Cambridge IGCSE · Additional Mathematics 0606 · Notes

Simultaneous equations

Read online at **pastopic.com/cambridge/additional-mathematics-0606/notes/simultaneous-equations**

Last checked 30 September 2026

Read first: **Quadratic functions**, **Factors of polynomials**

What you need to know

By the end of this topic you should be able to:

1. **Solve two equations in two unknowns at the same time**, by **elimination** (combine the equations so one letter disappears) or **substitution** (make one letter the subject of one equation and put it into the other). That includes:
   - two linear equations, even when the coefficients are surds;
   - one linear and one non-linear equation, where the non-linear one may have x^2, y^2 and xy terms, or fractions such as $\frac{x}{y}$;
   - two non-linear equations, such as $xy = k$ together with a quadratic curve.
2. **Give every solution as a matching pair** (x, y), and reject any that can't work.

Exam questions then use these methods in other topics: where a line meets a curve (followed by a mid-point, a perpendicular bisector, a length or an area), equations with logarithms or exponentials, and where a line meets a cubic. This note shows the simultaneous-equations part of each; the log laws, straight-line work and factor theorem are in their own notes.

Understand it

What "simultaneous" means

One equation in x and y, such as $3p + 2q = 16$, has infinitely many solutions: every point on its graph. Two equations together usually pin the letters down, because only a few pairs make **both** true. Graphically, the solutions are the points where the two graphs **cross**.

Elimination

When both equations are linear, line up the terms and add or subtract so one letter cancels. For $3p + 2q = 16$ and $5p - 2q = 0$, adding gives $8p = 16$, so $p = 2$; then $5(2) - 2q = 0$ gives $q = 5$. If no coefficients match, multiply one or both equations first to make them match.

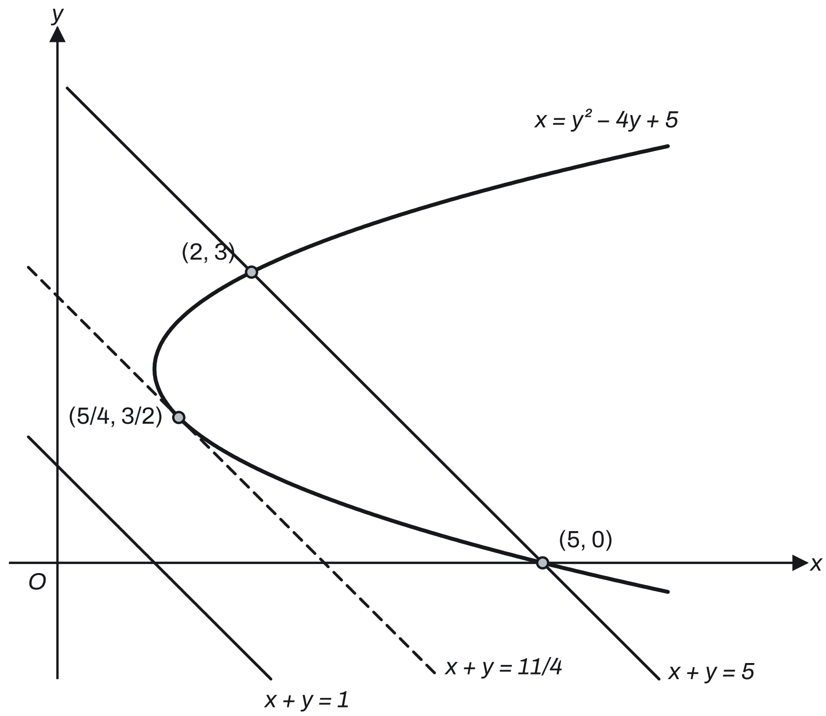
Substitution

Make one letter the subject of the simpler equation (usually the linear one), then replace that letter **everywhere** in the other equation, in brackets. You get one equation in one letter. For $3p + 2q = 16$ and $5p - 2q = 0$: from the second, $q = \frac{5}{2}p$, so $3p + 5p = 16$ and $p = 2$ again.

