

Vectors in two dimensions

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Read first: [Trigonometry](#), [Quadratic functions](#)

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What you need to know

By the end of this topic you should be able to:

1. **Read and write vectors in every form the paper uses:** a column vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, the $\mathbf{i}, \mathbf{j}$ form $4\mathbf{i} - 3\mathbf{j}$, a single bold letter $\mathbf{p}$ (underline it by hand), and $\overrightarrow{AB}$ for the step from A to B . Write them correctly yourself.
2. **Use position vectors and unit vectors.** The position vector of a point is the step from the origin O to it. A unit vector has length 1; the unit vector in the direction of $\mathbf{a}$ is $\frac{\mathbf{a}}{|\mathbf{a}|}$.
3. **Calculate with vectors:** find a vector's magnitude, add and subtract vectors, and multiply by a number (a scalar). This includes **equating like vectors** to find unknown numbers, and **vector geometry**: using vectors to find where lines meet and the ratios in which points divide lines (the diagram is given in the harder questions).
4. **Compose and resolve velocities.** Add two or more vectors to get a resultant, split a velocity into east and north (or x and y) parts, and use a velocity to find where a moving object is at time t , including whether two objects collide.

Scalar (dot) products and vectors in three dimensions are **not** in this syllabus.

Understand it

What a vector is

A **vector** has a size and a direction; a **scalar** has only a size. "Walk 5 m" is a scalar; "walk 4 m east and 3 m north" is a vector. In the $\mathbf{i}, \mathbf{j}$ form, $\mathbf{i}$ is one unit in the positive x -direction (often east) and $\mathbf{j}$ is one unit in the positive y -direction (often north):

$$4\mathbf{i} + 3\mathbf{j} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Both mean "4 across, 3 up". Use one form or the other: never write $\mathbf{i}$ or $\mathbf{j}$ inside a column vector.

The **magnitude** (length) of a vector comes from Pythagoras:

$$\left| \begin{pmatrix} x \\ y \end{pmatrix} \right| = \sqrt{x^2 + y^2}, \quad \text{so } |4\mathbf{i} + 3\mathbf{j}| = \sqrt{16 + 9} = 5.$$

A vector does not belong to any particular place. Two vectors are **equal** if they have the same size and direction, wherever they are drawn.

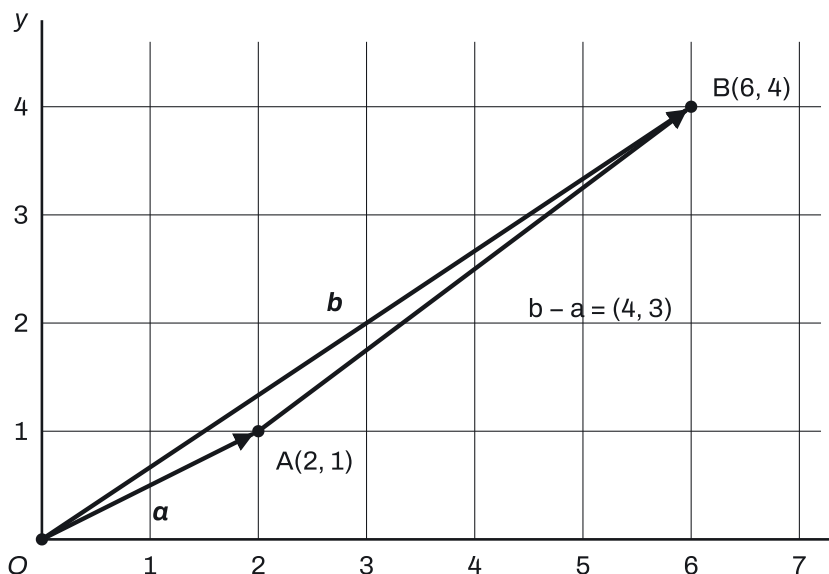
Position vectors and the step from A to B

The **position vector** of a point A is $\overrightarrow{OA}$, the step from the origin to A , usually called $\mathbf{a}$. If A is the point $(2, 1)$ then $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

To get from A to B , go back from A to O and then out to B :

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}.$$

"End minus start." Adding the two position vectors instead is the most common slip in this topic.



Here $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, and the **distance** AB is its magnitude, 5. The distance is a plain number (a scalar), never a vector.

Adding, subtracting and multiplying by a scalar

- **Add** by adding matching parts: $(3\mathbf{i} - \mathbf{j}) + (2\mathbf{i} + 5\mathbf{j}) = 5\mathbf{i} + 4\mathbf{j}$. On a diagram this is one step followed by the other (nose to tail), and the sum is the single step from the first start to the last end: the **resultant**.
- **Subtract** by subtracting matching parts. $\mathbf{p} - \mathbf{q}$ is the same as $\mathbf{p} + (-\mathbf{q})$, and $-\mathbf{q}$ is $\mathbf{q}$ turned round.
- **Multiply by a scalar** k by multiplying both parts: $3 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$. The new vector is **parallel** to the old one and $|k|$ times as long; if $k < 0$ it points the opposite way.

So **two vectors are parallel exactly when one is a scalar multiple of the other**. For example, $\begin{pmatrix} 6 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ are parallel because the second is $-\frac{2}{3}$ of the first (it points the opposite way).

Unit vectors and a vector of a given length

Divide a vector by its own magnitude and you get a **unit vector**: same direction, length 1. For $\mathbf{a} = 7\mathbf{i} - 24\mathbf{j}$, $|\mathbf{a}| = \sqrt{49 + 576} = 25$, so the unit vector in its direction is $\frac{1}{25}(7\mathbf{i} - 24\mathbf{j})$.

To make a vector of length L in the direction of $\mathbf{a}$, multiply the unit vector by L :

$$\text{vector of length } L \text{ along } \mathbf{a} = \frac{L}{|\mathbf{a}|} \mathbf{a}.$$

A vector of length 50 along $7\mathbf{i} - 24\mathbf{j}$ is $\frac{50}{25}(7\mathbf{i} - 24\mathbf{j}) = 14\mathbf{i} - 48\mathbf{j}$. For the **opposite** direction, use $-\frac{L}{|\mathbf{a}|}\mathbf{a}$.

