

Complex numbers A2

Read online at pastopic.com/cambridge/further-mathematics-9231/notes/complex-numbers

Last checked 30 September 2026

Read first: [Proof by induction](#)

© 2026 Pastopic. Free for personal study. Please share the link, not the file.

What you need to know

By the end of this topic you should be able to:

1. **Understand de Moivre's theorem**, $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$, for any integer n , positive or negative, and see it as repeated multiplication or division: multiplying complex numbers multiplies their moduli and adds their arguments, and dividing divides the moduli and subtracts the arguments.
2. **Prove de Moivre's theorem for a positive integer n** , for example by induction.
3. **Use de Moivre's theorem**, including with rational powers, to:
 - write $\cos n\theta$, $\sin n\theta$ and $\tan n\theta$ (and their reciprocals) as expressions in powers of $\cos \theta$, $\sin \theta$ or $\tan \theta$;
 - write powers of $\sin \theta$ and $\cos \theta$, such as $\cos^3 \theta$, as sums of $\cos k\theta$ or $\sin k\theta$ terms;
 - sum series of cosines or sines with the " $C + iS$ " method;
 - find the n th roots of unity, and use them, for example to find all the n th roots of any complex number.

From A Level Mathematics you already know the modulus and argument of a complex number, the forms $r(\cos \theta + i \sin \theta)$ and $re^{i\theta}$, and conjugates; this topic builds on them. Loci in the Argand diagram are not part of this topic.

Understand it

Multiplying and dividing in modulus–argument form

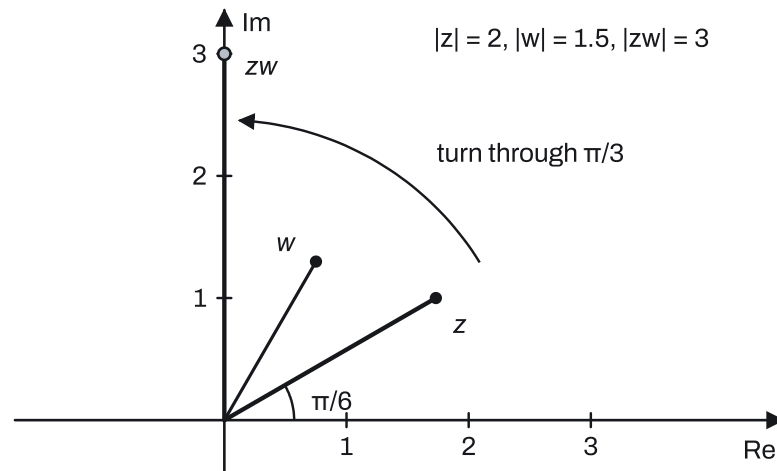
Write two complex numbers as $z = r(\cos \alpha + i \sin \alpha)$ and $w = s(\cos \beta + i \sin \beta)$. Multiplying out and using the compound-angle formulae,

$$zw = rs[(\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)] = rs(\cos(\alpha + \beta) + i \sin(\alpha + \beta)).$$

So **multiplying by w stretches by $|w|$ and turns anticlockwise through $\arg w$** about the origin. Dividing does the opposite:

$$\frac{z}{w} = \frac{r}{s}(\cos(\alpha - \beta) + i \sin(\alpha - \beta)).$$

In exponential form this is just the rules of indices: $re^{i\alpha} \times se^{i\beta} = rs e^{i(\alpha+\beta)}$.



For example, $z = 2e^{i\pi/6}$ and $w = \frac{3}{2}e^{i\pi/3}$ give $zw = 3e^{i\pi/2} = 3i$ and $\frac{z}{w} = \frac{4}{3}e^{-i\pi/6}$.

