

Circular motion

A2

Read online at pastopic.com/cambridge/further-mathematics-9231/notes/circular-motion

Last checked 1 October 2026

Read first: [Motion of a projectile](#)

© 2026 Pastopic. Free for personal study. Please share the link, not the file.

What you need to know

By the end of this topic you should be able to:

1. **Use angular speed.** Know that the angular speed ω of a particle moving in a circle is the angle (in radians) it turns through per second, and use $v = r\omega$ to link it to the speed v and the radius r .
2. **Know where the acceleration points.** A particle moving round a circle at constant speed accelerates **towards the centre**, with size $r\omega^2$ or $\frac{v^2}{r}$. Use these formulas; you do not need to prove them.
3. **Solve horizontal-circle problems** at constant speed: a particle on a string (a conical pendulum), on a smooth surface such as a bowl or cone, on a table joined to a string through a hole or ring, on an elastic string, or on a rough rotating disc.
4. **Solve vertical-circle problems** with no energy lost: find the speed at any point with conservation of energy, find the tension in a string or the normal contact force from a surface, find where that force becomes zero (the string goes slack or the particle leaves the surface), and find the conditions for the particle to go all the way round.

Proving the acceleration formulas is **not** required, and nothing here needs calculus. Motion after a string goes slack is ordinary projectile motion (see the Motion of a projectile note), and elastic strings use Hooke's law $T = \frac{\lambda x}{l}$ from the Hooke's law topic.

Understand it

Angular speed

Picture a particle P going round a circle with centre C . The line CP sweeps round. The **angular speed** ω is how fast that line turns, in **radians per second** (rad s^{-1}). One full turn is 2π radians, so a particle that makes one revolution every τ seconds has $\omega = \frac{2\pi}{\tau}$.

In one second the particle turns through ω radians, so it travels an arc of length $r\omega$ (arc = radius $\times$ angle in radians). That is its speed:

$$v = r\omega.$$

For example, a particle on a circle of radius 0.5 m that goes round once every 2 s has $\omega = \frac{2\pi}{2} = \pi \text{ rad s}^{-1}$ and speed $v = 0.5\pi \approx 1.57 \text{ m s}^{-1}$.

Questions sometimes give **revolutions per minute** (rpm). One revolution is 2π radians and one minute is 60 s, so n rpm is $\omega = \frac{2\pi n}{60} \text{ rad s}^{-1}$, and $\omega \text{ rad s}^{-1}$ is $\frac{60\omega}{2\pi} \text{ rpm}$.

Why the acceleration points to the centre

Acceleration is the rate of change of **velocity**, and velocity has a direction. A particle going round a circle at constant speed keeps turning, so its velocity keeps changing even though its speed does not. The change is always towards the centre, and its size is

$$a = r\omega^2 = \frac{v^2}{r} \quad (\text{towards the centre}).$$

The two forms are the same thing, because $v = r\omega$. For the particle above, $a = 0.5 \times \pi^2 \approx 4.93 \text{ m s}^{-2}$.

By Newton's second law, something must **pull the particle towards the centre** with a resultant force of $mr\omega^2 = \frac{mv^2}{r}$. That force is not a new force: it is the total of the real forces (tension, normal reaction, friction, weight) resolved **towards the centre**. So every circular-motion question starts the same way: draw the forces, then write

$$(\text{resultant force towards the centre}) = \frac{mv^2}{r} \quad \text{or} \quad mr\omega^2.$$

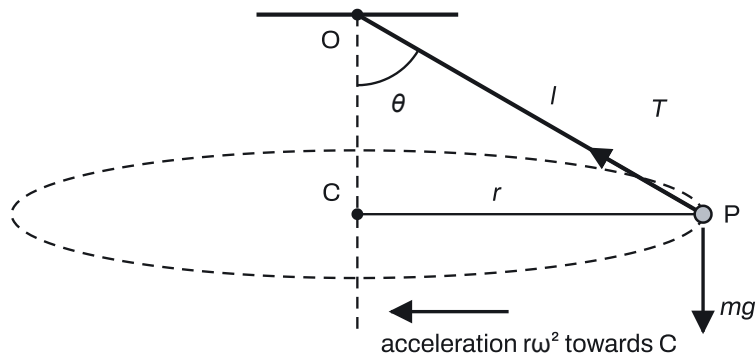
There is no outward "centrifugal force" to add. Never put $\frac{mv^2}{r}$ on the force diagram as if it were a force.

Horizontal circles

In a horizontal circle the particle stays at one height, so its vertical acceleration is zero and the speed is constant. Resolve in two directions:

- **vertically:** the forces balance (up = down);
- **horizontally, towards the centre:** resultant = $mr\omega^2$.

The classic case is the **conical pendulum**: a particle on a string of length l fixed at O , moving round a horizontal circle below O with the string at angle θ to the vertical.



The radius is **not** the string length: $r = l \sin \theta$. Then

$$T \cos \theta = mg, \quad T \sin \theta = mr\omega^2 = m(l \sin \theta)\omega^2.$$

Cancelling $\sin \theta$ in the second equation gives the neat result $T = ml\omega^2$, and then $\cos \theta = \frac{mg}{T} = \frac{g}{l\omega^2}$. Spinning faster makes $\cos \theta$ smaller, so the string swings out further.