Equating like vectors

If $x\mathbf{i} + y\mathbf{j} = 7\mathbf{i} - 2\mathbf{j}$, then $x = 7$ and $y = -2$, because no amount of $\mathbf{i}$ can make up for a difference in $\mathbf{j}$. The same holds for **any two vectors $\mathbf{a}$ and $\mathbf{b}$ that are not parallel**:

$$\lambda\mathbf{a} + \mu\mathbf{b} = \lambda'\mathbf{a} + \mu'\mathbf{b} \Rightarrow \lambda = \lambda' \text{ and } \mu = \mu'.$$

This one fact turns a vector equation into two ordinary equations. For example, to write $7\mathbf{i} + 4\mathbf{j}$ as $\lambda(2\mathbf{i} + \mathbf{j}) + \mu(\mathbf{i} - 2\mathbf{j})$: the $\mathbf{i}$ parts give $2\lambda + \mu = 7$ and the $\mathbf{j}$ parts give $\lambda - 2\mu = 4$. Solving, $\lambda = \frac{18}{5}$ and $\mu = -\frac{1}{5}$.

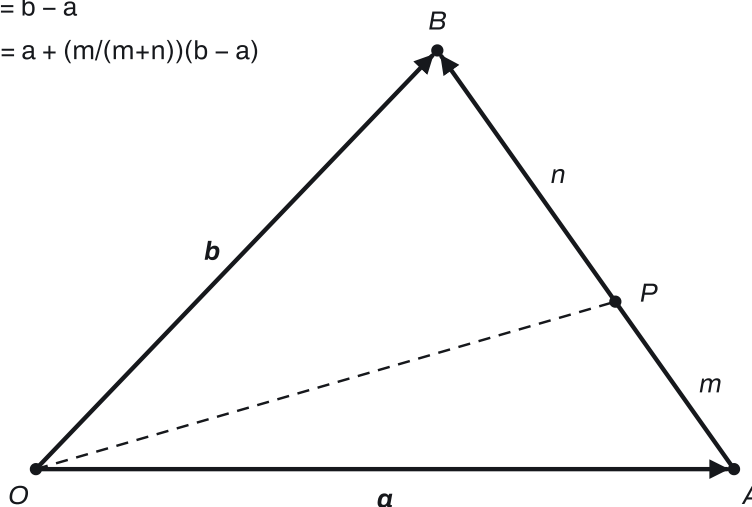
A point dividing a line in a ratio

If P lies on AB with $AP : PB = m : n$, then P is $\frac{m}{m+n}$ of the way from A to B :

$$\overrightarrow{OP} = \mathbf{a} + \frac{m}{m+n}(\mathbf{b} - \mathbf{a}) = \frac{n\mathbf{a} + m\mathbf{b}}{m+n}.$$

$$AB = \mathbf{b} - \mathbf{a}$$

$$OP = \mathbf{a} + (m/(m+n))(\mathbf{b} - \mathbf{a})$$



Read ratios carefully. $AP : PB = 1 : 3$ means P is $\frac{1}{4}$ of the way along; $AP : AB = 1 : 3$ means $\frac{1}{3}$ of the way. The midpoint of AB is $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

Collinear points. Three points A, B, C lie on one straight line when $\overrightarrow{AC} = k\overrightarrow{AB}$ for some number k : the two steps are parallel **and** share the point A . The value of k gives the ratio: $k = 3$ means $AC : AB = 3 : 1$, so $AB : BC = 1 : 2$.

Vector geometry: two routes to one point

When two lines cross at a point X and you do not know where, give each unknown fraction a letter, reach X by **two different routes**, and equate them. For example, if X is on the line AM and on the line OC , then $\overrightarrow{OX} = \overrightarrow{OA} + \lambda\overrightarrow{AM}$ and also $\overrightarrow{OX} = \mu\overrightarrow{OC}$. Write both in terms of the two basic vectors ($\mathbf{a}$ and $\mathbf{b}$, say), then equate the $\mathbf{a}$ parts and the $\mathbf{b}$ parts. That gives two equations for λ and μ , and these give the ratios. Example 3 works one through.

Velocity, speed and position

Velocity is a vector: how far and in which direction something moves each second (or hour). **Speed** is its magnitude. A velocity of $\begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ m s}^{-1}$ means 3 m east and 4 m north every second, a speed of 5 m s^{-1} .

An object that starts at position vector $\mathbf{r}_0$ and moves with constant velocity $\mathbf{v}$ is at

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

after time t . Each part is a simple expression in t : from $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ with velocity $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, the position is $\begin{pmatrix} 1 + 3t \\ 2 - t \end{pmatrix}$.

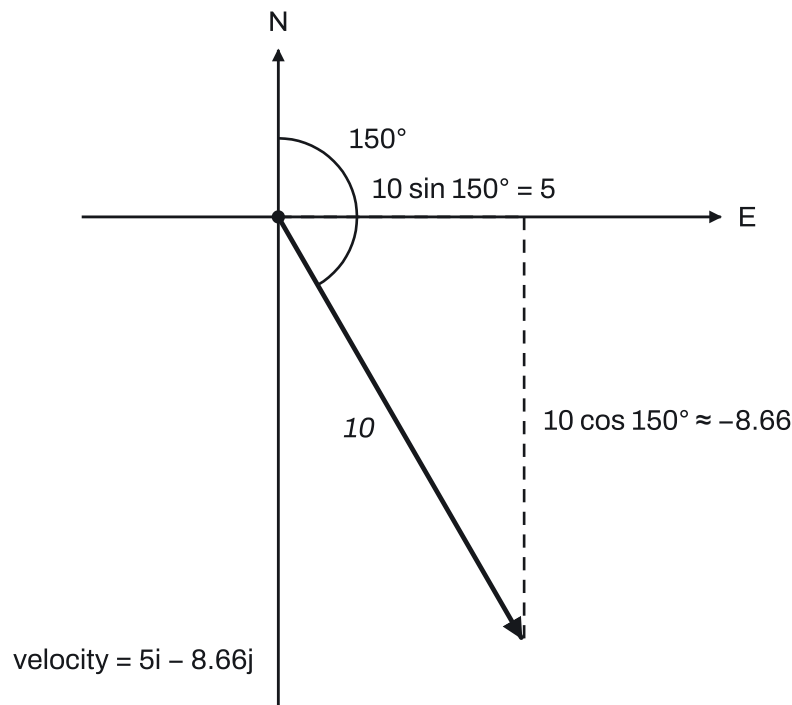
If you know the **speed** and a **direction vector** instead of the velocity, scale the direction vector to the right length, exactly as for "a vector of a given length": speed 10 along $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is $\frac{10}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$.