Substitution is the method to use when one equation is non-linear. Putting a linear expression into a quadratic gives a **quadratic** in one letter, which usually has two roots, so there are usually **two pairs** of answers.

How many answers? The discriminant tells you

After substituting a line into a quadratic curve, you get $ay^2 + by + c = 0$ (or the same in x). Its discriminant $b^2 - 4ac$ says how many times the line meets the curve.



For the curve $x = y^2 - 4y + 5$ and the line $x + y = c$, substitute $x = c - y$:

$$c - y = y^2 - 4y + 5 \;\Rightarrow\; y^2 - 3y + 5 - c = 0, \qquad b^2 - 4ac = 9 - 4(5 - c) = 4c - 11.$$

- $c = 5$: discriminant $9 > 0$, two roots $y = 0, 3$: the line **cuts** the curve at $(5, 0)$ and $(2, 3)$.
- $c = \frac{11}{4}$: discriminant 0, one repeated root $y = \frac{3}{2}$: the line **touches** the curve at $\left(\frac{5}{4}, \frac{3}{2}\right)$.
- $c = 1$: discriminant $-7 < 0$, no real roots: the line **misses** the curve, and the equations have **no solution**.

Pairing the answers

Each root gives **one** value of the other letter, found from the **linear** equation (it has only one answer for each root; the curve's equation may have two). Write the answers as pairs: $y = 0, x = 5$ and $y = 3, x = 2$. Mixing x from one pair with y from another is wrong.

Sometimes a root must be **rejected**: a length or $\ln$ of a negative number, or x^2 equal to a negative number. Say why when you reject it.

Equations that aren't polynomials

Logarithms, exponentials and powers can hide ordinary simultaneous equations:

- **Logs:** $\ln m + \ln(n+2) = \ln 6$ becomes $m(n+2) = 6$ using the log laws; $\log_2 p - \log_2 q = 3$ becomes $\frac{p}{q} = 2^3$.
- **Same base:** $3^m \div 9^n = 27$ is $3^{m-2n} = 3^3$, so $m - 2n = 3$.
- **A new letter:** in $3^a + 3^b = 12$, let $u = 3^a$ and $v = 3^b$; the equations become ordinary ones in u and v.

After that, solve as usual and then go back to the original letters.

Key facts and formulas

"Given" means it is in the List of formulas at the front of the exam paper. "Learn it" means you must remember it.

Formula	Status
Quadratic formula: $ax^2 + bx + c = 0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Given
After substituting a line into a quadratic curve: $b^2 - 4ac > 0$ two points, $= 0$ touches (one point), < 0 no points	Learn it
Mid-point of (x_1, y_1) and (x_2, y_2): $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$	Learn it
Length: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Learn it
Gradient: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$; perpendicular gradient $-\dfrac{1}{m}$	Learn it
Line through (x_1, y_1) with gradient m: $y - y_1 = m(x - x_1)$	Learn it
$\ln A + \ln B = \ln AB$, $\ln A - \ln B = \ln \dfrac{A}{B}$, $k \ln A = \ln A^k$ (any base)	Learn it
$\log_a A = k \iff A = a^k$; $\log_{a^2} A = \frac{1}{2} \log_a A$	Learn it
$a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(a^m)^n = a^{mn}$	Learn it
Rationalise: $\dfrac{k}{a + \sqrt{b}} = \dfrac{k(a - \sqrt{b})}{a^2 - b}$	Learn it

How to do it

In the Paper 1 and Paper 2 questions we have tagged for this topic (33 questions, 2021–2025), the types came up this often. 13 of the 33 questions mix two types (finding the points, then using them), so the counts add up to more than 33.

Question type	Questions
One linear and one non-linear equation	13
Using the intersection points: mid-point, perpendicular bisector, length, area	13
Equations with logarithms, exponentials or powers	10
Two non-linear equations (curve and curve)	6
A line meeting a cubic curve	2
Two linear equations with surd coefficients	2

1. One linear and one non-linear equation (13 questions)

1. **Rearrange the linear equation** to make one letter the subject. Pick the letter that avoids fractions: from $r - 2s = 3$ take $r = 2s + 3$.
2. **Substitute** into the non-linear equation, with brackets: $(2s+3)^2 + (2s+3)s - \ldots$. Write this line out; it earns the first method mark.
3. **Expand and collect** into $as^2 + bs + c = 0$.
4. **Solve** by factorising or with the formula. If there is no constant term, factorise out the letter: $4s^2 + 15s = 0 \Rightarrow s(4s+15) = 0$.
5. **Find the partner** of each root from the linear equation and **write the pairs**.