A particle moving round the **smooth inside of a hemispherical bowl** of radius a works the same way. The normal reaction R points along the radius to the bowl's centre O . If OP makes angle θ with the downward vertical, the circle has radius $a \sin \theta$, and

$$R \cos \theta = mg, \quad R \sin \theta = m(a \sin \theta)\omega^2 \Rightarrow R = m a \omega^2, \quad \cos \theta = \frac{g}{a \omega^2}.$$

The circle is $a \cos \theta = \frac{g}{\omega^2}$ below the level of O (the rim).

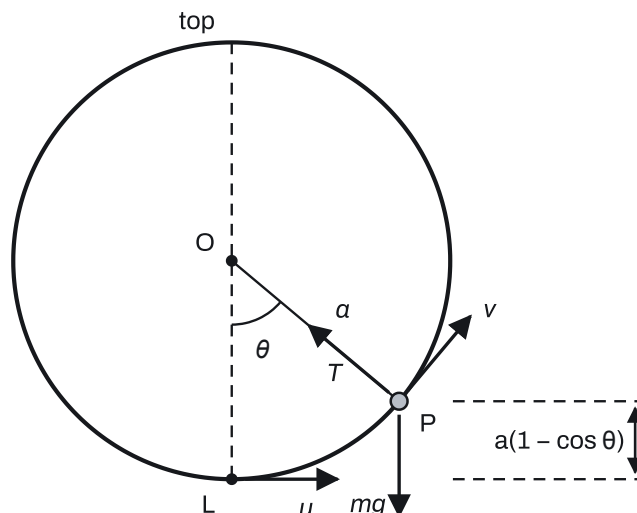
On the **smooth inside of a cone** fixed vertex down with semi-vertical angle β , the normal reaction is at right angles to the sloping surface, so it makes angle β with the **horizontal**: $R \sin \beta = mg$ vertically and $R \cos \beta = mr\omega^2$ horizontally. A particle at height h above the vertex moves on a circle of radius $r = h \tan \beta$.

Vertical circles: speed changes

In a vertical circle the particle rises and falls, so gravity speeds it up and slows it down. Its speed is **not** constant, and you cannot use suvat (the acceleration is not constant). Instead, with no energy lost (a smooth surface, no air resistance, and tension or a normal reaction that always acts at right angles to the motion, so does no work):

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgh,$$

where h is how far the particle has risen.



Take a particle on a string of length a (the radius of the circle), passing the lowest point L with speed u . When OP makes angle θ with the **downward** vertical, P is $a(1 - \cos \theta)$ above L , so

$$v^2 = u^2 - 2ga(1 - \cos \theta).$$

Vertical circles: the force towards the centre

At P the tension T points along PO , towards the centre. The weight mg points straight down; its part along the radius is $mg \cos \theta$, pointing **away** from O (when P is below O). Along the radius, towards O :

$$T - mg \cos \theta = \frac{mv^2}{a}.$$

The part of the weight along the tangent, $mg \sin \theta$, is what slows the particle down; you never need it, because the energy equation already accounts for it.

Two special points are worth knowing by heart:

- **Lowest point** ($\theta = 0$): $T - mg = \frac{mu^2}{a}$. The tension is greatest here.
- **Highest point** ($\theta = 180^\circ$, $\cos \theta = -1$): $T + mg = \frac{mv^2}{a}$. Weight now points towards the centre, and the tension is least here.

Putting the energy equation into the tension equation gives $T = \frac{mu^2}{a} - 2mg + 3mg \cos \theta$. The tension at the top is always exactly $6mg$ less than at the bottom.

For a point **above** the level of O , papers usually give the angle α between OP and the **upward** vertical. Then P is $a(1 + \cos \alpha)$ above L , and the weight's part along the radius, $mg \cos \alpha$, now points **towards** O :

$$v^2 = u^2 - 2ga(1 + \cos \alpha), \quad T + mg \cos \alpha = \frac{mv^2}{a}.$$

A particle moving on the **smooth inside** of a sphere or cylinder behaves exactly like one on a string, with the normal reaction R in place of T .

When the string goes slack or the particle leaves the surface

A string can only pull. When the equation needs T to be negative, the string cannot provide it, so it goes **slack** and the particle leaves the circle. The same happens on the inside of a surface when R would be negative. So:

The string goes slack (or the particle leaves the inside surface) where $T = 0$ (or $R = 0$). **The particle is still moving** there; only the force is zero.

With $T = 0$ at angle α from the upward vertical: $mg \cos \alpha = \frac{mv^2}{a}$, so $v^2 = ga \cos \alpha$. Put this into the energy equation:

$$ga \cos \alpha = u^2 - 2ga(1 + \cos \alpha) \Rightarrow \cos \alpha = \frac{u^2 - 2ga}{3ga}.$$

This only happens above the level of O : below it, T is at least $\frac{mv^2}{a} > 0$.