Two objects **collide** if they are at the same place **at the same time**. Equal x -parts give one value of t ; equal y -parts give another. They collide only if the two values are the same (and not negative).

Directions from angles and bearings

A velocity of speed v at angle α above the positive x -axis is $\begin{pmatrix} v \cos \alpha \\ v \sin \alpha \end{pmatrix}$. With bearings, the angle is measured clockwise from **north**, so a speed v on bearing θ has

$$\text{east part} = v \sin \theta, \quad \text{north part} = v \cos \theta.$$



The signs take care of themselves: on bearing 150° , $10 \sin 150^\circ = 5$ (east) and $10 \cos 150^\circ = -8.66$ (the minus sign means south), giving $5\mathbf{i} - 8.66\mathbf{j}$. Check against a sketch: 150° points south-east, so east positive, north negative. Going the other way, a vector $x\mathbf{i} + y\mathbf{j}$ has magnitude $\sqrt{x^2 + y^2}$ and makes the angle $\tan^{-1} \frac{|y|}{|x|}$ with the x -axis; turn that into a bearing with a sketch.

Key facts and formulas

"Given" means it is in the list of formulas printed in the exam paper. "Learn it" means you must remember it. No vector formula is in the list.

Formula	Status
$\begin{pmatrix} x \\ y \end{pmatrix} = x\mathbf{i} + y\mathbf{j}$; magnitude $\sqrt{x^2 + y^2}$	Learn it
$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$; distance $AB = \mathbf{b} - \mathbf{a} $	Learn it
Unit vector along $\mathbf{a}$: $\frac{\mathbf{a}}{ \mathbf{a} }$; vector of length L along $\mathbf{a}$: $\frac{L}{ \mathbf{a} }\mathbf{a}$	Learn it
Parallel: $\mathbf{p} = k\mathbf{q}$ ($k > 0$ same direction, $k < 0$ opposite)	Learn it
Like vectors: if $\lambda\mathbf{a} + \mu\mathbf{b} = \lambda'\mathbf{a} + \mu'\mathbf{b}$ ($\mathbf{a}, \mathbf{b}$ not parallel) then $\lambda = \lambda', \mu = \mu'$	Learn it
$AP : PB = m : n \Rightarrow \overrightarrow{OP} = \mathbf{a} + \frac{m}{m+n}(\mathbf{b} - \mathbf{a}) = \frac{n\mathbf{a} + m\mathbf{b}}{m+n}$; midpoint $\frac{1}{2}(\mathbf{a} + \mathbf{b})$	Learn it
Collinear: $\overrightarrow{AC} = k\overrightarrow{AB}$ with the point A in common	Learn it
Position at time t : $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$; speed $= \mathbf{v} $	Learn it
Speed v on bearing θ : $v \sin \theta \mathbf{i} + v \cos \theta \mathbf{j}$; at angle α to the x -axis: $v \cos \alpha \mathbf{i} + v \sin \alpha \mathbf{j}$	Learn it

Formula	Status
Sine rule and cosine rule (for a triangle of vectors)	Given

How to do it

In the Paper 1 and Paper 2 questions we have tagged for this topic (41 questions, 2021–2025), the types came up this often. 16 of the 41 questions mix two or three types, so the counts add up to more than 41.

Question type	Questions
Vector geometry: two routes to a point	13
Velocity, position and collisions	10
Magnitude, unit vectors and vectors of a given length	10
A point dividing a line in a ratio	9
Equating like vectors	7
Position vectors with numbers	6
Resultants with bearings and angles	3

1. Vector geometry: two routes to a point (13 questions)

- Mark the diagram.** Write the given vectors on it with arrows. Turn every ratio into a fraction of a line (for $CD : DB = 3 : 4$, $\overrightarrow{CD} = \frac{3}{7}\overrightarrow{CB}$).
- Build the vectors you need** from the given ones, "end minus start" where needed, such as $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.
- Route 1:** reach the unknown point along one line, using the first scalar: $\overrightarrow{OX} = \overrightarrow{OA} + \lambda\overrightarrow{AM}$. Expand and collect into (number) $\mathbf{a}$ + (number) $\mathbf{b}$.
- Route 2:** reach it along the other line with the second scalar, such as $\overrightarrow{OX} = \mu\overrightarrow{OC}$, and collect the same way.
- Equate** the $\mathbf{a}$ parts and the $\mathbf{b}$ parts. You now have two equations with no vectors in them. Solve them together.
- Answer what was asked:** the values, the vector (substitute back), or a ratio.

Variations you will meet:

- The question names the scalars for you**, such as " $\overrightarrow{AX} = \lambda\overrightarrow{AN}$ and $\overrightarrow{OX} = \mu\overrightarrow{OM}$ ", and asks for $\overrightarrow{OX}$ in terms of one scalar, then the other, then the values. These are steps 3, 4 and 5 in order. Use the letters they give.
- The question does not name them** (it just asks for two expressions for the vector to the crossing point, and then a ratio). Choose your own two letters and say what they stand for.

