

Trigonometry

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Read first: [Circular measure](#)

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What you need to know

By the end of this topic you should be able to:

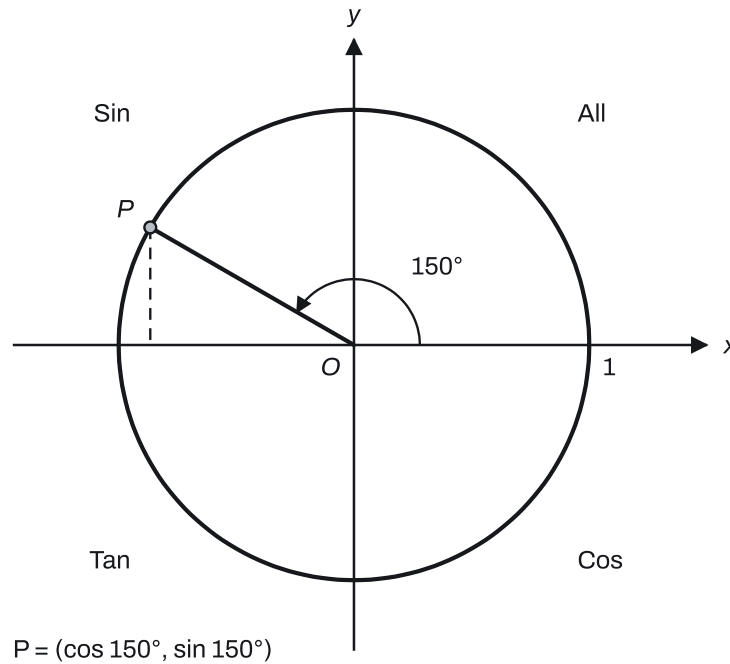
1. **Use all six trigonometric functions for any angle.** As well as $\sin$, $\cos$ and $\tan$, know $\sec = \frac{1}{\cos}$, $\operatorname{cosec} = \frac{1}{\sin}$ and $\cot = \frac{1}{\tan}$, and work with angles of any size, positive or negative, in degrees or radians.
2. **Understand amplitude and period.** Find the amplitude and the period of a function such as $y = 4 \sin 3x$, in degrees or in radians, and see how the graphs of related functions (for example $y = \cos x$ and $y = 2 \cos \frac{x}{2}$) are linked.
3. **Draw and use the graphs of $y = a \sin bx + c$, $y = a \cos bx + c$ and $y = a \tan bx + c$** over a given domain, where a is a positive integer, b is an integer or a simple fraction (denominator 2, 3, 4, 6 or 8) and c is an integer. For a $\tan$ graph, label the x -coordinate of every asymptote.
4. **Use the three identities** $\sin^2 A + \cos^2 A = 1$, $\sec^2 A = 1 + \tan^2 A$ and $\operatorname{cosec}^2 A = 1 + \cot^2 A$. They are printed in the list of formulas.
5. **Solve trigonometric equations** in any of the six functions, for a given domain, including equations that need one of the identities first.
6. **Prove identities** that involve the six functions.

Compound-angle and double-angle formulas (such as $\sin(A + B)$ or $\sin 2A$) are **not** in this syllabus. Every question can be done with the six definitions and the three identities above.

Understand it

Six functions for any angle

Draw a circle of radius 1 centred at the origin. Turn a radius anticlockwise from the positive x -axis through an angle θ . The point P where it ends is $(\cos \theta, \sin \theta)$. That is the definition of $\sin$ and $\cos$ for **any** angle, and $\tan \theta = \frac{\sin \theta}{\cos \theta}$. A negative angle means turning clockwise; one full turn is 360° or 2π radians.



Because P has coordinates $(\cos \theta, \sin \theta)$, the signs follow from the quadrant:

- first quadrant (0° to 90°): **all** positive;
- second (90° to 180°): only **sin** (and cosec) positive;
- third (180° to 270°): only **tan** (and cot) positive;
- fourth (270° to 360°): only **cos** (and sec) positive.

Read anticlockwise from the fourth quadrant this spells **CAST**. For 150° , P is in the second quadrant, so $\sin 150^\circ = \frac{1}{2}$ is positive and $\cos 150^\circ = -\frac{\sqrt{3}}{2}$ is negative.

The other three functions are reciprocals:

$$\sec \theta = \frac{1}{\cos \theta}, \quad \operatorname{cosec} \theta = \frac{1}{\sin \theta}, \quad \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}.$$

A reciprocal has the **same sign** as the function it comes from. Since $\sin$ and $\cos$ lie between -1 and 1 , $\sec$ and cosec are never between -1 and 1 : $\sec \theta \geq 1$ or $\sec \theta \leq -1$. They are undefined where $\cos \theta = 0$ (for $\sec$ and $\tan$) or $\sin \theta = 0$ (for cosec and $\cot$).