De Moivre's theorem

Multiplying $\cos \theta + i \sin \theta$ (modulus 1, argument θ) by itself n times turns through θ each time, so

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

This is **de Moivre's theorem**. In exponential form it says $(e^{i\theta})^n = e^{in\theta}$. For a general number, $(r(\cos \theta + i \sin \theta))^n = r^n(\cos n\theta + i \sin n\theta)$: raise the modulus to the power and multiply the argument.

Negative powers. Since $(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta) = \cos^2 \theta + \sin^2 \theta = 1$,

$$(\cos \theta + i \sin \theta)^{-1} = \cos \theta - i \sin \theta = \cos(-\theta) + i \sin(-\theta),$$

and raising this to the power n gives $(\cos \theta + i \sin \theta)^{-n} = \cos n\theta - i \sin n\theta$. So the theorem works for every integer. Dividing by $\cos \theta + i \sin \theta$ turns clockwise through θ .

For example, $1 + i = \sqrt{2}e^{i\pi/4}$, so $(1 + i)^8 = (\sqrt{2})^8 e^{2\pi i} = 16$ and $(1 + i)^{-4} = \frac{1}{4}e^{-i\pi} = -\frac{1}{4}$.

Rational powers give more than one answer. A number has q different q th roots, so $(\cos \theta + i \sin \theta)^{1/q}$ means any of $\cos \frac{\theta + 2k\pi}{q} + i \sin \frac{\theta + 2k\pi}{q}$ for $k = 0, 1, \dots, q - 1$. This is how roots are found below.

Proving it by induction

For positive integers n (the method is in the Proof by induction topic):

- **Base case**, $n = 1$: $(\cos \theta + i \sin \theta)^1 = \cos \theta + i \sin \theta$, which is true.
- **Assume** it is true for $n = k$: $(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta$.
- **Then for** $n = k + 1$:

$$\begin{aligned} (\cos \theta + i \sin \theta)^{k+1} &= (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta) \\ &= (\cos k\theta \cos \theta - \sin k\theta \sin \theta) + i(\sin k\theta \cos \theta + \cos k\theta \sin \theta) \\ &= \cos(k+1)\theta + i \sin(k+1)\theta. \end{aligned}$$

- **Conclude:** it is true for $n = 1$, and if it is true for $n = k$ it is true for $n = k + 1$, so by induction it is true for every positive integer n .

Multiple angles in powers of $\cos \theta$ and $\sin \theta$

Write $c = \cos \theta$ and $s = \sin \theta$. By de Moivre, $\cos 3\theta + i \sin 3\theta = (c + is)^3$. Expand the right-hand side with the binomial theorem, using $i^2 = -1$ and $i^3 = -i$:

$$(c + is)^3 = c^3 + 3c^2(is) + 3c(is)^2 + (is)^3 = (c^3 - 3cs^2) + i(3c^2s - s^3).$$

The real parts must be equal and the imaginary parts must be equal:

$$\begin{aligned}\cos 3\theta &= c^3 - 3cs^2 = c^3 - 3c(1 - c^2) = 4\cos^3 \theta - 3\cos \theta, \\ \sin 3\theta &= 3c^2s - s^3 = 3(1 - s^2)s - s^3 = 3\sin \theta - 4\sin^3 \theta.\end{aligned}$$

Dividing the two, then dividing top and bottom by c^3 , gives $\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$.

The signs of the terms in $(c + is)^n$ follow the powers of i : $1, i, -1, -i, 1, \dots$. So the real part is the terms with even powers of s , with signs $+, -, +, \dots$, and the imaginary part is the terms with odd powers of s , with signs $+, -, +, \dots$.