Going all the way round

- **String, or the inside of a surface.** The particle completes the circle if $T \geq 0$ at the top, where it is least: $\frac{mv^2}{a} - mg \geq 0$, so $v^2 \geq ga$ at the top. With $v^2 = u^2 - 4ga$, this means $u^2 \geq 5ga$.
- **A light rigid rod, or a bead threaded on a wire.** A rod can push as well as pull, and a wire holds the bead from both sides, so nothing goes slack. The particle only needs to **keep moving** at the top: $v^2 > 0$ there, so $u^2 > 4ga$.
- **Oscillating.** If the particle stops before it rises above the level of O ($u^2 \leq 2ga$), a string stays taut and the particle swings back and forth like a pendulum. For $2ga < u^2 < 5ga$ the string goes slack somewhere above the level of O .

On the outside of a sphere

A particle sliding over the **outside** of a smooth sphere of radius a is different: the normal reaction pushes **outwards**, away from the centre. At angle α from the upward vertical, towards O :

$$mg \cos \alpha - R = \frac{mv^2}{a}.$$

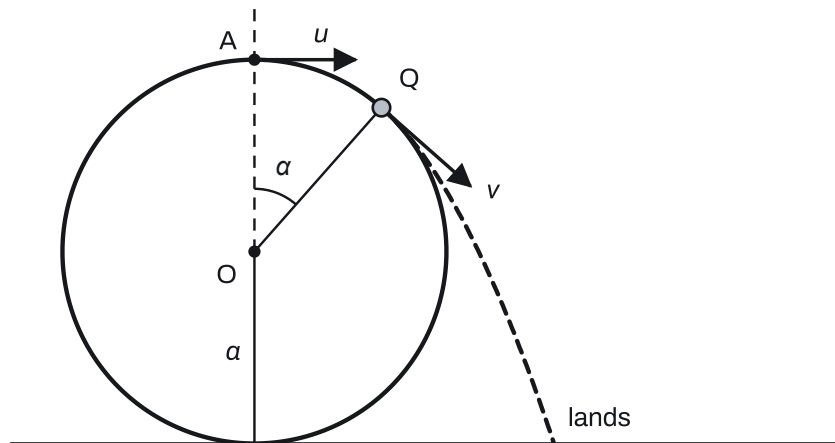
As the particle slides down, v grows and $\cos \alpha$ shrinks, so R falls to zero and the particle **leaves the surface** where $mg \cos \alpha = \frac{mv^2}{a}$. Starting at the top with speed u , it has dropped $a(1 - \cos \alpha)$, so $v^2 = u^2 + 2ga(1 - \cos \alpha)$, and

$$\cos \alpha = \frac{u^2 + 2ga}{3ga}.$$

Starting almost at rest ($u \approx 0$), it leaves where $\cos \alpha = \frac{2}{3}$. If $u^2 \geq ga$ it leaves the surface straight away at the top.

After leaving the circle

Once $T = 0$ or $R = 0$, the only force is gravity, so the particle is a **projectile**. It sets off along the **tangent** to the circle with the speed v it had there. At a point where the radius makes angle α with the vertical, the tangent makes angle α with the **horizontal**, so the velocity components are $v \cos \alpha$ horizontally and $v \sin \alpha$ vertically (upwards when a string goes slack on the way up, downwards when leaving the outside of a sphere).



Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember or derive it. Take $g = 10 \text{ m s}^{-2}$ unless told otherwise.

Formula	Status
Uniform circular motion: acceleration $r\omega^2$ or $\frac{v^2}{r}$, towards the centre	Given
$v = r\omega$	Learn it
One revolution = 2π rad; n rpm $\Rightarrow \omega = \frac{2\pi n}{60}$; period $\frac{2\pi}{\omega}$	Learn it
Newton's second law towards the centre: resultant = $mr\omega^2 = \frac{mv^2}{r}$	Learn it
Conical pendulum (string length l , angle θ to vertical): $r = l \sin \theta$, $T \cos \theta = mg$, $T = ml\omega^2$	Learn it
Energy: $\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgh$ (KE $\frac{1}{2}mv^2$, GPE mgh)	Learn it

Formula	Status
Vertical circle, θ from downward vertical: $v^2 = u^2 - 2ga(1 - \cos \theta)$, $T - mg \cos \theta = \frac{mv^2}{a}$	Learn it
Vertical circle, α from upward vertical: $v^2 = u^2 - 2ga(1 + \cos \alpha)$, $T + mg \cos \alpha = \frac{mv^2}{a}$	Learn it
Slack string or leaving the surface: $T = 0$ or $R = 0$ (speed is not zero)	Learn it
Complete circles: string or inside a surface $u^2 \geq 5ga$; rod or bead on a wire $u^2 > 4ga$	Learn it
Outside a sphere: $mg \cos \alpha - R = \frac{mv^2}{a}$; from the top with speed u , leaves where $\cos \alpha = \frac{u^2 + 2ga}{3ga}$	Learn it
Hooke's law $T = \frac{\lambda x}{l}$	Given
Limiting friction $F = \mu R$ (and $F \leq \mu R$ always)	Learn it
suvat equations and the equation of a projectile's trajectory (for motion after leaving the circle)	Given

How to do it

In the Paper 3 questions we have tagged for this topic (61 questions, 2021–2025), the types came up this often. 24 of the 61 questions mix more than one type, so the counts add up to more than 61.