Exact values. From half an equilateral triangle of side 2 (angles 30° , 60° , 90° , sides 1, $\sqrt{3}$, 2) and a right-angled isosceles triangle with shorter sides 1 (hypotenuse $\sqrt{2}$):

θ	0	$30^\circ = \frac{\pi}{6}$	$45^\circ = \frac{\pi}{4}$	$60^\circ = \frac{\pi}{3}$	$90^\circ = \frac{\pi}{2}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

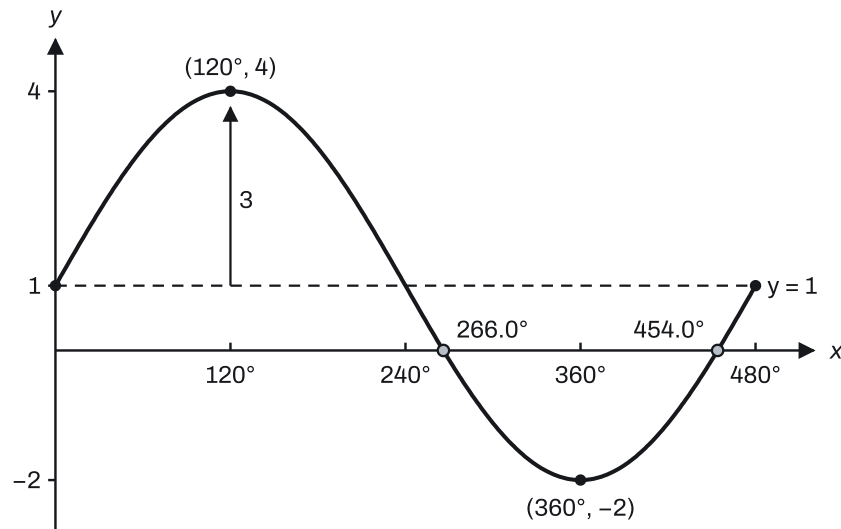
You need these whenever a question says "exact" or "in terms of π ".

Graphs, amplitude and period

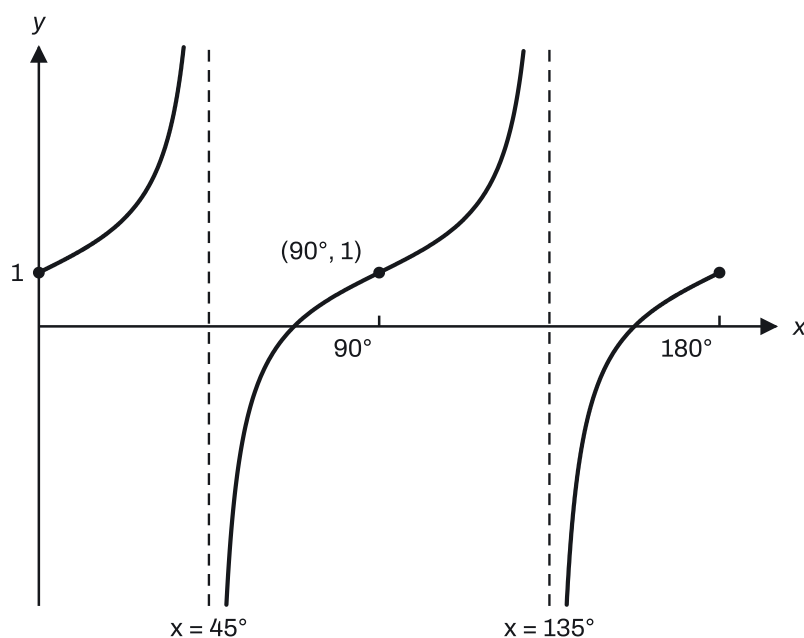
$y = \sin x$ and $y = \cos x$ are waves between -1 and 1 that repeat every 360° (2π). $y = \tan x$ repeats every 180° (π) and has vertical **asymptotes** where $\cos x = 0$, at $x = 90^\circ, 270^\circ, \dots$

In $y = a \sin bx + c$ (and the same for $\cos$):

- a stretches the wave vertically. The **amplitude** is a : the distance from the middle line to a peak.
- b squashes it horizontally. The wave now fits b cycles into 360° , so the **period** is $\frac{360^\circ}{b}$ (or $\frac{2\pi}{b}$ in radians).
- c moves it up by c . The middle line is $y = c$, the maximum is $c + a$ and the minimum is $c - a$.



For $y = a \tan bx + c$ there is no amplitude (the graph has no top or bottom). The period is $\frac{180^\circ}{b}$ (or $\frac{\pi}{b}$), and the asymptotes are where $bx = 90^\circ, 270^\circ, \dots$ (or -90° , and so on). The graph crosses its middle value $y = c$ halfway between two asymptotes.