Powers of $\cos \theta$ and $\sin \theta$ in multiple angles

This runs the other way. Let $z = \cos \theta + i \sin \theta$. Then $z^{-1} = \cos \theta - i \sin \theta$, so

$$z + \frac{1}{z} = 2 \cos \theta, \quad z - \frac{1}{z} = 2i \sin \theta,$$

and in the same way, from de Moivre,

$$z^n + \frac{1}{z^n} = 2 \cos n\theta, \quad z^n - \frac{1}{z^n} = 2i \sin n\theta.$$

To write $\cos^3 \theta$ in multiple angles, expand $(z + \frac{1}{z})^3$ and pair each power with its reciprocal:

$$(2 \cos \theta)^3 = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} = \left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right) = 2 \cos 3\theta + 6 \cos \theta,$$

so $\cos^3 \theta = \frac{1}{4}(\cos 3\theta + 3 \cos \theta)$. In the same way $(2i \sin \theta)^3 = -8i \sin^3 \theta$ and $(z - \frac{1}{z})^3 = (z^3 - \frac{1}{z^3}) - 3(z - \frac{1}{z}) = 2i \sin 3\theta - 6i \sin \theta$, so $\sin^3 \theta = \frac{1}{4}(3 \sin \theta - \sin 3\theta)$.

An odd power of $\sin \theta$ gives sines; an even power of $\sin \theta$, or any power of $\cos \theta$, gives cosines. The point of this is integration: $\cos 3\theta$ is easy to integrate, $\cos^3 \theta$ is not.

The n th roots of unity

The equation $z^n = 1$ has exactly n roots. Write $1 = e^{2k\pi i}$ for any integer k ; then $z = e^{2k\pi i/n}$, and $k = 0, 1, \dots, n-1$ give n different values (after that they repeat). These are the **n th roots of unity**:

$$1, \omega, \omega^2, \dots, \omega^{n-1}, \quad \text{where } \omega = e^{2\pi i/n}.$$

They all have modulus 1, and their arguments are spaced $\frac{2\pi}{n}$ apart, so on an Argand diagram they are the vertices of a regular n -sided polygon inside the unit circle, with one vertex at 1. For $n = 6$ they are 1, $e^{\pm i\pi/3}$, $e^{\pm 2i\pi/3}$ and -1 .

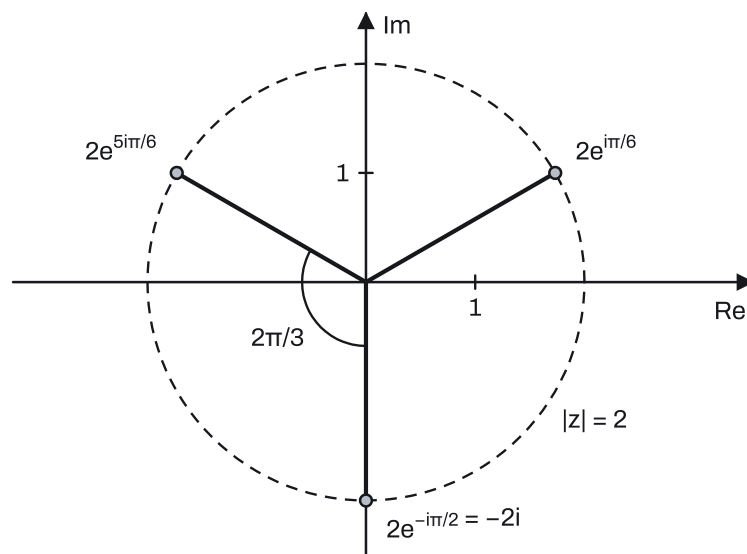
Two facts are used often:

- **They add up to 0.** For $\omega \neq 1$, $1 + \omega + \dots + \omega^{n-1} = \frac{\omega^n - 1}{\omega - 1} = 0$, because $\omega^n = 1$.
- **Non-real roots come in conjugate pairs**, $e^{\pm 2k\pi i/n}$, and each pair multiplies out to a real quadratic factor: $z^2 - 2 \cos \frac{2k\pi}{n} z + 1$.

Roots of any complex number. To solve $z^n = w$, write $w = Re^{i\alpha}$. One root is $R^{1/n}e^{i\alpha/n}$, and multiplying it by each n th root of unity gives all of them:

$$z = R^{1/n} e^{i(\alpha+2k\pi)/n}, \quad k = 0, 1, \dots, n-1.$$

They all lie on a circle of radius $R^{1/n}$, spaced $\frac{2\pi}{n}$ apart.