Question type	Questions
Vertical circle: tension or reaction at a point	22
Vertical circle: string slack, leaving the surface, complete circles	16
Horizontal circle: string or smooth surface	11
Horizontal circle with an elastic string	9
Projectile after leaving the circle	9
Connected particles: string through a ring or hole	8
Speed by energy only	8
Rough turntable: limiting friction	4

Start every question with a **diagram**: mark the centre, the radius, the angles, the speeds at the points you care about and every force. Examiners say candidates who draw one make far fewer errors.

1. Vertical circle: tension or reaction at a point (22 questions)

1. **Energy equation** between a point where you know the speed and the point you want. Write the height difference from the diagram: $a(1 - \cos \theta)$ above the lowest point when θ is from the downward vertical, $a(1 + \cos \alpha)$ when α is from the upward vertical. If the start is **not** the lowest point, measure both heights from the same level (the level of O is easiest).
2. **Newton's second law along the radius** at that point: the forces towards O minus the forces away from O equal $\frac{mv^2}{a}$. Weight contributes $mg \cos(\text{angle with the vertical})$, towards O above the level of O and away from it below.
3. **Substitute** v^2 from step 1 into step 2 and solve for T (or R).

For example, a particle of mass 0.5 kg on a string of length 0.5 m passes the lowest point at 6 m s^{-1} . At the lowest point $T = 0.5 \times 10 + \frac{0.5 \times 36}{0.5} = 41 \text{ N}$. At the top $v^2 = 36 - 2 \times 10 \times 1 = 16$, so $T = \frac{0.5 \times 16}{0.5} - 5 = 11 \text{ N}$. The difference is $30 \text{ N} = 6mg$, as it should be.

Variations you will meet:

- **Ratio of tensions or reactions**, such as "the greatest tension is 4 times the least". The greatest is at the lowest point and the least at the top. Write both in terms of u (or the unknown angle), set up the ratio **the right way round** and solve. With $T_{\text{low}} = \frac{mu^2}{a} + mg$ and $T_{\text{top}} = \frac{mu^2}{a} - 5mg$, a ratio of 4 gives $\frac{u^2}{ga} + 1 = 4\left(\frac{u^2}{ga} - 5\right)$, so $u^2 = 7ga$.
- **The speed is given somewhere in the middle** (at an angle θ), and the unknown is $\cos \theta$. Write two energy equations (that point to the lowest point, and to the highest point), then the two tensions, then the ratio. Every expression keeps $\cos \theta$ in it until the last line.
- **Two different points**, such as "the reaction at B is three times the reaction at A ": energy between A and B , Newton's law at each, then the given link.
- **Greatest reaction** on the inside of a surface: at the lowest point.
- **Show a general result**, such as "the tension at the highest point is $6mg$ less than at the lowest point": write Newton's law at both points and the energy equation between them, then eliminate the speeds. The same steps link the tension at any two points.
- **A string that catches on a peg** below O : the speed does not change at that instant, but the radius shrinks to the distance from the peg, so the tension jumps. Use the same v with the old radius (just before) and the new radius (just after).
- **A light rod with a particle at each end**, pivoted at a point on the rod: both particles turn with the same ω , so their speeds are in the ratio of their distances from the pivot. Include **both** particles in the energy equation.
- **A bead on a wire** or a particle on a rod: the force can point either way, so if your answer comes out negative, it simply points outwards.

2. Vertical circle: string slack, leaving the surface, complete circles (16 questions)

1. **Energy equation** from the known point to the point where the string goes slack (angle α from the upward vertical).

2. **Newton's law there with $T = 0$ (or $R = 0$):** $mg \cos \alpha = \frac{mv^2}{a}$, so $v^2 = ga \cos \alpha$.

3. **Eliminate v^2** and solve for $\cos \alpha$ (or for the starting speed if α is given).

For example, for a string to go slack where $\cos \alpha = \frac{1}{4}$: $v^2 = \frac{ga}{4}$ there, and $u^2 = v^2 + 2ga \left(1 + \frac{1}{4}\right) = 2.75ga$ at the lowest point.

Variations you will meet:

- **"Show that the particle completes circles if $u^2 \geq \dots$ ":** at the top, $T \geq 0$ (or $R \geq 0$), so $v^2 \geq ga$ there; then use energy. Keep the **inequality** from the first line, or say clearly why an equality at the limit gives the least value; the mark scheme wants the inequality to appear convincingly, not to jump in at the end.
- **A bead on a wire or a particle on a rod:** the condition is $v > 0$ at the top, not $T \geq 0$.
- **Leaving the outside of a sphere or cylinder:** towards O , $mg \cos \alpha - R = \frac{mv^2}{a}$; put $R = 0$. Show-that forms such as $\cos \alpha = \frac{u^2 + 2ga}{3ga}$ come from this and the energy equation.
- **Inside a cylinder or sphere with the start not at the bottom**, for example projected downwards from a point level with O : measure heights from the level of O (P is $a \cos \alpha$ above O), so $\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mga \cos \alpha$.
- **Given where it goes slack, find the start:** same equations, solve for u or the starting angle instead.