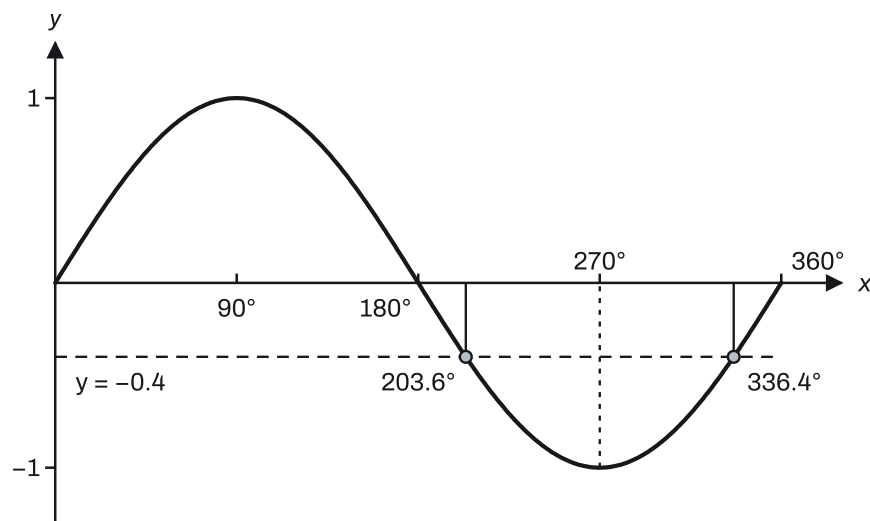
Related graphs. $y = \cos 2x$ is $y = \cos x$ squashed to half its width; $y = 4 \cos 2x$ is that stretched to four times its height; $y = 4 \cos 2x - 1$ is that moved down 1. Each change affects one number only.

Solving an equation: one angle gives many

A calculator gives only one answer, the **principal value**: $\sin^{-1}$ and $\tan^{-1}$ give an angle between -90° and 90° , and $\cos^{-1}$ gives one between 0° and 180° . The graph is symmetric, so there are others:

- $\sin x = k$: if α is one answer, so is $180^\circ - \alpha$;
- $\cos x = k$: if α is one answer, so is $-\alpha$ (or $360^\circ - \alpha$);
- $\tan x = k$: if α is one answer, so is $\alpha + 180^\circ$;

and every answer can have 360° added or taken away as often as you like (180° for tan).



For $\sin x = -0.4$ with $0^\circ \leq x \leq 360^\circ$: the calculator gives $\alpha = -23.58^\circ$. Then $180^\circ - \alpha = 203.58^\circ$ and $\alpha + 360^\circ = 336.42^\circ$, so $x = 203.6^\circ$ or 336.4° .

Multiple and compound angles. For $\sin 2x = k$ with $0^\circ \leq x \leq 360^\circ$, the angle being "sined" is $2x$, and it runs from 0° to 720° : two full cycles, so up to four answers. Find all values of $2x$ in that range first, then halve them. For $\cos(x + 50^\circ)$, find the range of $x + 50^\circ$, solve for it, then take away 50° .

The three identities

Divide $\sin^2 \theta + \cos^2 \theta = 1$ (Pythagoras on the unit circle) by $\cos^2 \theta$:

$$\tan^2 \theta + 1 = \sec^2 \theta.$$

Divide it by $\sin^2 \theta$ instead:

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta.$$

An **identity** is true for every angle (where both sides exist), unlike an equation, which is true only for some. The identities let you swap a squared function for another one: $\cos^2 \rightarrow 1 - \sin^2$, $\sec^2 \rightarrow 1 + \tan^2$, $\tan^2 \rightarrow \sec^2 - 1$, $\cot^2 \rightarrow \operatorname{cosec}^2 - 1$, and so on. That is how you turn an equation with two different functions into a

quadratic in one of them, and how you simplify an expression in a proof.

The notation $\sin^2 \theta$ means $(\sin \theta)^2$, and $\sin^2 \theta$ is **not** $\sin(\theta^2)$.

Key facts and formulas

"Given" means it is in the list of formulas printed in the exam paper. "Learn it" means you must remember it.

Formula	Status
$\sin^2 A + \cos^2 A = 1$	Given
$\sec^2 A = 1 + \tan^2 A$	Given
$\operatorname{cosec}^2 A = 1 + \cot^2 A$	Given
$\sec A = \frac{1}{\cos A}$, $\operatorname{cosec} A = \frac{1}{\sin A}$, $\cot A = \frac{1}{\tan A}$	Learn it
$\tan A = \frac{\sin A}{\cos A}$, $\cot A = \frac{\cos A}{\sin A}$	Learn it
Exact values of $\sin$, $\cos$, $\tan$ at 0° , 30° , 45° , 60° , 90° (table above)	Learn it
Signs by quadrant: CAST	Learn it
Second solutions: $\sin: 180^\circ - \alpha$ ($\pi - \alpha$); $\cos: 360^\circ - \alpha$ ($2\pi - \alpha$); $\tan: \alpha + 180^\circ$ ($\alpha + \pi$)	Learn it
$y = a \sin bx + c$ or $a \cos bx + c$: amplitude a , period $\frac{360^\circ}{b} = \frac{2\pi}{b}$, maximum $c + a$, minimum $c - a$	Learn it
$y = a \tan bx + c$: period $\frac{180^\circ}{b} = \frac{\pi}{b}$; asymptotes where $bx = \pm 90^\circ, \pm 270^\circ, \dots$	Learn it
Sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$; cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$; area $\frac{1}{2}ab \sin C$	Given
π radians = 180°	Learn it

How to do it

In the Paper 1 and Paper 2 questions we have tagged for this topic (103 questions, 2021–2025), the types came up this often. 36 of the 103 questions mix two or more types (most often "show that ... hence solve"), so the counts add up to more than 103.