For example, $8i = 8e^{i\pi/2}$, so the cube roots of $8i$ are $2e^{i(\pi/2+2k\pi)/3}$: $2e^{i\pi/6}$, $2e^{5i\pi/6}$ and $2e^{3i\pi/2} = 2e^{-i\pi/2} = -2i$.

Summing series: the " $C + iS$ " method

To add up a series of cosines, C , pair it with the matching series of sines, S . Then $C + iS$ is a sum of powers of $e^{i\theta}$, usually a geometric series or a binomial expansion, which you can sum; C is the real part of the answer and S is the imaginary part.

For example, take $C = \sum_{r=0}^n \binom{n}{r} \cos r\theta$ and $S = \sum_{r=0}^n \binom{n}{r} \sin r\theta$. With $z = e^{i\theta}$,

$$C + iS = \sum_{r=0}^n \binom{n}{r} z^r = (1 + z)^n.$$

To take the real part, take out a half angle: $1 + e^{i\theta} = e^{i\theta/2} (e^{-i\theta/2} + e^{i\theta/2}) = 2 \cos \frac{\theta}{2} e^{i\theta/2}$. So

$$C + iS = 2^n \cos^n \frac{\theta}{2} e^{in\theta/2}, \quad C = 2^n \cos^n \frac{\theta}{2} \cos \frac{n\theta}{2}, \quad S = 2^n \cos^n \frac{\theta}{2} \sin \frac{n\theta}{2}.$$

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it. MF19 has no complex-number formulae at all.

Formula	Status
$z_1 z_2$: multiply the moduli, add the arguments; $\frac{z_1}{z_2}$: divide the moduli, subtract the arguments	Learn it
De Moivre: $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$, for any integer n	Learn it
$e^{i\theta} = \cos \theta + i \sin \theta$, so $(re^{i\theta})^n = r^n e^{in\theta}$	Learn it
$(\cos \theta + i \sin \theta)^{-n} = \cos n\theta - i \sin n\theta$	Learn it
With $z = \cos \theta + i \sin \theta$: $z^n + z^{-n} = 2 \cos n\theta$, $z^n - z^{-n} = 2i \sin n\theta$	Learn it
n th roots of unity: $e^{2k\pi i/n}$, $k = 0, 1, \dots, n-1$; they add up to 0	Learn it
Roots of $z^n = Re^{i\alpha}$: $R^{1/n} e^{i(\alpha+2k\pi)/n}$, $k = 0, 1, \dots, n-1$	Learn it
Binomial series: $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + b^n$	Given
Geometric series: $S_n = \frac{a(1-r^n)}{1-r}$, $S_\infty = \frac{a}{1-r}$ for $ r < 1$ (true for complex r too)	Given
$\cos^2 \theta + \sin^2 \theta \equiv 1$, $1 + \tan^2 \theta \equiv \sec^2 \theta$, $\cot^2 \theta + 1 \equiv \operatorname{cosec}^2 \theta$	Given
$\sin(A \pm B)$, $\cos(A \pm B)$, $\sin 2A$, $\cos 2A$	Given
$1 + e^{i\theta} = 2 \cos \frac{\theta}{2} e^{i\theta/2}$ and $e^{i\theta} - 1 = 2i \sin \frac{\theta}{2} e^{i\theta/2}$ (half-angle trick)	Learn it

Results you can rebuild quickly, and use to check your working: $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$, $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$, $\cos^3 \theta = \frac{1}{4}(\cos 3\theta + 3 \cos \theta)$, $\sin^3 \theta = \frac{1}{4}(3 \sin \theta - \sin 3\theta)$. In the exam you must still derive any result the question asks you to show.

