = 1350$ seats. For more than 60 seats: $18 + 3(n - 1) > 60$ gives $n - 1 > 14$, so $n > 15$: the **16th row** (it has 63 seats).
4. $a + ar = 20$ and $ar = 12$, so $a = 20 - 12 = 8$ and $r = \frac{12}{8} = \frac{3}{2}$. Since $r = \frac{3}{2} > 1$, the terms keep growing, so there is no sum to infinity. $S_6 = \frac{8\left(\left(\frac{3}{2}\right)^6 - 1\right)}{\frac{3}{2} - 1} = 16 \times \left(\frac{729}{64} - 1\right) = \frac{665}{4} = 166.25$.
5. $r = 1 - 2 \cos \theta$. A sum to infinity needs $-1 < 1 - 2 \cos \theta < 1$. Subtract 1: $-2 < -2 \cos \theta < 0$; divide by -2 (reversing the signs): $0 < \cos \theta < 1$. In $0^\circ < \theta < 360^\circ$, $\cos \theta < 1$ always holds, and $\cos \theta > 0$ for $0^\circ < \theta < 90^\circ$ and $270^\circ < \theta < 360^\circ$.

Past questions to try

- **0606/23/M/J/25 Q7**: the term independent of x , then an unknown from two coefficients.
- **0606/11/M/J/25 Q11**: an arithmetic progression with its terms written in terms of d .
- **0606/21/M/J/25 Q11**: a geometric progression from two sums of terms, then its sum to infinity.
- **0606/12/M/J/24 Q10**: progressions with trigonometry and logarithm terms, and convergence.