Question type	Questions
Solving an equation in one function, with multiple or compound angles	37
Proving identities	28
Amplitude, period and sketching graphs	18
Finding a , b and c from a graph or from facts about it	17
Equations that become a quadratic through an identity	14

Question type	Questions
Equations solved by factorising or by changing to $\tan$	12
Exact values: one function from another, and exact triangles	10
Eliminating θ : y in terms of x	7

1. Solving an equation in one function, with multiple or compound angles (37 questions)

- Get the function on its own**, for example $2 \cos(\dots) = -\sqrt{3} \Rightarrow \cos(\dots) = -\frac{\sqrt{3}}{2}$. Turn sec, cosec or cot into cos, sin or tan by taking the reciprocal: $\cot u = 2 \Rightarrow \tan u = \frac{1}{2}$.
- Change the range**. Call the whole angle u (such as $u = 2x$, $u = \frac{x}{2}$ or $u = 3x - \frac{\pi}{4}$) and work out the range of u from the range of x .
- Find every value of u in that range**: the principal value, the symmetric partner, then add or take away 360° (2π) until you leave the range.
- Turn each u back into x** , and check each answer is in the original range.
- Give the answers in the form asked: degrees to 1 d.p., radians to 3 s.f., or exact multiples of π .

For example, $\sin 2x = 0.6$ for $0^\circ \leq x \leq 360^\circ$: $0^\circ \leq 2x \leq 720^\circ$. The principal value is 36.87° , and $180^\circ - 36.87^\circ = 143.13^\circ$; adding 360° gives 396.87° and 503.13° . Halving: $x = 18.4^\circ, 71.6^\circ, 198.4^\circ, 251.6^\circ$.

And $\tan(x - 30^\circ) = 2$ for $0^\circ \leq x \leq 360^\circ$: $-30^\circ \leq x - 30^\circ \leq 330^\circ$, so $x - 30^\circ = 63.43^\circ$ or 243.43° , giving $x = 93.4^\circ$ or 273.4° .

Variations you will meet:

- Radians**. If the range is in radians, the answers must be too. Set the calculator to radians, or work in degrees and convert at the end. Never mix the two in one line: $3x + 1.2$ has 1.2 radians in it.
- "In terms of π "**: the function value will be one from the exact-values table. Work with fractions of π throughout (see Example 2).
- A squared function**, such as $9 \cos^2 u = 4$: square-root with $\pm$, so $\cos u = \frac{2}{3}$ or $-\frac{2}{3}$, and solve both.
- The equation comes from another topic**: a derivative set to a value, the terms of a progression, or a function $f(x) = k$. Once it is in the form "function of an angle = number", it is this type.
- A value that cannot happen**: $\sin u = 1.4$ or $\sec u = 0.5$ has no solutions; say so and move on.

2. Proving identities (28 questions)

- Start from the more complicated side** (usually the left) and work towards the other. Never write the identity as an equation and do the same thing to both sides.
- Write everything in sin and cos** if you can't see another way: $\sec = \frac{1}{\cos}$, $\tan = \frac{\sin}{\cos}$, and so on.
- Combine fractions** over a common denominator. If a denominator is $1 - \sin x$ or $1 + \cos x$, multiplying by the "partner" ($1 + \sin x$, $1 - \cos x$) often helps, because $(1 - \sin x)(1 + \sin x) = 1 - \sin^2 x = \cos^2 x$.
- Use an identity** to replace $\sin^2 + \cos^2$ by 1, or $1 - \sin^2$ by $\cos^2$, and so on.
- Factorise and cancel**, then write the final line exactly as the question does.

For example, to show $\frac{\tan x}{\sec x} = \sin x$: $\frac{\sin x}{\cos x} \div \frac{1}{\cos x} = \frac{\sin x}{\cos x} \times \cos x = \sin x$.

Variations you will meet:

- **"Hence solve"**: the part you proved turns a complicated equation into a simple one. Replace the left side with the right side of your identity, then solve (usually type 1 or type 5). The angle may be 2θ , $3x$ or $\frac{\theta}{2}$ in the "hence" part even though the identity used x .
- **"Show that ... can be written as $a \tan^2 x + b$ "**: find the constants by getting everything over $\cos^2 x$ and splitting.
- **Square roots**, such as $\sqrt{1 + \tan^2 \theta} = \sec \theta$ for $0 \leq \theta < \frac{\pi}{2}$: the range is given so that the square root is the positive one.
- **Expanding brackets** such as $(\sin \theta + \cos \theta)^2 = \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta = 1 + 2 \sin \theta \cos \theta$.
- **Undefined values in the "hence" part**. If a factor such as $\cos \theta = 0$ appears but the original expression has $\sec \theta$ or $\tan \theta$ in it, those angles must be rejected (see Example 4).