How to do it

In the Paper 2 questions we have tagged for this topic (39 questions, 2021–2025; 12 of them appear in two papers of the same session, so there are 27 different questions), the types came up this often. 4 of the 39 questions mix two types, so the counts add up to more than 39. Several also have a part on integration (integrating the result, a reduction formula or rectangles); that is taught in the Integration topic, and only the complex-number parts are counted here.

Question type	Questions
Roots of complex numbers and roots of unity	13
Multiple angles in powers, then roots of a polynomial	11

Question type	Questions
Powers of $\sin \theta$ and $\cos \theta$ in multiple angles	10
Summing series with $C + iS$	9

1. Roots of complex numbers and roots of unity (13 questions)

1. **Write the right-hand side in exponential (or modulus–argument) form**, $Re^{i\alpha}$. Find R with Pythagoras, and α from a quick sketch so it is in the right quadrant.
2. **Take the n th root of the modulus** and divide the argument by n : $z = R^{1/n}e^{i\alpha/n}$ is one root.
3. **Add $\frac{2\pi}{n}$ to the argument** again and again to get the other $n - 1$ roots.
4. **Bring every argument into the range asked for** (such as $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$) by adding or subtracting 2π , and write each root in the form asked for, $re^{i\theta}$ or $r(\cos \theta + i \sin \theta)$, with $r > 0$.

For example, $z^4 = -81$: $-81 = 81e^{i\pi}$, so $z = 3e^{i(\pi+2k\pi)/4}$. In $-\pi < \theta \leq \pi$ the roots are $3e^{i\pi/4}$, $3e^{3i\pi/4}$, $3e^{-3i\pi/4}$ and $3e^{-i\pi/4}$.

Variations you will meet:

- **"Write down the n th roots of unity"**: $e^{2k\pi i/n}$ for $k = 0, \dots, n - 1$, all n of them. For fourth roots they are $\pm 1, \pm i$.
- **A shifted equation**, such as $(z - 2)^3 = w$: find the cube roots u of w as usual, then $z = u + 2$. If the answer form is something like $r \cos \theta + i(r \sin \theta + 1)$, that is a root of w with i added: write it in that shape rather than combining it.
- **A polynomial that factorises** into two such equations, such as $z^6 - 3z^4 + 4z^2 - 12 = (z^4 + 4)(z^2 - 3)$: expand the brackets to check the factorising, then solve each factor and list all the roots together (here $\pm\sqrt{3}$ and $\sqrt{2}e^{\pm i\pi/4}$, $\sqrt{2}e^{\pm 3i\pi/4}$).
- **The sum of the roots of unity is 0**. "Given $z^n = 1$ and $z \neq 1$, show $1 + z + \dots + z^{n-1} = 0$ ": use the geometric series sum and $z^n = 1$.
- **Real roots of a polynomial** built from roots of unity, such as $x^n - 1$ times something: the only real n th roots of unity are 1, and -1 when n is even.

2. Multiple angles in powers, then roots of a polynomial (11 questions)

Showing the identity:

1. **Write** $\cos n\theta + i \sin n\theta = (c + is)^n$, where $c = \cos \theta$, $s = \sin \theta$, and say you are using de Moivre's theorem.
2. **Expand** with the binomial theorem. Keep only the part you need: the **real** part for $\cos n\theta$, the **imaginary** part for $\sin n\theta$, both for $\tan n\theta$ or $\cot n\theta$.
3. **Replace** s^2 by $1 - c^2$ (for an answer in $\cos \theta$) or c^2 by $1 - s^2$ (for $\sin \theta$). Expand each bracket fully on its own line, then collect terms.

4. **For $\tan n\theta$** divide the imaginary part by the real part, then divide the top and bottom by c^n so each term becomes a power of $\tan \theta$. For $\cot n\theta$ divide by s^n instead.