Variations you will meet:

- **"Find the points where the line meets the curve"** is the same thing: the answers are coordinates, (x, y).
- **Fractions** such as $\frac{2}{p} + \frac{1}{q} = 1$: multiply through by every denominator (pq) first, giving $2q + p = pq$. With $p = q + 1$ this becomes $q^2 - 2q - 1 = 0$, so $q = 1 \pm \sqrt{2}$ and $p = 2 \pm \sqrt{2}$ (signs matching).
- **Exact answers** or **"DO NOT USE A CALCULATOR"**: use the formula and simplify the surd, for example $\frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$.
- **Only one solution:** if the squared terms cancel when you substitute, you get a linear equation and just one pair. If the quadratic has a repeated root, the line touches the curve.

2. Using the intersection points: mid-point, perpendicular bisector, length, area (13 questions)

These questions start with type 1 or type 4 to find two points A and B, then ask for more. Using $P(5, 0)$ and $Q(2, 3)$ from the diagram:

1. **Mid-point** $M = \left(\frac{5+2}{2}, \frac{0+3}{2}\right) = \left(\frac{7}{2}, \frac{3}{2}\right)$.
2. **Gradient** of $PQ = \frac{3-0}{2-5} = -1$, so the **perpendicular gradient** is $-\frac{1}{-1} = 1$.
3. **Perpendicular bisector:** through M with the perpendicular gradient: $y - \frac{3}{2} = 1\left(x - \frac{7}{2}\right)$, that is $y = x - 2$.
4. **Length** $PQ = \sqrt{(5-2)^2 + (0-3)^2} = \sqrt{18} = 3\sqrt{2}$.

Variations you will meet:

- **A point with an unknown coordinate lies on the bisector**, such as $(k, 5)$: substitute it into the bisector's equation and solve for k.
- **Where the bisector meets the axes**, then the area of the triangle with the origin: put $x = 0$ and $y = 0$, then area $= \frac{1}{2} \times |x\text{-intercept}| \times |y\text{-intercept}|$.
- **A point D on the bisector twice as far from AB as C is:** the bisector is perpendicular to AB, so distance from AB is distance from M along it. Find the step $\overrightarrow{MC}$, then $D = M \pm 2\,\overrightarrow{MC}$. With $C(5, 3)$ on $y = x - 2$: $\overrightarrow{MC} = \left(\frac{3}{2}, \frac{3}{2}\right)$, so $D = \left(\frac{13}{2}, \frac{9}{2}\right)$ or $\left(\frac{1}{2}, -\frac{3}{2}\right)$. There are **two** answers, one each side.
- **A point reflected in AB:** it is on the other side of M, the same distance away: $D = M - \overrightarrow{MC}$.
- **Show that B is the mid-point of AC**, or that a triangle is right-angled: compute the mid-point, or show two gradients multiply to -1.
- **Tangent at one of the points** (needs differentiation): find $\frac{dy}{dx}$ at that point, write the tangent, then solve it with another line.