3. Amplitude, period and sketching graphs (18 questions)

1. **Amplitude** = a . **Period** = $\frac{360^\circ}{b}$ or $\frac{2\pi}{b}$ ($\frac{180^\circ}{b}$ or $\frac{\pi}{b}$ for $\tan$). Give the units the question uses.
2. **Key values**: middle line $y = c$, maximum $c + a$, minimum $c - a$.
3. **Mark the quarter-periods**. A $\cos$ graph starts at its maximum when $x = 0$; a $\sin$ graph starts on its middle line going up. Each quarter-period after that is the next key point (middle, minimum, middle, maximum for $\cos$).
4. **Plot the ends of the domain** (substitute them) and draw a smooth wave through the key points: rounded peaks, not sharp points, and the right number of cycles.
5. **Tan**: draw the asymptotes as vertical dashed lines and label their x -values; the curve gets closer to them but never touches.

Variations you will meet:

- **State the intercept with the y -axis**: substitute $x = 0$.
- **"The line $y = k$ is a tangent to the curve"**: horizontal tangents are at the maximum and the minimum, so $k = c + a$ or $k = c - a$.
- **Least or greatest value** of the curve, and the x where it happens: minimum is $c - a$, at the x where $\cos bx = -1$ (or $\sin bx = -1$).
- **How many solutions** does an equation have: sketch both sides and count the crossings.

4. Finding a , b and c from a graph or from facts about it (17 questions)

1. **Read the maximum and the minimum** from the graph.
2. $a = \frac{\max - \min}{2}$ and $c = \frac{\max + \min}{2}$.
3. **Find the period** (the x -distance for one full cycle, peak to peak), then $b = \frac{360^\circ}{\text{period}}$ or $\frac{2\pi}{\text{period}}$.

4. **Check with one point:** substitute a point you can read, such as the y -intercept.

For example, a sine curve with highest value 5 and lowest value -3 that fits two full cycles into 360° has $a = 4$, $c = 1$ and period 180° , so $b = 2$: $y = 4 \sin 2x + 1$.

Variations you will meet:

- **Given points and the period, no graph:** find b from the period first, then substitute the points to get two equations in a and c , and solve them together.
- **Tan graph:** b comes from the gap between asymptotes (that gap is the period); c is the y -value halfway between asymptotes; a second point gives a .
- **Sine or cosine?** A graph with a peak on the y -axis is a cos graph; one that crosses its middle line going up at $x = 0$ is a sin graph. Use the form the question gives.
- **Then use the equation,** for example to find where the curve meets the x -axis or to find its minimum point.

5. Equations that become a quadratic through an identity (14 questions)

1. **Spot the squared function** and use an identity to change it into the other function in the equation: $\sec^2 \leftrightarrow \tan^2$, $\operatorname{cosec}^2 \leftrightarrow \cot^2$, $\cos^2 \leftrightarrow \sin^2$.
2. **Collect terms** into a three-term quadratic $= 0$ in one function.
3. **Factorise** (or use the quadratic formula).
4. **Reject impossible values:** sin and cos must lie between -1 and 1 ; sec and cosec must not lie strictly between -1 and 1 . tan and cot can take any value.
5. **Solve** each remaining equation as in type 1.

Variations you will meet:

- **Reciprocals mixed in**, such as $\operatorname{cosec} u$ and $\sin u$ in one equation: replace $\operatorname{cosec} u$ by $\frac{1}{\sin u}$ and multiply through by $\sin u$.
- **The quadratic is already there**, such as $p \tan^2 x + q \tan x + r = 0$: go straight to step 3.
- **A compound angle inside**, such as $\cos^2(2x + 1)$: treat $2x + 1$ as u throughout.

6. Equations solved by factorising or by changing to tan (12 questions)

1. **Never divide by a function** that could be zero. Move everything to one side and **factorise** instead. For example, $\sin x \tan x = 2 \sin x$ becomes $\sin x(\tan x - 2) = 0$, so $\sin x = 0$ **or** $\tan x = 2$. For $0^\circ \leq x \leq 360^\circ$ that gives $x = 0^\circ, 180^\circ, 360^\circ$ and $x = 63.4^\circ, 243.4^\circ$.
2. **sin and cos to the same power on both sides** (for example $4 \sin x = 5 \cos x$): divide by $\cos x$ to get $\tan x = \frac{5}{4}$, so $x = 51.3^\circ$ or 231.3° . This is safe because $\cos x = 0$ doesn't satisfy the equation.
3. **Reciprocals equal to each other**, such as $p \sec u = q \operatorname{cosec} u$: write them as $\frac{p}{\cos u} = \frac{q}{\sin u}$ and cross-multiply to get $\tan u$.
4. Solve each factor as in type 1 and list **all** the answers together.

Variations you will meet:

- **A polynomial in $\sin x$** (from the Factors of polynomials topic): factorise it with the factor theorem, then solve each factor.
- **Grouping four terms:** $pq + pr + sq + sr = (p + s)(q + r)$, with p, s being functions.
- **A factor gives an undefined value:** if the equation contained $\cot x$ or $\operatorname{cosec} x$, angles where $\sin x = 0$ are not allowed.

7. Exact values: one function from another, and exact triangles (10 questions)

One function from another.

1. **Draw a right-angled triangle** for the acute angle with the given value, or use an identity: $\sin^2 = 1 - \cos^2$, $\tan^2 = \sec^2 - 1$, $\cot^2 = \operatorname{cosec}^2 - 1$.
2. **Decide the sign** from the quadrant (CAST).
3. **Simplify** surds and rationalise if asked.