For example, $\cot 3\theta = \frac{c^3 - 3cs^2}{3c^2s - s^3}$; dividing top and bottom by s^3 gives $\cot 3\theta = \frac{\cot^3 \theta - 3 \cot \theta}{3 \cot^2 \theta - 1}$.

Reciprocal functions. For $\sec n\theta$ or $\operatorname{cosec} n\theta$, first get $\cos n\theta$ or $\sin n\theta$ as a polynomial, take the reciprocal, then divide the top and bottom by the highest power so everything becomes $\sec \theta$ or $\operatorname{cosec} \theta$. For example $\sin 3\theta = 3s - 4s^3$, so

$$\operatorname{cosec} 3\theta = \frac{1}{3s - 4s^3} = \frac{1/s^3}{3/s^2 - 4} = \frac{\operatorname{cosec}^3 \theta}{3 \operatorname{cosec}^2 \theta - 4}.$$

"Hence" solving a polynomial:

1. **Put $x = \cos \theta$** (or $\sin \theta$, $\tan \theta$, $\sec \theta$, ..., whichever matches part (a)) and rearrange the equation until one side is exactly your identity. State the link, for example "with $x = \sec \theta$, the equation is $\sec 3\theta = 2$ ".
2. **Solve the simple equation for $n\theta$:** write the general solution, such as $n\theta = \pm\alpha + 2k\pi$.
3. **Pick θ values that give different values of x .** A polynomial of degree n has at most n roots, so you need that many distinct values. $\cos \theta$ takes each value once for $0 \leq \theta \leq \pi$; $\sin \theta$ and $\tan \theta$ once for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
4. **Give the roots in the form asked**, such as $\cos(q\pi)$, and check none is repeated.

For example, with $\cos 3\theta = 4c^3 - 3c$, the equation $8x^3 - 6x - 1 = 0$ is $2 \cos 3\theta = 1$, so $\cos 3\theta = \frac{1}{2}$, $3\theta = \pm\frac{\pi}{3} + 2k\pi$, and the three roots are $\cos \frac{\pi}{9}$, $\cos \frac{5\pi}{9}$ and $\cos \frac{7\pi}{9}$.

Variations you will meet:

- **An equation in $\sin n\theta / \sin \theta$** (the polynomial has only even powers after dividing by $\sin \theta$): solve $\sin n\theta = 0$ but leave out $\sin \theta = 0$, because you divided by it.
- **A squared identity**, such as $(\text{polynomial})^2 = \frac{1}{4}$: that gives $\cos n\theta = \pm\frac{1}{2}$. Remember **both signs**; it doubles the number of roots.
- **A polynomial that factorises** into a root-of-unity factor times the identity: solve each factor and combine the distinct real roots.
- **Roots in the form $\sec(q\pi)$ or $\cot(q\pi)$** : turn the equation into $\cos n\theta = \dots$ or $\tan n\theta = \dots$ first, solve, then write each root back as $\sec$ or $\cot$ of the angle.

3. Powers of $\sin \theta$ and $\cos \theta$ in multiple angles (10 questions)

1. **Start from $z + \frac{1}{z} = 2 \cos \theta$ or $z - \frac{1}{z} = 2i \sin \theta$** , with $z = \cos \theta + i \sin \theta$. Write this line down.
2. **Expand $(z \pm \frac{1}{z})^n$** with the binomial theorem.
3. **Group** each z^k with its z^{-k} , showing the brackets clearly: $(z^k + z^{-k})$ or $(z^k - z^{-k})$.
4. **Replace** each bracket with $2 \cos k\theta$ or $2i \sin k\theta$, then divide both sides to leave the power on its own.