- **Lines that meet outside the shape**, where one line has to be extended past a vertex to reach the other: the scalar is then more than 1 (or negative), but the method is the same.
- **A straight line of three points** such as X, Y, Z with $\overrightarrow{YZ} = m\overrightarrow{XY}$: that is one route; build the other around the shape.
- **Turning a scalar into a ratio**. If $\overrightarrow{AE} = \frac{4}{7}\overrightarrow{AB}$ then $AE : EB = 4 : 3$, not $4 : 7$. Give the ratio in the form asked.
- **Show that three points are collinear**, such as O, U and W : write $\overrightarrow{OW}$ as a number times $\overrightarrow{OU}$, then say the two vectors are parallel **and** share the point O .
- **Trapezium or parallelogram**. Opposite sides of a parallelogram are equal vectors; in a trapezium the parallel sides are multiples of the same vector (for example $\overrightarrow{CB} = \frac{2}{3}\overrightarrow{OA}$).

2. Velocity, position and collisions (10 questions)

1. **Velocity vector**. If you are given a speed and a direction vector, find the direction vector's magnitude, then scale: speed 10 along $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ is $\frac{10}{5} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}$. If you are given an angle or bearing, resolve (section 7).
2. **Position at time t** : $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. Keep the t .
3. **Collide or not?** Equate the x -parts and solve for t ; equate the y -parts and solve for t . The same non-negative t from both means they collide (find where by substituting it). Different values mean they do not; say so.
4. **Distance between them at time t** : subtract the position vectors ($\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$), then use Pythagoras on its two parts. Expand and simplify to $\sqrt{at^2 + bt + c}$.

Variations you will meet:

- **Speed from a position vector**. In $\begin{pmatrix} 4 + 8t \\ -1 - 15t \end{pmatrix}$ the velocity is the part multiplied by t , $\begin{pmatrix} 8 \\ -15 \end{pmatrix}$, so the speed is 17. The part without t is the starting position.
- **Velocity from two positions**. If an object moves from $\mathbf{r}_1$ to $\mathbf{r}_2$ in 2 hours, its velocity is $\frac{1}{2}(\mathbf{r}_2 - \mathbf{r}_1)$.
- **Does it pass through a point?** Equate its position to the point. Two different values of t from the two parts mean it does not.
- **"Show that they never collide" using the distance**. If $d^2 = at^2 + bt + c$, then $d = 0$ needs $at^2 + bt + c = 0$. Show it has no real roots ($b^2 - 4ac < 0$), or that its only root is a negative time, which is impossible because $t \geq 0$. Completing the square helps: $d^2 = 5t^2 + 20t + 30 = 5(t + 2)^2 + 10 \geq 10$, so the distance is never 0.
- **When is the distance a given value?** Set $at^2 + bt + c$ equal to the distance **squared** and solve the quadratic. Reject negative times.
- **One object starts later**. If B starts 30 s after A , and t is A 's time, then B has been moving for $(t - 30)$ s, so $\mathbf{r}_B = \mathbf{r}_0 + (t - 30)\mathbf{v}_B$. With clock times, count hours from the start time (from 08 30 to 10 00 is $t = 1.5$).

- **Collide at an unknown point:** sometimes you must find an unknown speed or velocity from the fact that they meet; equate the parts and solve for the unknown as well as t .

3. Magnitude, unit vectors and vectors of a given length (10 questions)

1. Find the magnitude of the direction vector: $|x\mathbf{i} + y\mathbf{j}| = \sqrt{x^2 + y^2}$.
2. **Unit vector:** divide by it. **Vector of length L :** multiply the direction vector by $\frac{L}{\text{magnitude}}$.
3. For the **opposite** direction, put a minus sign in front.

Variations you will meet:

- **Build the vectors first, then combine.** If $\mathbf{q}$ has magnitude 10 along $3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{r}$ has magnitude 13 along $-12\mathbf{i} + 5\mathbf{j}$, then $\mathbf{q} = 6\mathbf{i} + 8\mathbf{j}$ and $\mathbf{r} = -12\mathbf{i} + 5\mathbf{j}$, so $\mathbf{q} + \mathbf{r} = -6\mathbf{i} + 13\mathbf{j}$ with magnitude $\sqrt{205}$. Never add the magnitudes: $10 + 13$ is only right when the directions are the same.
- **Unit vector along $\overrightarrow{AB}$ or $\overrightarrow{OR}$:** find that vector first ("end minus start"), then divide by its magnitude. Don't stop at the magnitude.
- **Magnitude and angle.** For $\mathbf{p} + \mathbf{q}$, find the vector, its magnitude, then the angle with the x -axis from $\tan^{-1}$ of the parts, checking the quadrant with a sketch.
- **An unknown in a vector of known length:** $|k\mathbf{i} + (k - 7)\mathbf{j}| = 13$ gives $k^2 + (k - 7)^2 = 169$, a quadratic (Example 1).
- **Same direction from a line:** " $\overrightarrow{OR}$ is in the same direction as $\overrightarrow{PQ}$ " means $\overrightarrow{OR} = k\overrightarrow{PQ}$ with $k > 0$.

4. A point dividing a line in a ratio (9 questions)

1. Draw or mark the line with the ratio on it.
2. Write the step as a fraction of the whole: $AP : PB = 1 : 3$ gives $\overrightarrow{AP} = \frac{1}{4}\overrightarrow{AB}$.
3. $\overrightarrow{OP} = \mathbf{a} + \frac{1}{4}(\mathbf{b} - \mathbf{a}) = \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$. Simplify: collect the $\mathbf{a}$ terms and the $\mathbf{b}$ terms.