For example, if $\cos \theta = -\frac{3}{5}$ and $180^\circ < \theta < 270^\circ$: the triangle is 3, 4, 5, and θ is in the third quadrant, where only $\tan$ is positive, so $\sin \theta = -\frac{4}{5}$ and $\tan \theta = \frac{4}{3}$.

Exact triangles ("do not use a calculator"): use the exact values you are given or know, with the sine rule, the cosine rule and $\text{area} = \frac{1}{2}ab \sin C$ (all given). Keep surds, expand brackets carefully, and rationalise denominators by multiplying by the conjugate, such as $\frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2}$.

Variations you will meet:

- **In terms of a letter:** if $\sec x = k$, then $\cos x = \frac{1}{k}$ and $\sin x = \pm \sqrt{1 - \frac{1}{k^2}}$, with the sign from the quadrant.
- **Exact values from a special triangle**, such as $\sec 30^\circ$ from half an equilateral triangle: find the missing side with Pythagoras, then use the definition.

8. Eliminating θ : y in terms of x (7 questions)

1. **Make each function the subject:** from each equation, write one function of θ in terms of x or y .
2. **Pick the identity** that links those two functions, or turn one into a reciprocal first (for example $\sin \theta$ into $\operatorname{cosec} \theta$ to pair with $\cot \theta$).
3. **Substitute** and rearrange to $y = \dots$, with no fractions inside fractions.

For example, if $x = 2 + 3 \cos \theta$ and $y = 1 - 3 \sin \theta$, then $\cos \theta = \frac{x-2}{3}$ and $\sin \theta = \frac{1-y}{3}$, so $\sin^2 \theta + \cos^2 \theta = 1$ gives $(x-2)^2 + (1-y)^2 = 9$.

Variations you will meet:

- **Two functions with no direct identity**, such as $\sin$ and $\tan$: find $\cos$ from $\sin$ first, then use $\tan = \frac{\sin}{\cos}$ (Quick check question 3).
- **Then differentiate or find a tangent:** once you have y in terms of x , the question may carry on into calculus.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

The function g is defined by $g(x) = 3 \sin \frac{3x}{4} + 1$ for $0^\circ \leq x \leq 480^\circ$.

(a) Write down the amplitude and the period of g . **[2]**

(b) Sketch the graph of $y = g(x)$. **[3]**

(c) Find the x -coordinates of the points where the graph meets the x -axis. **[3]**

(a) Amplitude = 3. **B1**

Period = $\frac{360^\circ}{\frac{3}{4}} = 480^\circ$. **B1**

(b) The middle line is $y = 1$, the maximum is $1 + 3 = 4$ and the minimum is $1 - 3 = -2$. A sin graph starts on its middle line going up, and each quarter-period is 120° , so it goes $(0, 1)$, the maximum $(120^\circ, 4)$, the middle line at 240° , the minimum $(360^\circ, -2)$ and back to $(480^\circ, 1)$. See the second diagram in "Understand it".

B1 for one full sine-shaped cycle, **B1** for the maximum 4 and minimum -2 , **B1** for the correct end points $(0, 1)$ and $(480^\circ, 1)$ with the turning points at 120° and 360° .

(c) $3 \sin \frac{3x}{4} + 1 = 0 \Rightarrow \sin \frac{3x}{4} = -\frac{1}{3}$. **M1**

The calculator gives -19.47° , outside the range $0^\circ \leq \frac{3x}{4} \leq 360^\circ$. Sine is negative in the third and fourth quadrants, so $\frac{3x}{4} = 180^\circ + 19.47^\circ = 199.47^\circ$ or $360^\circ - 19.47^\circ = 340.53^\circ$. **M1** for both angles in the new range.

$x = \frac{4}{3} \times 199.47^\circ = 266.0^\circ$ and $x = \frac{4}{3} \times 340.53^\circ = 454.0^\circ$. **A1**

Example 2

Solve the equation $2 \cos \left(3x - \frac{\pi}{4} \right) + \sqrt{3} = 0$ for $-\pi \leq x \leq \pi$. Give your answers in terms of π . **[5]**

Let $u = 3x - \frac{\pi}{4}$. Then $\cos u = -\frac{\sqrt{3}}{2}$. **B1**

Range: $-3\pi - \frac{\pi}{4} \leq u \leq 3\pi - \frac{\pi}{4}$, that is $-\frac{13\pi}{4} \leq u \leq \frac{11\pi}{4}$. **M1** for changing the range.