Variations you will meet:

- **A product**, such as $\sin^2 \theta \cos^4 \theta$: multiply $(z - \frac{1}{z})^2 (z + \frac{1}{z})^4$. It is quicker to write it as $(z^2 - z^{-2})^2 (z + \frac{1}{z})^2$ first, since $(z - \frac{1}{z})(z + \frac{1}{z}) = z^2 - z^{-2}$.
- **An odd power of $\sin \theta$** : the left side has i^n in it, such as $(2i \sin \theta)^5 = 32i \sin^5 \theta$, and every bracket on the right is $2i \sin k\theta$, so the i cancels. Show that it does.
- **A reciprocal**, such as " $\operatorname{cosec}^3 \theta = \frac{a}{\dots}$ ": find $\sin^3 \theta$ in multiple angles, then take the reciprocal.
- **$\tan^n \theta$ or $\cot^n \theta$** : expand both $(z - \frac{1}{z})^n$ and $(z + \frac{1}{z})^n$ and divide. For example $\tan^3 \theta = \frac{\sin^3 \theta}{\cos^3 \theta} = \frac{3 \sin \theta - \sin 3\theta}{\cos 3\theta + 3 \cos \theta}$.
- **Then integrate** or use it in a reduction formula: that part belongs to the Integration topic, but you need the identity exactly right first.

4. Summing series with $C + iS$ (9 questions)

1. **Pair the series** you want with its partner: if you want $\sum \cos r\theta$, write $C + iS = \sum (\cos r\theta + i \sin r\theta) = \sum z^r$ with $z = e^{i\theta}$. A coefficient such as k^r goes into z : $z = ke^{i\theta}$.
2. **Sum it**: usually a geometric series (first term, common ratio z , number of terms), sometimes a binomial expansion.
3. **Make the denominator real**. Either multiply the top and bottom by the conjugate of the denominator, or take out a half angle, such as $e^{i\theta} - 1 = e^{i\theta/2} (e^{i\theta/2} - e^{-i\theta/2}) = 2i \sin \frac{\theta}{2} e^{i\theta/2}$. Useful: $(ke^{i\theta} - 1)(ke^{-i\theta} - 1) = k^2 - 2k \cos \theta + 1$.
4. **Take the real part** for C or the **imaginary part** for S , and simplify to the given form with the compound-angle formulae.

Variations you will meet:

- **An infinite series** with $|z| < 1$, such as $\sum k^r \cos r\theta$ with $0 < k < 1$: use $S_\infty = \frac{a}{1 - z}$. Keep only z in the working, no n .
- **A ratio that is not $e^{i\theta}$** , such as $\sum (1 + i \tan \theta)^r$: first write the ratio in modulus-argument form, $1 + i \tan \theta = \sec \theta (\cos \theta + i \sin \theta)$, so its r th power is $\sec^r \theta (\cos r\theta + i \sin r\theta)$.
- **Multiples in the angle**, such as $\sum \sin 3r\theta$: the ratio is z^3 .
- **A sum with r in front**, such as $\sum r k^r \cos r\theta$: differentiate the known result for $\sum k^r \sin r\theta$ with respect to θ , since $\frac{d}{d\theta} \sin r\theta = r \cos r\theta$. Use the quotient rule; you need not simplify unless asked. For the infinite version, let $n \rightarrow \infty$ in that result, using a given limit such as $\frac{n}{2^n} \rightarrow 0$.
- **Special values**: substitute a given θ (and a multiple for n) into your result, and simplify exactly.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

Find the roots of the equation $z^5 = -16\sqrt{2} + 16\sqrt{2}i$, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. **[5]**

Modulus: $\sqrt{(16\sqrt{2})^2 + (16\sqrt{2})^2} = \sqrt{512 + 512} = 32$. The point $(-16\sqrt{2}, 16\sqrt{2})$ is in the second quadrant with $\tan^{-1} 1 = \frac{\pi}{4}$, so the argument is $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$. **B1** for $32e^{3\pi i/4}$.

$$z = 32^{1/5} e^{i(3\pi/4 + 2k\pi)/5} = 2e^{i(3\pi/20 + 2k\pi/5)}.$$

$k = 0$ gives $z = 2e^{3\pi i/20}$. **M1** for the fifth root of the modulus and dividing the argument by 5, **A1**.