3. Horizontal circle: string or smooth surface (11 questions)

1. **Find the radius** of the circle from the geometry: $r = l \sin \theta$ for a string of length l at θ to the vertical; $r = a \sin \theta$ in a bowl of radius a ; $r = h \tan \beta$ at height h in a cone.
2. **Resolve vertically:** the forces balance.
3. **Resolve horizontally, towards the centre:** resultant $= mr\omega^2$ or $\frac{mv^2}{r}$.
4. **Combine** to find what is asked (ω , v , T , the angle, a depth).

Variations you will meet:

- **Two strings** from the particle to two fixed points, one above the other. The tensions are **different**, so call them T_1 and T_2 ; using one T for both scores nothing. The upper string's tension has an upward part and the lower string's a downward part: $T_1 \cos \theta_1 - T_2 \cos \theta_2 = mg$, and both pull inwards: $T_1 \sin \theta_1 + T_2 \sin \theta_2 = mr\omega^2$.
- **Two particles on a chain of strings** (O to P to Q , both circling with the same ω): resolve vertically for the **whole system** to get the top tension quickly, then do each particle horizontally with its own radius.
- **Speeding up**, such as "find the ratio $\frac{\omega_2}{\omega_1}$ ": write the equations for each case; many things cancel.
- **Revolutions per minute:** $\text{rpm} = \frac{60\omega}{2\pi}$. Don't forget the 60.
- **A string held on a smooth table** (particle circling a fixed point on the table): the tension alone gives the central force, $T = mr\omega^2$, and the table's reaction balances the weight.

4. Horizontal circle with an elastic string (9 questions)

1. **Hooke's law:** $T = \frac{\lambda x}{l}$, with x the extension.

- The radius includes the extension:** on a table with the string fixed at the centre, $r = l + x$. For an elastic conical pendulum, $r = (l + x) \sin \theta$.
- Newton's law towards the centre:** $T = m(l + x)\omega^2$ or $\frac{mv^2}{l+x}$ (table), or resolve both ways (conical).
- Equate and solve.** With a speed given this is usually a **quadratic** in x ; keep the positive root only.

For example, $\lambda = 20 \text{ N}$, $l = 0.5 \text{ m}$, $m = 0.25 \text{ kg}$, $\omega = 6 \text{ rad s}^{-1}$ on a smooth table: $40x = 0.25 \times 36 \times (0.5 + x) = 4.5 + 9x$, so $x = \frac{4.5}{31} \approx 0.145 \text{ m}$.

Variations you will meet:

- Two situations with the same string** (two speeds or two masses): write the equation for each; the two equations give two unknowns such as λ and l . Checking that both give the same λ is a good test.
- A hanging particle below a hole** gives the tension (type 6), which then gives the extension from Hooke's law.
- Elastic conical pendulum:** $T \cos \theta = mg$ and $T = m(l + x)\omega^2$ (the same cancelling as for a string), with $T = \frac{\lambda x}{l}$.

5. Projectile after leaving the circle (9 questions)

- Find where it leaves** ($\cos \alpha$) and its speed v there (types 1 and 2).
- Split the velocity:** the velocity is along the tangent, at angle α to the horizontal: $v \cos \alpha$ across and $v \sin \alpha$ up or down.
- Use suvat or the trajectory** with g downwards and no horizontal acceleration.

Variations you will meet:

- Greatest height** after a string goes slack: the particle is moving up at $v \sin \alpha$, so it rises a further $\frac{(v \sin \alpha)^2}{2g}$ above the slack point, which is $a \cos \alpha$ above O .
- Landing speed:** use energy from the start to the ground (the speed doesn't care about the path); the horizontal part stays $v \cos \alpha$, so the vertical part at landing is $\sqrt{V^2 - (v \cos \alpha)^2}$, and the angle with the ground follows from $\tan$ of vertical over horizontal.
- Time to the ground:** from the vertical motion alone, $s = ut + \frac{1}{2}at^2$ with the leaving height and $v \sin \alpha$ downwards (or use the vertical velocity at landing).
- Crossing a gap**, such as a sphere with its top cut off: the particle leaves at one edge and must land on the other, so its horizontal range equals the chord between them. Use the range with the launch angle equal to α .

6. Connected particles: string through a ring or hole (8 questions)

- One string, one tension:** a light string through a smooth ring or hole has the same tension T all along.
- The hanging particle is in equilibrium**, so T equals its weight. Write this down explicitly: the mark scheme needs to see $T = Mg$.

3. **Use the geometry** for the circling particle: the two parts of the string add up to the total length. If the circle's centre is directly below the ring, the circling part makes angle θ with the downward vertical, and $r = (\text{its length}) \sin \theta$. If the hanging particle **is** at the centre of the circle, the circling particle is level with it, so the hanging part = (circling part) $\cos \theta$.
4. **Resolve** for the circling particle (vertically and towards the centre) and solve.