3. Equations with logarithms, exponentials or powers (10 questions)

1. **Turn each equation into one without logs or powers**:
   - log laws first: $\lg a + 3\lg b = \lg(ab^3)$, then $\lg(ab^3) = 2$ means $ab^3 = 100$; $\ln(\ldots) = 0$ means $(\ldots) = 1$;
   - powers: write every number as a power of the **same base** ($9 = 3^2$, $27 = 3^3$) and equate the powers; $e^{\cdots} = 1$ means the power is 0;
   - different log bases: change to one base first, for example $\log_4 v = \frac{1}{2}\log_2 v$.
2. **Two equations linear in $\ln y$ (or $\log_2 x$)**, such as $2x + \ln y = 7$: treat $\ln y$ as a single unknown and eliminate as usual, then undo the log at the end ($\ln y = k \Rightarrow y = e^k$).
3. **Sums of exponentials** such as $3^a + 3^b = 12$ with $3^{a+b} = 27$: let $u = 3^a$, $v = 3^b$. Then $u + v = 12$ and $uv = 27$, so $u^2 - 12u + 27 = 0$, $u = 3$ or 9, giving $(a, b) = (1, 2)$ or $(2, 1)$. **Say what the new letters stand for**, and change back at the end.
4. **Solve** the resulting equations (usually type 1), then **check every answer** in the original: a log needs a positive input, and e^x or 3^x can't be negative or zero.

4. Two non-linear equations (curve and curve) (6 questions)

1. **Make one letter (or one square) the subject** of the simpler equation: $y = \frac{k}{x}$ from $xy = k$, or $x^2 = y + 13$ from $y = x^2 - 13$.
2. **Substitute**, then clear any fractions by multiplying by the highest power of x in a denominator.

3. **Spot a disguised quadratic**: an equation in x^4 and x^2 is a quadratic in x^2. For $x^2y = 12$ and $y = x^2 + 1$: $x^2(x^2+1) = 12$, so $x^4 + x^2 - 12 = 0$, $(x^2+4)(x^2-3) = 0$. Reject $x^2 = -4$ and take **both** square roots of $x^2 = 3$: $x = \pm\sqrt{3}$, $y = 4$.
4. **Pair the answers**, finding each partner from the simpler equation.

Variations you will meet:

- **Both curves in the form $y = \ldots$:** set them equal. For $y = \frac{4}{x^2}$ and $y = x + 3$, $x^3 + 3x^2 - 4 = 0$, which is $(x-1)(x+2)^2 = 0$: the curves cross at $(1, 4)$ and touch at $(-2, 1)$.
- **Answers to 3 decimal places:** solve the disguised quadratic exactly first, then round only the final x-values.
- **Where a curve cuts an axis:** that is the curve solved with $x = 0$ (or $y = 0$).

5. A line meeting a cubic curve (2 questions)

1. **Set the two expressions for y equal** and collect everything on one side: a cubic $= 0$.
2. **Find one root** with the factor theorem (try factors of the constant term), or take out a common factor of x. If a factor is given in an earlier part, use it.
3. **Divide** to get the quadratic factor, then factorise or use the formula.
4. Find each y from the **line**.

For $y = x^3 - 2x^2 - 3x + 5$ and $y = x + 5$: $x^3 - 2x^2 - 4x = 0$, so $x(x^2 - 2x - 4) = 0$, giving $x = 0$ or $x = 1 \pm \sqrt{5}$. The points are $(0, 5)$, $(1+\sqrt{5}, 6+\sqrt{5})$ and $(1-\sqrt{5}, 6-\sqrt{5})$. A squared factor means the line touches the curve there.

6. Two linear equations with surd coefficients (2 questions)

1. **Eliminate** as usual (substitution is often easiest).
2. You get something like $(2+\sqrt{2})p = 4$. Divide, then **rationalise** by multiplying top and bottom by the conjugate: $p = \frac{4}{2+\sqrt{2}} \times \frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{4(2-\sqrt{2})}{2} = 4 - 2\sqrt{2}$.
3. Find the other letter by substituting into the **simpler** equation, not by rationalising again.

This came from $(1+\sqrt{2})p + q = 3$ and $p - q = 1$: adding gives $(2+\sqrt{2})p = 4$, so $p = 4 - 2\sqrt{2}$ and $q = p - 1 = 3 - 2\sqrt{2}$. No calculator is allowed for these; the rationalising step must be shown.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

Solve the simultaneous equations

$$r - 2s = 3, \qquad r^2 + rs - 2s^2 = 9. \quad [\mathbf{5}]$$

From the first equation, $r = 2s + 3$. Substitute:

$$(2s+3)^2 + (2s+3)s - 2s^2 = 9.$$

M1 for a correct equation in one letter.

$$4s^2 + 12s + 9 + 2s^2 + 3s - 2s^2 = 9 \ \Rightarrow\ 4s^2 + 15s = 0.$$

A1

$s(4s+15) = 0$, so $s = 0$ or $s = -\frac{15}{4}$. **M1** for solving their quadratic.