Variations you will meet:

- **Show a result linking $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$** when B divides AC in the ratio $m : n$. Start from a fact about steps, such as $n\overrightarrow{AB} = m\overrightarrow{BC}$, so $n(\mathbf{b} - \mathbf{a}) = m(\mathbf{c} - \mathbf{b})$, and rearrange with every line shown.
- **Find the far point.** If B lies on AC with $AB : AC = 3 : 5$, then $\mathbf{b} - \mathbf{a} = \frac{3}{5}(\mathbf{c} - \mathbf{a})$, so $5\mathbf{b} - 5\mathbf{a} = 3\mathbf{c} - 3\mathbf{a}$ and $\mathbf{c} = \frac{1}{3}(5\mathbf{b} - 2\mathbf{a})$.
- **A point beyond the end of a line.** If $OA : AD = 3 : 2$ with D on OA extended, then OD is $\frac{5}{3}$ times OA , so $\overrightarrow{OD} = \frac{5}{3}\mathbf{a}$.
- **" R lies on PQ ; show that $m + n = 1$ "** where $\overrightarrow{OR} = m\mathbf{p} + n\mathbf{q}$: write $\overrightarrow{OR} = \mathbf{p} + k(\mathbf{q} - \mathbf{p}) = (1 - k)\mathbf{p} + k\mathbf{q}$, so $m + n = (1 - k) + k = 1$. If R is **between** P and Q , then $0 < k < 1$.
- **The intersection of the diagonals** of a quadrilateral: the ratio on one diagonal gives $\overrightarrow{OP}$, and the ratio on the other then links $\mathbf{b}$ to $\mathbf{a}$ and $\mathbf{c}$.

5. Equating like vectors (7 questions)

1. Multiply out and add the vectors on each side, so each side is one column vector (or one **i** part and one **j** part).
2. Equate the top (the **i**) parts, and equate the bottom (the **j**) parts. That is two equations.
3. Solve them together. Check by substituting into the original equation.

Variations you will meet:

- **Unknowns inside the vectors**, such as $\mathbf{a} = \alpha\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = \mathbf{i} + \beta\mathbf{j}$ with $\mathbf{c} = 2\mathbf{a} + \mathbf{b}$: build $2\mathbf{a} + \mathbf{b}$ in terms of the unknowns first, then equate. Terms such as $m \times (n - 1)$ just stay as they are until you equate.
- **Unknown lengths along known directions**. " $\overrightarrow{OP}$ is in the direction of $3\mathbf{i} + \mathbf{j}$ and $\overrightarrow{PQ}$ is in the direction of $\mathbf{i} - 2\mathbf{j}$ ": write $\overrightarrow{OP} = s(3\mathbf{i} + \mathbf{j})$ and $\overrightarrow{PQ} = t(\mathbf{i} - 2\mathbf{j})$, then use $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$ and equate.
- **Parallel vectors with an unknown**: $\mathbf{p} = k\mathbf{q}$ gives two equations, one for each part; eliminate k (Example 1).

6. Position vectors with numbers (6 questions)

1. Turn each point into a column vector.
2. For a step, use "end minus start": $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$. For the distance, take its magnitude.
3. For a point defined by a vector fact (a midpoint, a ratio, $3\overrightarrow{AB} = 2\overrightarrow{AX}$), write that fact with position vectors and solve for the unknown position vector.

Variations you will meet:

- **Midpoint**. " A is the midpoint of BC " means $\mathbf{a} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, so $\mathbf{c} = 2\mathbf{a} - \mathbf{b}$.
- **A point on a line with an unknown**. If C , with $\overrightarrow{OC} = k \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, lies on the line AB , then $\overrightarrow{AC}$ is a multiple of $\overrightarrow{AB}$: $\frac{\text{top of } \overrightarrow{AC}}{\text{top of } \overrightarrow{AB}} = \frac{\text{bottom of } \overrightarrow{AC}}{\text{bottom of } \overrightarrow{AB}}$. Solve for k , then compare $\overrightarrow{AC}$ with $\overrightarrow{AB}$ for the ratio. The question may say "use a vector method": a drawing or gradients alone won't earn the marks.
- **A point on a line extended**. R on PQ extended with $4\overrightarrow{QR} = \overrightarrow{PR}$: write $4(\mathbf{r} - \mathbf{q}) = \mathbf{r} - \mathbf{p}$ and solve for $\mathbf{r}$.
- **Parallel to an axis**. A vector is parallel to the x -axis when its **j** part is 0 (and to the y -axis when its **i** part is 0).
- **Parallel to another step**. " $\overrightarrow{RS}$ is parallel to $\overrightarrow{PQ}$ " means $\overrightarrow{RS} = k\overrightarrow{PQ}$: equate parts, or set the ratios of the parts equal.
- **Fourth vertex of a parallelogram**. In $ABCD$, $\overrightarrow{AD} = \overrightarrow{BC}$, so $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$ (Example 2).

7. Resultants with bearings and angles (3 questions)

1. **Resolve** each vector into **i** (east) and **j** (north) parts: speed v on bearing θ gives $v \sin \theta \mathbf{i} + v \cos \theta \mathbf{j}$. Check the signs against a sketch. Keep 4 or more significant figures.

2. **Add** the parts to get the resultant.

3. **Magnitude** by Pythagoras. **Bearing**: sketch the resultant, find the angle it makes with north (or east) using $\tan^{-1}$, then turn that into a three-figure bearing.

Variations you will meet:

- **Find the missing vector** in $\mathbf{a} + \mathbf{b} = \mathbf{c}$: resolve $\mathbf{a}$ and $\mathbf{c}$, then $\mathbf{b} = \mathbf{c} - \mathbf{a}$.
- **Use the triangle instead**. Two vectors placed nose to tail make a triangle with the resultant. If you know the angle between them, the cosine rule gives the magnitude and the sine rule gives an angle, which is often quicker. The angle inside the triangle is **not** the difference of the bearings; work it out from a sketch.
- **Exact components**. At 60° or 30° , use $\cos 60^\circ = \frac{1}{2}$, $\sin 60^\circ = \frac{\sqrt{3}}{2}$ and keep surds when the question asks for exact values.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

(a) The vector $\mathbf{v} = k\mathbf{i} + (k - 7)\mathbf{j}$, where k is a positive constant, has magnitude 13. Find the value of k and the unit vector in the direction of $\mathbf{v}$. **[4]**

(b) The vectors $\mathbf{p} = (s + 1)\mathbf{i} + 2\mathbf{j}$ and $\mathbf{q} = 6\mathbf{i} + s\mathbf{j}$ are parallel and in the same direction. Find the value of s . **[4]**

(a) $k^2 + (k - 7)^2 = 169$. **M1** for using Pythagoras with the magnitude.

$$2k^2 - 14k + 49 = 169 \Rightarrow k^2 - 7k - 60 = 0 \Rightarrow (k - 12)(k + 5) = 0. \text{ Since } k > 0, k = 12. \text{ **A1** }$$

So $\mathbf{v} = 12\mathbf{i} + 5\mathbf{j}$, and the unit vector is $\frac{1}{13}(12\mathbf{i} + 5\mathbf{j})$. **M1 A1**

(b) Parallel, so $\mathbf{p} = \lambda\mathbf{q}$: $s + 1 = 6\lambda$ and $2 = \lambda s$. **M1** for writing one vector as a multiple of the other and equating parts.