Variations you will meet:

- **The circling particle is on a table or a cone surface:** add the normal reaction R to the vertical equation, such as $R + T \cos \theta = mg$ on a table below a ring. The particle stays on the table only while $R \geq 0$.
- **Two speeds giving two angles:** the radius changes with the angle, so don't reuse the same r .
- **Both ends circling** (a string through two holes, a particle on each end): the tension is the same, so write T from one particle and use it in the other's equation.
- **On a smooth table:** the hanging particle's weight provides the whole central force, $Mg = \frac{mv^2}{r}$.

7. Speed by energy only (8 questions)

1. **Choose two points** and a level for zero potential energy.
2. **Write KE + GPE** at each and set them equal (no energy is lost). Every term has m , and the heights come from the geometry.
3. **Solve** for the speed.

Variations you will meet:

- **Greatest speed:** at the lowest point of the circle.
- **Least starting speed to just reach a point:** set the speed there to zero for a rod, bead or a point at or below the level of O . For a string above the level of O , use $T = 0$ instead (type 2).
- **Rigid rod with two particles:** write each speed as its distance from the pivot times ω , and include both particles' kinetic and potential energies.

8. Rough turntable: limiting friction (4 questions)

1. **Friction provides (some of) the central force.** On the point of slipping **outwards**, friction points **towards the centre** and equals μR .
2. **Resolve vertically** for R , including any vertical part of a string's tension: with a string pulling up at angle θ to the vertical, $R = mg - T \cos \theta$.
3. **Towards the centre:** (inward parts of tension) $+\mu R = mr\omega^2$.

On its own on a rough disc, $\mu mg = mr\omega^2$ at the limit, so the greatest angular speed is $\omega = \sqrt{\frac{\mu g}{r}}$: further out slips first.

Variations you will meet:

- **Two particles joined by a string along a radius:** the outer one pulls the inner one outwards, so for the inner particle friction minus tension equals its $mr\omega^2$, and for the outer particle friction plus tension equals its $mr\omega^2$ (each with its own radius).
- **Answer in rpm:** convert ω at the end.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

A particle of mass 0.4 kg is attached to one end of a light inextensible string of length 1.25 m. The other end is attached to a fixed point O . The particle moves in a horizontal circle below O with constant angular speed 4 rad s⁻¹.

- (a) Find the tension in the string. **[3]**
- (b) Find the angle between the string and the vertical. **[2]**
- (c) Find the speed of the particle and the number of revolutions it makes per minute. **[3]**
- (a)** Let the string make angle θ with the vertical. The radius is $r = 1.25 \sin \theta$. **B1**

Towards the centre: $T \sin \theta = 0.4 \times (1.25 \sin \theta) \times 4^2$. **M1**

Cancel $\sin \theta$: $T = 0.4 \times 1.25 \times 16 = 8 \text{ N}$. **A1**

(b) Vertically: $T \cos \theta = 0.4 \times 10$, so $\cos \theta = \frac{4}{8} = \frac{1}{2}$. **M1**

$\theta = 60^\circ$. **A1**

(c) $r = 1.25 \sin 60^\circ = 1.0825 \dots \text{ m}$, so $v = r\omega = 1.25 \times \frac{\sqrt{3}}{2} \times 4 = 2.5\sqrt{3} = 4.33 \text{ m s}^{-1}$ (3 s.f.). **B1**

Revolutions per minute = $\frac{4 \times 60}{2\pi} = \frac{120}{\pi} = 38.2$ (3 s.f.). **M1 A1**

The first diagram in "Understand it" is drawn for this pendulum.

Example 2

A particle P of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity $4mg$. The other end of the string is attached to a fixed point O on a smooth horizontal table. P moves on the table in a horizontal circle with centre O and constant speed $\sqrt{3ga}$.

(a) Show that the extension of the string is $\frac{1}{2}a$. **[4]**

(b) Find the angular speed of P in terms of g and a . **[2]**

(a) Let the extension be x . Hooke's law: $T = \frac{4mgx}{a}$. **B1**

The radius is $a + x$, so towards O : $T = \frac{m(3ga)}{a + x}$. **B1**

Equate: $\frac{4mgx}{a} = \frac{3mga}{a + x}$, so $4x(a + x) = 3a^2$, that is $4x^2 + 4ax - 3a^2 = 0$. **M1**

$(2x - a)(2x + 3a) = 0$, and $x > 0$, so $x = \frac{1}{2}a$. **A1**

(b) $r = a + \frac{1}{2}a = \frac{3}{2}a$, so $\omega = \frac{v}{r} = \frac{\sqrt{3ga}}{\frac{3}{2}a}$. **M1**

$\omega^2 = \frac{3ga}{\frac{9}{4}a^2} = \frac{4g}{3a}$, so $\omega = 2\sqrt{\frac{g}{3a}}$. **A1**

Example 3

A particle P of mass m is attached to one end of a light inextensible string of length a . The other end is attached to a fixed point O . P hangs at rest below O and is then projected horizontally with speed $\sqrt{3ga}$, so that it starts to move in a vertical circle.