$s = 0$: $r = 3$. $s = -\frac{15}{4}$: $r = -\frac{15}{2} + 3 = -\frac{9}{2}$. **A1 A1**, one for each correct pair.

The solutions are $r = 3$, $s = 0$ and $r = -\frac{9}{2}$, $s = -\frac{15}{4}$.

Example 2

Solve the simultaneous equations

$$\frac{3^m}{9^n} = 27, \qquad \ln m + \ln(n+2) = \ln 6. \quad [\mathbf{6}]$$

$\frac{3^m}{3^{2n}} = 3^3$, so $m - 2n = 3$. **B1**

$\ln\big(m(n+2)\big) = \ln 6$, so $m(n+2) = 6$. **B1**

Substitute $m = 2n + 3$: **M1**

$$(2n+3)(n+2) = 6 \ \Rightarrow\ 2n^2 + 7n + 6 = 6 \ \Rightarrow\ 2n^2 + 7n = 0.$$

A1

$n(2n+7) = 0$, so $n = 0$ or $n = -\frac{7}{2}$. **M1**

$n = 0$ gives $m = 3$. $n = -\frac{7}{2}$ gives $m = -4$, but $\ln(-4)$ is not defined (and $n + 2 = -\frac{3}{2}$ is negative too), so reject it.

The only solution is $m = 3, n = 0$. **A1**, which needs the rejection.

Check: $\frac{3^3}{9^0} = 27$ and $\ln 3 + \ln 2 = \ln 6$.

Example 3

The curve $x^2 + y^2 = 25$ and the parabola $y = x^2 - 13$ meet at four points.

(a) Find the coordinates of the four points. **[5]**

(b) The four points are the corners of a trapezium. Find its area. **[2]**

(a) The parabola gives $x^2 = y + 13$. Substitute this for x^2 in the first equation (there is no need to find x first): **M1**

$$(y + 13) + y^2 = 25 \;\Rightarrow\; y^2 + y - 12 = 0.$$

A1

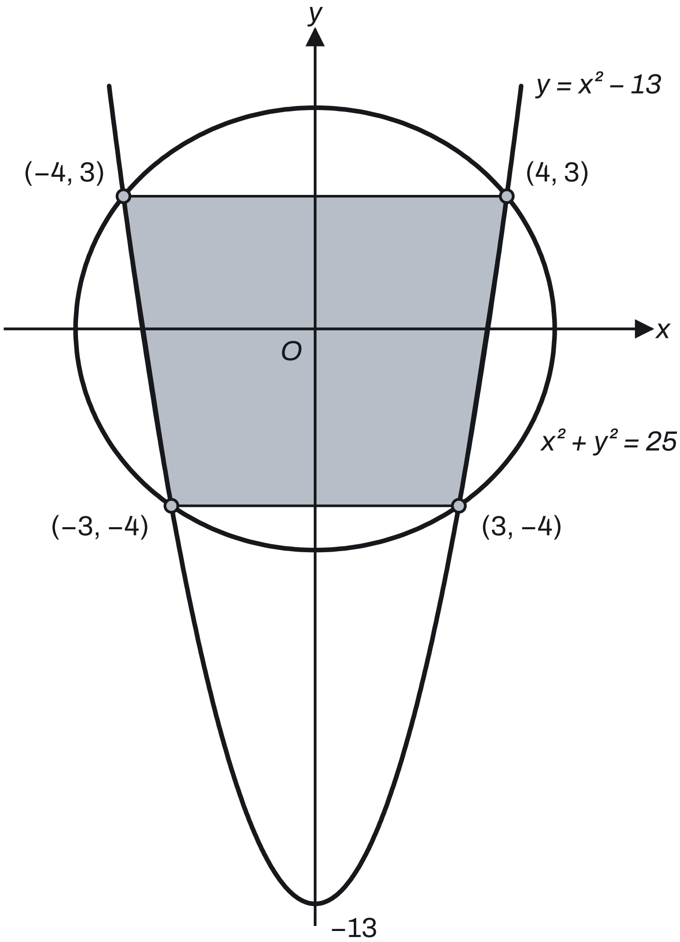
$(y + 4)(y - 3) = 0$, so $y = 3$ or $y = -4$. **M1**