From the second, $\lambda = \frac{2}{s}$, so $s + 1 = \frac{12}{s}$, giving $s^2 + s - 12 = 0$. **M1 A1**

$(s + 4)(s - 3) = 0$, so $s = 3$ or $s = -4$. With $s = 3$, $\lambda = \frac{2}{3} > 0$: $\mathbf{p} = 4\mathbf{i} + 2\mathbf{j}$ and $\mathbf{q} = 6\mathbf{i} + 3\mathbf{j}$ point the same way. With $s = -4$, $\lambda = -\frac{1}{2} < 0$: they point opposite ways. So $s = 3$. **A1**

Example 2

Relative to an origin O , the points A , B and C have position vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ -1 \end{pmatrix}$. $ABCD$ is a parallelogram.

(a) Find the position vector of D . [2]

(b) Find the length of each diagonal, AC and BD , giving your answers as surds. [3]

(c) The point F lies on BD such that $BF : FD = 3 : 1$. Find the position vector of F . [2]

(a) In a parallelogram $\overrightarrow{AD} = \overrightarrow{BC} = \begin{pmatrix} 9 \\ -1 \end{pmatrix} - \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$. **M1**

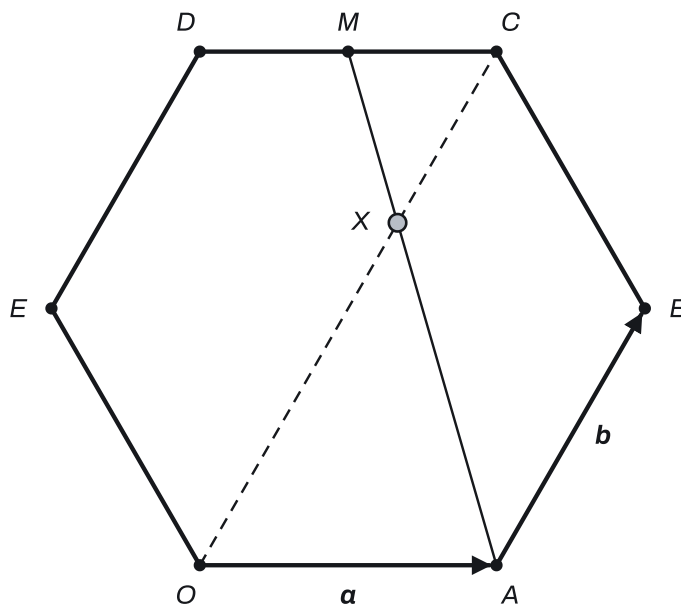
$$\overrightarrow{OD} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}. \text{A1}$$

(b) $\overrightarrow{AC} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$, so $AC = \sqrt{64 + 9} = \sqrt{73}$. **M1 A1**

$$\overrightarrow{BD} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ -9 \end{pmatrix}, \text{ so } BD = \sqrt{16 + 81} = \sqrt{97}. \text{A1}$$

(c) $\overrightarrow{BF} = \frac{3}{4}\overrightarrow{BD}$, so $\overrightarrow{OF} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} -4 \\ -9 \end{pmatrix} = \begin{pmatrix} 4 \\ -\frac{7}{4} \end{pmatrix}$. **M1 A1**

Example 3



$OABCDE$ is a regular hexagon with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{AB} = \mathbf{b}$. The side CD is parallel to OA and the same length, and the diagonal OC is parallel to AB and twice as long. M is the midpoint of CD . The lines AM and OC meet at X , where $\overrightarrow{AX} = \lambda \overrightarrow{AM}$ and $\overrightarrow{OX} = \mu \overrightarrow{OC}$.

(a) Find $\overrightarrow{OM}$ and $\overrightarrow{AM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

(b) Find $\overrightarrow{OX}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and λ , and in terms of $\mathbf{b}$ and μ . [2]

(c) Hence find the values of λ and μ , and state the ratio $OX : XC$. [4]

(a) $\overrightarrow{OC} = 2\mathbf{b}$ and $\overrightarrow{CD} = -\mathbf{a}$ (it points the opposite way to OA). **B1**

$$\overrightarrow{OM} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CD} = 2\mathbf{b} - \frac{1}{2}\mathbf{a}. \text{ B1}$$

$$\overrightarrow{AM} = \overrightarrow{OM} - \overrightarrow{OA} = 2\mathbf{b} - \frac{3}{2}\mathbf{a}. \text{ B1}$$

(b) Route 1: $\overrightarrow{OX} = \mathbf{a} + \lambda(2\mathbf{b} - \frac{3}{2}\mathbf{a}) = (1 - \frac{3}{2}\lambda)\mathbf{a} + 2\lambda\mathbf{b}. \text{ B1}$

Route 2: $\overrightarrow{OX} = \mu(2\mathbf{b}) = 2\mu\mathbf{b}. \text{ B1}$

(c) Equate the two expressions. **M1** for equating like vectors to get equations without vectors.

$\mathbf{a}$ parts: $1 - \frac{3}{2}\lambda = 0$, so $\lambda = \frac{2}{3}. \text{ A1}$

$\mathbf{b}$ parts: $2\lambda = 2\mu$, so $\mu = \frac{2}{3}. \text{ A1}$

$\overrightarrow{OX} = \frac{2}{3}\overrightarrow{OC}$, so $OX : XC = 2 : 1. \text{ B1}$ (Likewise $AX : XM = 2 : 1.$)

Example 4

In this question $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north. Distances are in kilometres and time t is in hours.