(a) Find the tension in the string when OP is horizontal. **[3]**

(b) Show that the string becomes slack when OP makes an angle α with the upward vertical, where $\cos \alpha = \frac{1}{3}$. **[4]**

(c) Find the greatest height above O reached by P in its subsequent motion. **[3]**

(a) When OP is horizontal, P is a above its start. Energy: $\frac{1}{2}m(3ga) = \frac{1}{2}mv^2 + mga$, so $v^2 = ga$. **M1 A1**

The weight is at right angles to OP there, so towards O : $T = \frac{mv^2}{a} = mg$. **B1**

(b) At angle α from the upward vertical, P is $a(1 + \cos \alpha)$ above its start:

$$\frac{1}{2}m(3ga) = \frac{1}{2}mv^2 + mga(1 + \cos \alpha) \Rightarrow v^2 = ga - 2ga \cos \alpha.$$

M1 A1

Towards O : $T + mg \cos \alpha = \frac{mv^2}{a}$. The string is slack when $T = 0$, so $v^2 = ga \cos \alpha$. **B1**

$$ga \cos \alpha = ga - 2ga \cos \alpha \Rightarrow 3 \cos \alpha = 1 \Rightarrow \cos \alpha = \frac{1}{3}. \text{ A1 (the given answer, every step shown)}$$

(c) At that point $v^2 = \frac{1}{3}ga$ and the velocity is along the tangent, at angle α above the horizontal. Its vertical part is $v \sin \alpha$, with $\sin^2 \alpha = 1 - \frac{1}{9} = \frac{8}{9}$. **B1**

$$\text{Extra height} = \frac{(v \sin \alpha)^2}{2g} = \frac{\frac{1}{3}ga \times \frac{8}{9}}{2g} = \frac{4a}{27}. \text{ M1}$$

The slack point is $a \cos \alpha = \frac{1}{3}a$ above O , so the greatest height above O is $\frac{1}{3}a + \frac{4}{27}a = \frac{13}{27}a$. **A1**

Example 4

A smooth sphere of radius a and centre O is fixed to horizontal ground. A particle is projected horizontally from the highest point A of the sphere with speed $\frac{1}{2}\sqrt{ga}$. It moves on the outside of the sphere and leaves the surface at Q , where OQ makes an angle α with the upward vertical.

(a) Show that $\cos \alpha = \frac{3}{4}$. **[4]**

(b) Find the speed with which the particle hits the ground, and the angle its path then makes with the ground. **[5]**

(a) Energy from A to Q (a drop of $a(1 - \cos \alpha)$):

$$v^2 = \frac{1}{4}ga + 2ga(1 - \cos \alpha).$$

M1 A1

Towards O at Q : $mg \cos \alpha - R = \frac{mv^2}{a}$, and $R = 0$ as it leaves, so $v^2 = ga \cos \alpha$. **B1**

$$ga \cos \alpha = \frac{1}{4}ga + 2ga - 2ga \cos \alpha \Rightarrow 3 \cos \alpha = \frac{9}{4} \Rightarrow \cos \alpha = \frac{3}{4}. \text{ A1}$$

(b) A is $2a$ above the ground. Energy from A to the ground: $V^2 = \frac{1}{4}ga + 2g(2a) = \frac{17}{4}ga$, so $V = \frac{1}{2}\sqrt{17ga}$.

M1 A1

At Q , $v^2 = \frac{3}{4}ga$ and the velocity is at angle α below the horizontal. The horizontal part never changes: $v \cos \alpha$, with $(v \cos \alpha)^2 = \frac{3}{4}ga \times \frac{9}{16} = \frac{27}{64}ga$. **B1**

Vertical part at the ground: $\frac{17}{4}ga - \frac{27}{64}ga = \frac{245}{64}ga$. **M1**

$$\tan(\text{angle}) = \sqrt{\frac{245}{27}}, \text{ so the path makes } 71.6^\circ \text{ with the ground (1 d.p.)}. \text{ A1}$$

The third diagram in "Understand it" shows this sphere and path.

Common mistakes

- **Saying the particle stops when the string goes slack.** It is still moving; only the tension is zero. Examiners report this error again and again: use $T = 0$ in Newton's law, never $v = 0$ in the energy equation (unless the question is about a rod or bead that just reaches the top).

- **Using the wrong radius.** In a conical pendulum the radius is $l \sin \theta$, not l . With a ring, the radius comes from the circling **part** of the string, not the whole string. When the angle changes, the radius changes too.
- **Leaving the extension out of the radius.** For an elastic string on a table the radius is $l + x$. Using l gives a linear equation instead of the correct quadratic.
- **Mixing up $\sin$ and $\cos$ when resolving.** Mark the angle on the diagram and check: the vertical part of a string at θ to the vertical is $T \cos \theta$, the horizontal part $T \sin \theta$.
- **Using one tension for two strings.** Two separate strings have two separate tensions. Only a single string through a smooth ring or hole has the same tension throughout.
- **Using a speed from one point in Newton's law at another.** Each point has its own speed: find it with energy first.
- **Energy equations with a missing or wrong term.** Every term needs m (it's $\frac{1}{2}mv^2$ and mgh), and heights must be measured from one level. Check that each term is an energy.
- **Treating a horizontal circle as a vertical one**, or adding energy equations when the height doesn't change.
- **Forgetting the 60 in rpm.** $\frac{\omega}{2\pi}$ is revolutions per **second**.
- **Swapping masses** in connected-particle questions: write which particle each equation is for.
- **Putting a ratio the wrong way round.** "Greatest to least is 4 : 1" means $T_{\text{low}} = 4T_{\text{top}}$.