$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, and $\cos$ is negative in the second and third quadrants, so $u = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ or $u = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$. Taking away 2π : $-\frac{7\pi}{6}, -\frac{5\pi}{6}$; taking away 4π : $-\frac{19\pi}{6}$ (in range, since $-\frac{19\pi}{6} > -\frac{13\pi}{4}$) and $-\frac{17\pi}{6}$. Adding 2π gives $\frac{17\pi}{6}$, which is more than $\frac{11\pi}{4}$, so it is out. **M1** for finding values beyond the first two.

$$u = -\frac{19\pi}{6}, -\frac{17\pi}{6}, -\frac{7\pi}{6}, -\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}.$$

Now $x = \frac{u + \frac{\pi}{4}}{3}$. For example, $u = \frac{5\pi}{6}$ gives $3x = \frac{10\pi}{12} + \frac{3\pi}{12} = \frac{13\pi}{12}$, so $x = \frac{13\pi}{36}$. The six answers are

$$x = -\frac{35\pi}{36}, -\frac{31\pi}{36}, -\frac{11\pi}{36}, -\frac{7\pi}{36}, \frac{13\pi}{36}, \frac{17\pi}{36}.$$

A2 (**A1** for any three correct), with no extra answers in the range.

Example 3

Solve the equation $2 \tan^2 2x = 5 \sec 2x + 1$ for $0^\circ \leq x \leq 180^\circ$. **[6]**

Use $\tan^2 2x = \sec^2 2x - 1$: **M1**

$$2(\sec^2 2x - 1) = 5 \sec 2x + 1 \Rightarrow 2 \sec^2 2x - 5 \sec 2x - 3 = 0.$$

A1

$(2 \sec 2x + 1)(\sec 2x - 3) = 0$, so $\sec 2x = -\frac{1}{2}$ or $\sec 2x = 3$. **M1** for factorising a three-term quadratic.

$\sec 2x = -\frac{1}{2}$ would mean $\cos 2x = -2$, which is impossible. **B1** for rejecting it.

$\sec 2x = 3 \Rightarrow \cos 2x = \frac{1}{3}$. With $0^\circ \leq 2x \leq 360^\circ$: $2x = 70.53^\circ$ or $360^\circ - 70.53^\circ = 289.47^\circ$. **M1**

$x = 35.3^\circ$ or $x = 144.7^\circ$. **A1**, with no other answers in the range.

Example 4

(a) Show that $(\operatorname{cosec} \theta - \sin \theta)(\sec \theta - \cos \theta) = \sin \theta \cos \theta$. **[4]**

(b) Hence solve the equation $(\operatorname{cosec} \theta - \sin \theta)(\sec \theta - \cos \theta) = 3 \cos^2 \theta$ for $0^\circ < \theta < 360^\circ$. **[4]**

(a) Write each bracket over one denominator: **M1**

$$\operatorname{cosec} \theta - \sin \theta = \frac{1}{\sin \theta} - \sin \theta = \frac{1 - \sin^2 \theta}{\sin \theta} = \frac{\cos^2 \theta}{\sin \theta}.$$

M1 for using $1 - \sin^2 \theta = \cos^2 \theta$. In the same way

$$\sec \theta - \cos \theta = \frac{1 - \cos^2 \theta}{\cos \theta} = \frac{\sin^2 \theta}{\cos \theta}.$$

A1 for both brackets simplified.

$$\frac{\cos^2 \theta}{\sin \theta} \times \frac{\sin^2 \theta}{\cos \theta} = \sin \theta \cos \theta.$$

A1, the given answer reached with every step shown.

(b) By part (a), $\sin \theta \cos \theta = 3 \cos^2 \theta$, so $\sin \theta \cos \theta - 3 \cos^2 \theta = 0$ and

$$\cos \theta (\sin \theta - 3 \cos \theta) = 0.$$

M1 for factorising rather than dividing by $\cos \theta$.

$\cos \theta = 0$ gives $\theta = 90^\circ$ or 270° , but $\sec \theta$ is undefined there, so the left side of the equation doesn't exist: reject both. **B1**

$$\sin \theta = 3 \cos \theta \Rightarrow \tan \theta = 3. \text{ **M1**}$$

$$\theta = 71.6^\circ \text{ or } 71.57^\circ + 180^\circ = 251.6^\circ. \text{ **A1**}$$

Common mistakes

- **Dividing by a function and losing answers.** Cancelling $\sin x$ from both sides of an equation throws away every answer of $\sin x = 0$. Examiners see this every year. Factorise instead.
- **Answers missing, or extra answers outside the range.** Many candidates stop at the calculator's value or its partner, or halve only the first two values of $2x$. Change the range for the whole angle first, list every value in it, and check each final answer against the range. Extra answers in the range lose the last accuracy mark.
- **Mixing degrees and radians.** In $\sin(3x + 1.2)$ the 1.2 is in radians, so the calculator must be in radians. A range given in radians needs answers in radians, and one given in degrees needs degrees.
- **Rounding too early.** Keep at least 4 or 5 significant figures for the working angle, then divide or subtract, then round. Rounding $2x$ first can move the final answer.
- **Getting an identity wrong,** for example writing $\sec^2 x = 1 - \tan^2 x$, or $\cot^2 x = \operatorname{cosec}^2 x + 1$. The three identities are in the list of formulas: copy them from there.
- **Treating a proof like an equation.** Cross-multiplying both sides, or ending with " $1 = 1$ ", does not prove the identity. Start from one side and reach the other.
- **Dropping the angle.** Writing $\sin$ instead of $\sin x$, or $\sin^2$ on its own, loses the last mark of a proof. Every function needs its angle on every line.
- **Confusing amplitude and period, and getting the period wrong.** For $y = a \sin bx + c$ the period is $\frac{360^\circ}{b}$, not $360^\circ \times b$ and not b . For $\tan$ it is $\frac{180^\circ}{b}$ ($\frac{\pi}{b}$).
- **Poor sketches.** Common faults: pointed peaks, the wrong number of cycles, not starting and ending at the right values at the ends of the domain, and $\tan$ graphs without labelled asymptotes. Mark the maximum and minimum values on the y -axis.
- **Wrong sign for an exact value.** $\sin x = \pm \sqrt{\dots}$: choose the sign from the quadrant, and say which you chose.
- **Keeping impossible values.** $\cos u = -2$ or $\sec u = \frac{1}{2}$ has no solutions; trying $\cos^{-1}(-2)$ gives an error, not an answer.