A patrol boat P leaves the point with position vector $2\mathbf{i} - 3\mathbf{j}$ at time $t = 0$. It travels at a constant speed of 6.5 km/h in the direction of $12\mathbf{i} + 5\mathbf{j}$.

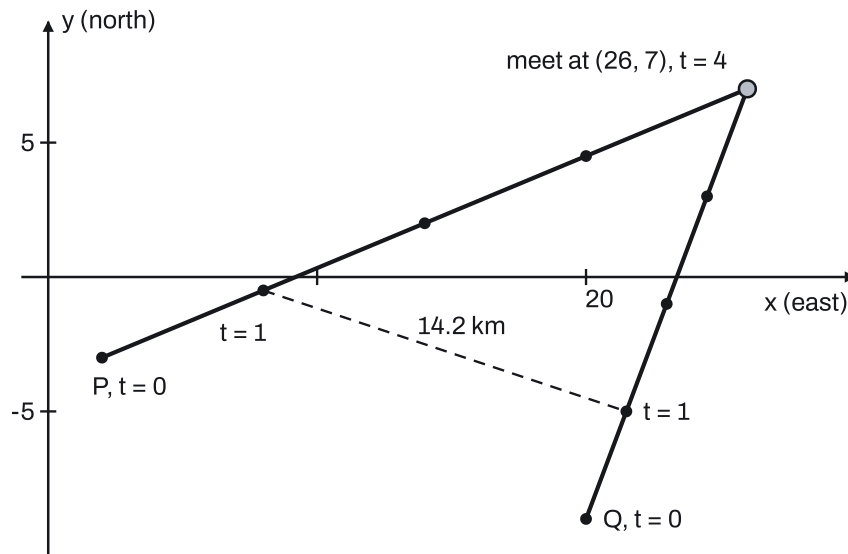
(a) Find the velocity vector of P , and its position vector at time t . [3]

A launch Q leaves the point with position vector $20\mathbf{i} - 9\mathbf{j}$, also at $t = 0$, and moves with constant velocity. It meets P when $t = 4$.

(b) Find the velocity vector of Q . [3]

(c) Find the speed of Q and the bearing on which it travels. [3]

(d) Find the distance between the two boats when $t = 1$. [3]



(a) $|12\mathbf{i} + 5\mathbf{j}| = \sqrt{144 + 25} = 13$. **B1**

Velocity = $\frac{6.5}{13}(12\mathbf{i} + 5\mathbf{j}) = 6\mathbf{i} + 2.5\mathbf{j}$. **B1**

Position: $\mathbf{r}_P = (2 + 6t)\mathbf{i} + (-3 + 2.5t)\mathbf{j}$. **B1** (follow through from the velocity)

(b) At $t = 4$, P is at $26\mathbf{i} + 7\mathbf{j}$. **B1**

Q covers $(26\mathbf{i} + 7\mathbf{j}) - (20\mathbf{i} - 9\mathbf{j}) = 6\mathbf{i} + 16\mathbf{j}$ in 4 hours. **M1**

Velocity of $Q = \frac{1}{4}(6\mathbf{i} + 16\mathbf{j}) = 1.5\mathbf{i} + 4\mathbf{j}$. **A1**

(c) Speed = $\sqrt{1.5^2 + 4^2} = \sqrt{18.25} = 4.27$ km/h (3 s.f.). **B1**

Q goes 1.5 east for every 4 north, so its direction is $\tan^{-1} \frac{1.5}{4} = 20.6^\circ$ east of north. **M1** Bearing 020.6° . **A1**

(d) At $t = 1$: $\mathbf{r}_P = 8\mathbf{i} - 0.5\mathbf{j}$ and $\mathbf{r}_Q = 21.5\mathbf{i} - 5\mathbf{j}$. **B1**

$\overrightarrow{PQ} = 13.5\mathbf{i} - 4.5\mathbf{j}$. **M1** for subtracting the position vectors.

Distance = $\sqrt{13.5^2 + 4.5^2} = \sqrt{202.5} = 14.2$ km (3 s.f.). **A1**

Common mistakes

- **Adding position vectors instead of subtracting.** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, "end minus start". Examiners report this in almost every session, and it wrecks everything after it.
- **Using the direction vector as the velocity.** "Speed 15 in the direction $4\mathbf{i} - 3\mathbf{j}$ " is **not** $15(4\mathbf{i} - 3\mathbf{j})$, nor $4\mathbf{i} - 3\mathbf{j}$. Find the magnitude (5) first and multiply by $\frac{15}{5}$, giving $12\mathbf{i} - 9\mathbf{j}$.
- **Stopping at the magnitude** when a unit vector was asked for, or giving the magnitude as "the unit vector". A unit vector is a vector, divided by its magnitude.

- **Swapping $\mathbf{i}$ and $\mathbf{j}$ with bearings.** $\mathbf{j}$ is north. On a bearing, the east part uses $\sin$ and the north part uses $\cos$. Say why a part is negative (it points south or west).
- **Misreading a ratio.** $AB : BC = 1 : 4$ and $AB : AC = 1 : 4$ put the point in different places. Mark the ratio on the diagram before writing any vector.
- **Arrows the wrong way.** $\overrightarrow{BA} = -\overrightarrow{AB}$. Draw the direction of every vector you use on the diagram.
- **Equating before the vectors are gone.** You can't solve an equation that still has $\mathbf{a}$ and $\mathbf{b}$ in it. Collect terms, then equate the $\mathbf{a}$ parts and the $\mathbf{b}$ parts to get equations in the scalars only.
- **Using one letter for two different fractions.** Each route needs its own scalar (λ on one line, μ on the other).
- **Leaving out t , or putting it in the wrong place.** The position is $\text{start} + t \times \text{velocity}$; a final answer with no t in it can't be right.
- **Deciding a collision from one equation.** Solve both the x and y equations; the times must match. Two different times means no collision, so don't write "they collide".
- **Subtracting the squares** when finding a distance. $\sqrt{x^2 + y^2}$ always adds them, even when a part is negative.
- **Mixing forms.** Don't write $\mathbf{i}$ or $\mathbf{j}$ inside a column vector, and write vectors as vectors (bold, underlined or with an arrow), not as coordinates, when the question asks for a vector.
- **Rounding too early** in a resultant or bearing question. Keep 4 or more significant figures until the end.