Both roots work, and each gives **two** values of x:

- $y = 3$: $x^2 = 16$, so $x = \pm 4$;
- $y = -4$: $x^2 = 9$, so $x = \pm 3$.

A1 A1, one for each pair of points.

The points are $(4, 3)$, $(-4, 3)$, $(3, -4)$ and $(-3, -4)$.



(b) The top side (on $y = 3$) has length 8, the bottom side (on $y = -4$) has length 6, and they are parallel, $3 - (-4) = 7$ apart. **M1**

Area $= \frac{1}{2}(8 + 6) \times 7 = 49$. **A1**

Example 4

The line $x + y = 5$ meets the curve $x = y^2 - 4y + 5$ at the points P and Q, where P is on the x-axis.

(a) Find the coordinates of P and Q. **[4]**

(b) The point R has coordinates $(1, 2)$. Show that angle PQR is a right angle, and find the area of triangle PQR. **[4]**

(c) Show that the line $x + y = 1$ does not meet the curve. **[2]**

(a) The line gives $x = 5 - y$; substitute into the curve: **M1**

$$5 - y = y^2 - 4y + 5 \;\Rightarrow\; y^2 - 3y = 0.$$

A1

$y(y - 3) = 0$, so $y = 0$ or $y = 3$. **M1**

$y = 0$: $x = 5$. $y = 3$: $x = 2$. So $P(5, 0)$ and $Q(2, 3)$. **A1**

(b) Gradient of $QP = \frac{0-3}{5-2} = -1$ and gradient of $QR = \frac{2-3}{1-2} = 1$. **M1**

$(-1) \times 1 = -1$, so QP is perpendicular to QR: angle $PQR = 90^\circ$. **A1**

$QP = \sqrt{3^2 + 3^2} = 3\sqrt{2}$ and $QR = \sqrt{1^2 + 1^2} = \sqrt{2}$. **M1**

Area $= \frac{1}{2} \times 3\sqrt{2} \times \sqrt{2} = 3$. **A1**

(c) Substitute $x = 1 - y$: $1 - y = y^2 - 4y + 5$, so $y^2 - 3y + 4 = 0$. **M1**

Discriminant $= (-3)^2 - 4(1)(4) = -7 < 0$, so there are no real solutions and the line does not meet the curve. **A1**

Common mistakes

- **Not showing the elimination.** Examiners report every year that the key step, an equation in one unknown, must be seen. Answers typed into a calculator's equation solver earn little when there is a slip, and nothing on "show that" or no-calculator questions.
- **Dropping brackets when substituting.** $x^2 - 2xy$ with $x = 3 - 2y$ is $(3 - 2y)^2 - 2(3 - 2y)y$, not $3 - 2y^2 - 6 - 4y^2$. Put the whole expression in a bracket first, then expand.
- **Mismatching the pairs.** Find each partner from the linear (or simpler) equation and write the answers as pairs. Two x-values and two y-values listed separately, or in the wrong order, lose the final mark.
- **Losing the negative square root.** In a disguised quadratic, $x^2 = 16$ gives $x = 4$ **and** $x = -4$. Candidates often stop at the positive one, or reject the negative one for no reason.
- **Wrong log and index laws.** $\ln(e^x + e^y)$ is **not** $x + y$, and $e^x + e^y$ is **not** e^{x+y}. Only products and quotients combine.
- **Using x and y as the new letters.** When you let $u = e^x$, choose new letters, say so, and remember to change back at the end.
- **Keeping an impossible answer.** A value that makes a log input negative, or e^x negative, must be rejected with a reason.
- **Rounding too early**, or to the wrong accuracy. Work with exact values until the last line, and read whether the question wants 3 significant figures, 3 decimal places or an exact form.
- **Mid-point and gradient slips.** Use the points you found, keep the signs, and use $-\frac{1}{m}$ for the perpendicular, not $\frac{1}{m}$ or $-m$.