How to get the marks

- **Two equations, two marks.** Most questions give one method mark for an energy equation that includes every kinetic and potential energy term, each with the right units, and a separate mark for Newton's second law towards the centre. Write each one out in full before you substitute or simplify, even if a sign or a $\sin/\cos$ slip creeps in: those are often still allowed for the method mark.
- **Say what you set to zero.** "String slack, so $T = 0$ " or "leaves the surface, so $R = 0$ " must be seen.
- **State the tension from a hanging particle:** $T = mg$ written down explicitly, or the final mark can be lost.
- **"Show that":** the answer is given, so every step must be visible and the last line must match. For an inequality ("show that $u^2 \geq 5ga$ "), start from $T \geq 0$ or $R \geq 0$ and carry the inequality through.
- **"In terms of m , g and a ":** give an exact expression in just those letters, such as $\omega = 2\sqrt{\frac{g}{3a}}$. Leave surds unsimplified if you like, but no decimals.
- **Numerical answers** to 3 significant figures (angles to 1 decimal place) unless the question says otherwise, with $g = 10 \text{ m s}^{-2}$.
- **Follow-through:** a wrong angle from part (a) used correctly in part (b) still earns the method marks, so always carry on.
- **Only one root:** when a quadratic gives two values (such as an extension), reject the impossible one and say why.

Quick check

1. A particle moves in a circle of radius 0.6 m at 45 revolutions per minute. Find its angular speed and the size of its acceleration.
2. A particle of mass 0.2 kg moves in a horizontal circle on the smooth inner surface of a fixed hemispherical bowl of radius 0.5 m, with angular speed 5 rad s^{-1} . Find the normal reaction on the particle and the depth of the circle below the rim.
3. A small coin lies on a rough horizontal disc, 0.2 m from its centre. The coefficient of friction is 0.4. Find the greatest angular speed at which the disc can turn without the coin slipping.
4. A bead is threaded on a smooth circular wire of radius a fixed in a vertical plane. It is projected from the lowest point with speed u . Find the condition on u for the bead to go all the way round, and compare it with a particle on a string of length a .
5. A particle of mass 0.3 kg on a smooth horizontal table is attached to a light inextensible string that passes through a small smooth hole in the table. A particle of mass 0.6 kg hangs at rest on the other end. The particle on the table moves in a circle of radius 0.5 m about the hole. Find its speed and its angular speed.

Answers

1. $\omega = \frac{2\pi \times 45}{60} = \frac{3\pi}{2} \approx 4.71 \text{ rad s}^{-1}$. Acceleration $= r\omega^2 = 0.6 \times \left(\frac{3\pi}{2}\right)^2 = 1.35\pi^2 \approx 13.3 \text{ m s}^{-2}$, towards the centre.
2. With OP at θ to the downward vertical: $R \sin \theta = m(0.5 \sin \theta)\omega^2$, so $R = 0.2 \times 0.5 \times 25 = 2.5 \text{ N}$. Vertically $R \cos \theta = 0.2 \times 10 = 2$, so $\cos \theta = 0.8$. Depth below the rim $= 0.5 \times 0.8 = 0.4 \text{ m}$.
3. At the limit $\mu mg = mr\omega^2$, so $\omega^2 = \frac{0.4 \times 10}{0.2} = 20$ and $\omega = \sqrt{20} \approx 4.47 \text{ rad s}^{-1}$.
4. The wire holds the bead from both sides, so it only has to keep moving at the top: $v^2 = u^2 - 4ga > 0$, so $u > 2\sqrt{ga}$. On a string it needs $T \geq 0$ at the top, so $u \geq \sqrt{5ga}$, which is faster.
5. The hanging particle is in equilibrium, so $T = 0.6 \times 10 = 6 \text{ N}$. Towards the hole: $6 = \frac{0.3v^2}{0.5}$, so $v^2 = 10$ and $v = \sqrt{10} \approx 3.16 \text{ m s}^{-1}$. $\omega = \frac{v}{r} = \frac{\sqrt{10}}{0.5} \approx 6.32 \text{ rad s}^{-1}$.

Past questions to try

- **9231/33/O/N/25 Q2**: a particle in a bowl at two angular speeds, finding their ratio.
- **9231/34/O/N/25 Q4**: complete circles inside a sphere, then the greatest reaction.
- **9231/32/O/N/24 Q6**: a string that catches on a peg, then goes slack.
- **9231/32/M/J/24 Q5**: two particles on a rough turntable joined by a string.