How to get the marks

- **Show the vector route.** Write $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ (or similar) before substituting. Method marks are for the route, and they follow through if a vector used in it is wrong.
- **"Use a vector method"** means your working must use vectors. Coordinate geometry, gradients or accurate drawing score nothing, even with the right answer.
- **"Find in terms of $\mathbf{a}$ and $\mathbf{b}$. Simplify your answer."** Collect terms into one multiple of $\mathbf{a}$ plus one multiple of $\mathbf{b}$.
- **"Hence find the values of m and n ":** equate the two expressions you found in the earlier parts. Write the two scalar equations clearly, then solve them.
- **"Show that"** means the answer is given. Every step must be shown, ending with the printed result.
- **Ratios** must be given as ratios, such as $4 : 3$, not as a fraction.
- **Collisions:** finish with a sentence. For "show that they collide", show the same t satisfies both parts (or that both are at the same point at that time). For "show that they never collide", give the reason: different times, a negative discriminant, or a negative time.
- **Distances** are scalars: give a number with units, not a vector.
- **Accuracy:** exact answers (surds) when asked, otherwise 3 significant figures, and bearings to 0.1° as three figures (020.6°).
- **Write vectors in a correct form** consistently: $\begin{pmatrix} x \\ y \end{pmatrix}$ or $x\mathbf{i} + y\mathbf{j}$, never a mixture.

Quick check

1. $\mathbf{a} = 6\mathbf{i} - 8\mathbf{j}$. Find the vector $\mathbf{b}$ of magnitude 25 in the same direction as $\mathbf{a}$, then find $|\mathbf{a} + \mathbf{b}|$.
2. The points A and B have position vectors $\mathbf{i} + 4\mathbf{j}$ and $12\mathbf{i} - 4\mathbf{j}$. Find the constants λ and μ such that $\overrightarrow{AB} = \lambda(3\mathbf{i} - \mathbf{j}) + \mu(-2\mathbf{i} + 5\mathbf{j})$.
3. In triangle OAB , $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. P lies on OA with $OP : PA = 3 : 1$, and Q is the midpoint of OB . Find $\overrightarrow{PQ}$, and the position vector of the midpoint of PQ , in terms of $\mathbf{a}$ and $\mathbf{b}$.
4. A boat can move at 8 m s^{-1} through still water. A current carries the water due south at 3 m s^{-1} . The boat must actually travel due east. Find the bearing on which it must steer, and its resultant speed.
5. A particle has position vector $(5 - 2t)\mathbf{i} + (3t - 7)\mathbf{j}$ metres at time t seconds. Find its speed, and the position vector of the particle when it is due north of the origin.

Answers

1. $|\mathbf{a}| = \sqrt{36 + 64} = 10$, so $\mathbf{b} = \frac{25}{10}(6\mathbf{i} - 8\mathbf{j}) = 15\mathbf{i} - 20\mathbf{j}$. Then $\mathbf{a} + \mathbf{b} = 21\mathbf{i} - 28\mathbf{j}$ and $|\mathbf{a} + \mathbf{b}| = \sqrt{441 + 784} = 35$. (Same direction, so here the lengths do add: $10 + 25$.)
2. $\overrightarrow{AB} = 11\mathbf{i} - 8\mathbf{j}$. $\mathbf{i}$: $3\lambda - 2\mu = 11$; $\mathbf{j}$: $-\lambda + 5\mu = -8$. From the second, $\lambda = 5\mu + 8$, so $15\mu + 24 - 2\mu = 11$, $\mu = -1$ and $\lambda = 3$.
3. $\overrightarrow{OP} = \frac{3}{4}\mathbf{a}$ and $\overrightarrow{OQ} = \frac{1}{2}\mathbf{b}$, so $\overrightarrow{PQ} = \frac{1}{2}\mathbf{b} - \frac{3}{4}\mathbf{a}$. The midpoint of PQ has position vector $\frac{1}{2}(\frac{3}{4}\mathbf{a} + \frac{1}{2}\mathbf{b}) = \frac{3}{8}\mathbf{a} + \frac{1}{4}\mathbf{b}$.
4. Steering on bearing θ , the boat's own velocity is $8 \sin \theta \mathbf{i} + 8 \cos \theta \mathbf{j}$; the current adds $-3\mathbf{j}$. To travel due east the $\mathbf{j}$ parts must cancel: $8 \cos \theta - 3 = 0$, so $\cos \theta = \frac{3}{8}$ and $\theta = 68.0^\circ$. Steer on bearing 068.0° . The resultant is $8 \sin \theta \mathbf{i} = \sqrt{64 - 9} \mathbf{i} = \sqrt{55} \mathbf{i}$, a speed of 7.42 m s^{-1} .
5. Velocity $-2\mathbf{i} + 3\mathbf{j}$, so speed $\sqrt{13} = 3.61 \text{ m s}^{-1}$. Due north of O means the $\mathbf{i}$ part is 0: $5 - 2t = 0$, $t = 2.5$. Then $3(2.5) - 7 = 0.5$, so the position vector is $0.5\mathbf{j}$.

Past questions to try

- **0606/12/O/N/22 Q5**: a vector of given length, a point dividing a line, and equating like vectors.
- **0606/12/F/M/25 Q11**: two routes to where two lines in a triangle cross.
- **0606/23/M/J/25 Q10**: velocities from a speed and a direction, then whether two particles collide.
- **0606/22/M/J/22 Q6**: a velocity from a bearing, then the magnitude and bearing of a resultant.