How to get the marks

- **The method marks:** one for a correct equation in one unknown, one for solving your quadratic (factorising or the formula, with working seen), then accuracy marks for the answers. A slip early on still lets you earn the later method marks if your working is shown.

- **"Solve the simultaneous equations"** needs **all** the pairs, and no extra ones. Give them as $x = \ldots, y = \ldots$ pairs or as coordinates.
- **"Exact"** means surds, fractions, $\ln$ or e forms, such as $y = e^{2/5}$, not 1.49. **"In the form $a + b\sqrt{3}$"** means rationalise and simplify.
- **"DO NOT USE A CALCULATOR"** means show every step: the factor theorem check, the division, the formula with the numbers in, and the rationalising.
- **"Show that"**: the answer is given, so each step must be visible and the last line must match exactly.
- **"Hence"**: use the result of the part before, such as the linear equation you have just shown.
- **Rejecting answers:** say why, for example "$x^2 = -\frac{1}{2}$ has no real roots" or "$\ln$ of a negative number is undefined".
- **Accuracy:** unless told otherwise, non-exact answers to 3 significant figures (angles to 1 decimal place).

Quick check

1. Show that the line $y = 2x - 5$ touches the curve $x^2 + y^2 = 5$, and find the point where it touches.
2. Solve $\sqrt{3}\,a + b = 4$ and $a - \sqrt{3}\,b = 2$, giving each answer in the form $h + k\sqrt{3}$, where h and k are rational.
3. Solve $\log_2 p - \log_2 q = 3$ and $p - q = 14$.
4. Find the coordinates of the points where the line $y = x$ meets the curve $y = x(x-3)^2$. Why must you not divide both sides by x?
5. Solve $x - 2y = 1$ and $x^2 - 4y^2 = 5$, and explain why there is only one solution.

Answers

1. Substitute $y = 2x - 5$: $x^2 + (2x-5)^2 = 5 \Rightarrow 5x^2 - 20x + 20 = 0 \Rightarrow 5(x-2)^2 = 0$. The repeated root $x = 2$ (discriminant 0) means the line touches the curve, at $(2, -1)$.
2. From the second, $a = 2 + \sqrt{3}\,b$. Then $\sqrt{3}(2 + \sqrt{3}\,b) + b = 4 \Rightarrow 2\sqrt{3} + 4b = 4 \Rightarrow b = 1 - \frac{1}{2}\sqrt{3}$. Then $a = 2 + \sqrt{3} - \frac{3}{2} = \frac{1}{2} + \sqrt{3}$.
3. $\log_2 \frac{p}{q} = 3$, so $p = 8q$. Then $8q - q = 14$, so $q = 2$ and $p = 16$. Both are positive, so the logs are defined.
4. $x(x-3)^2 = x \Rightarrow x\big((x-3)^2 - 1\big) = 0 \Rightarrow x(x-2)(x-4) = 0$, so $x = 0, 2, 4$. The points are $(0,0)$, $(2,2)$ and $(4,4)$. Dividing by x would lose the root $x = 0$, the point at the origin; take x out as a factor instead.
5. $x = 2y + 1$, so $(2y+1)^2 - 4y^2 = 5 \Rightarrow 4y + 1 = 5 \Rightarrow y = 1, x = 3$. The y^2 terms cancel, so the equation in y is linear, not quadratic, and has only one root.

Past questions to try

- **0606/21/O/N/22 Q1**: two linear equations with surd coefficients, answers in the form $a + b\sqrt{7}$.
- **0606/22/M/J/21 Q9**: two non-linear equations giving a disguised quadratic, four pairs of answers.
- **0606/13/M/J/24 Q11**: simultaneous log equations with different bases, exact answers.
- **0606/12/M/J/24 Q7**: a line meets a curve, then a perpendicular bisector and a reflection.