How to get the marks

- **Show the identity you use.** The first mark in a quadratic equation is usually for replacing the squared function correctly, so write that line.
- **Show the quadratic and how you solved it:** factorised brackets or the formula with numbers in. A method mark is for "factorising or solving a three-term quadratic".
- **One correct angle** in the right range for the multiple or compound angle usually earns a method mark on its own, even before you divide or subtract. The final accuracy marks need **all** the answers and **no extras** in the range.
- **Accuracy:** degrees to 1 decimal place, radians to 3 significant figures, unless the question says otherwise. "Exact" or "in terms of π " means no decimals at all.
- **"Show that":** the answer is given, so every step must be visible, including the line where you use an identity. Your last line must match the printed answer exactly.
- **"Hence":** use the part before. Solving the original expression another way usually earns nothing.
- **"Write down"** (amplitude, period, values of k) needs no working, but "find" does.
- **"Sketch":** the right shape and the right number of cycles, with the key values (maximum, minimum, end points, intercepts and asymptotes) marked or clear from the axes. It doesn't need to be to scale, but it must be a smooth curve.
- **"Do not use a calculator":** every step of surd arithmetic must be shown; a decimal check earns nothing.
- **Method marks follow through.** A wrong root of the quadratic, solved correctly, still earns the method mark for finding an angle, so keep going.

Quick check

1. The curve $y = a \sin bx + c$, where a and b are positive, has a maximum value of 7, a minimum value of -1 and a period of 120° . Find a , b and c .
2. Given that $\sin \theta = \frac{2}{3}$ and that θ is obtuse, find the exact value of $\tan \theta + \sec \theta$.
3. Given that $x = 2 \sin \theta$ and $y = \tan \theta$, where $0 < \theta < \frac{\pi}{2}$, find y in terms of x .
4. Show that $(\sin x + \operatorname{cosec} x)^2 = \sin^2 x + \cot^2 x + 3$.
5. Solve the equation $\cot x = 2 \cos x$ for $0^\circ \leq x \leq 360^\circ$.

Answers

1. $a = \frac{7-(-1)}{2} = 4$, $c = \frac{7+(-1)}{2} = 3$, and $\frac{360^\circ}{b} = 120^\circ$ gives $b = 3$. So $y = 4 \sin 3x + 3$.
2. $\cos^2 \theta = 1 - \frac{4}{9} = \frac{5}{9}$. θ is obtuse (second quadrant), so $\cos \theta = -\frac{\sqrt{5}}{3}$. Then $\tan \theta = \frac{2/3}{-\sqrt{5}/3} = -\frac{2}{\sqrt{5}}$ and $\sec \theta = -\frac{3}{\sqrt{5}}$, so $\tan \theta + \sec \theta = -\frac{5}{\sqrt{5}} = -\sqrt{5}$.
3. $\sin \theta = \frac{x}{2}$, and $\cos \theta = \sqrt{1 - \frac{x^2}{4}} = \frac{\sqrt{4-x^2}}{2}$ (positive, since θ is acute). So $y = \frac{\sin \theta}{\cos \theta} = \frac{x}{\sqrt{4-x^2}}$.
4. $(\sin x + \operatorname{cosec} x)^2 = \sin^2 x + 2 \sin x \operatorname{cosec} x + \operatorname{cosec}^2 x = \sin^2 x + 2 + (1 + \cot^2 x) = \sin^2 x + \cot^2 x + 3$, since $\sin x \operatorname{cosec} x = 1$.
5. $\frac{\cos x}{\sin x} = 2 \cos x \Rightarrow \cos x = 2 \sin x \cos x \Rightarrow \cos x(1 - 2 \sin x) = 0$. $\cos x = 0$ gives $x = 90^\circ, 270^\circ$ (both allowed, as $\cot x = 0$ there); $\sin x = \frac{1}{2}$ gives $x = 30^\circ, 150^\circ$. So $x = 30^\circ, 90^\circ, 150^\circ, 270^\circ$.

Past questions to try

- **0606/13/M/J/24 Q12**: a sec equation with a compound angle, answers exact.
- **0606/12/F/M/25 Q2**: finding a , b and c from the graph of a cos curve.
- **0606/22/O/N/24 Q8**: a quadratic in cosec with a multiple angle, using $\operatorname{cosec}^2 = 1 + \cot^2$.
- **0606/11/M/J/24 Q7**: eliminating θ to write y in terms of x .