The others add $\frac{2\pi}{5} = \frac{8\pi}{20}$ each time: $\frac{11\pi}{20}$, $\frac{19\pi}{20}$, $\frac{27\pi}{20}$ and $\frac{35\pi}{20}$. The last two are more than π , so subtract 2π : $-\frac{13\pi}{20}$ and $-\frac{5\pi}{20} = -\frac{\pi}{4}$.

$$z = 2e^{3\pi i/20}, \quad 2e^{11\pi i/20}, \quad 2e^{19\pi i/20}, \quad 2e^{-13\pi i/20}, \quad 2e^{-\pi i/4}.$$

A1 for two more, **A1** for the last two, all with arguments in range.

Example 2

(a) Use de Moivre's theorem to show that $\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$. **[4]**

(b) Hence obtain the roots of the equation $64x^6 - 96x^4 + 36x^2 - 3 = 0$ in the form $\cos(q\pi)$, where q is a rational number. **[4]**

(a) With $c = \cos \theta$, $s = \sin \theta$, de Moivre gives $\cos 6\theta = \operatorname{Re}(c + is)^6$:

$$\cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6.$$

M1 A1 for expanding and taking the real part.

Now use $s^2 = 1 - c^2$ and expand each bracket:

$$\begin{aligned} \cos 6\theta &= c^6 - 15c^4(1 - c^2) + 15c^2(1 - 2c^2 + c^4) - (1 - 3c^2 + 3c^4 - c^6) \\ &= c^6 - 15c^4 + 15c^6 + 15c^2 - 30c^4 + 15c^6 - 1 + 3c^2 - 3c^4 + c^6 \\ &= 32c^6 - 48c^4 + 18c^2 - 1. \end{aligned}$$

M1 for $s^2 = 1 - c^2$, **A1** for the given answer with the expanded line shown.

(b) Let $x = \cos \theta$. Then $64x^6 - 96x^4 + 36x^2 - 2 = 2 \cos 6\theta$, so the equation is $2 \cos 6\theta - 1 = 0$, that is $\cos 6\theta = \frac{1}{2}$. **B1**

$6\theta = \pm \frac{\pi}{3} + 2k\pi$, so $\theta = \pm \frac{\pi}{18} + \frac{k\pi}{3}$. **M1**

In $0 \leq \theta \leq \pi$: $\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$, and these give six different values of $\cos \theta$. **A1** for one correct root.

$$x = \cos\left(\frac{1}{18}\pi\right), \cos\left(\frac{5}{18}\pi\right), \cos\left(\frac{7}{18}\pi\right), \cos\left(\frac{11}{18}\pi\right), \cos\left(\frac{13}{18}\pi\right), \cos\left(\frac{17}{18}\pi\right).$$

A1 for the other five, with no repeats. (As decimals: $\pm 0.985, \pm 0.643, \pm 0.342$; the equation has only even powers, so its roots come in $\pm$ pairs.)

Example 3

(a) By considering the binomial expansions of $\left(z + \frac{1}{z}\right)^2$ and $\left(z^2 - \frac{1}{z^2}\right)^2$, where $z = \cos \theta + i \sin \theta$, use de Moivre's theorem to show that

$$\sin^2 \theta \cos^4 \theta = \frac{1}{32} (2 + \cos 2\theta - 2 \cos 4\theta - \cos 6\theta). \quad [5]$$

(b) Hence find the exact value of $\int_0^{\pi/2} \sin^2 \theta \cos^4 \theta d\theta$. **[3]**

(a) $z + \frac{1}{z} = 2 \cos \theta$ and $z^2 - \frac{1}{z^2} = 2i \sin 2\theta = 4i \sin \theta \cos \theta$. **B1** for these.

So $\left(z + \frac{1}{z}\right)^2 \left(z^2 - \frac{1}{z^2}\right)^2 = (2 \cos \theta)^2 (4i \sin \theta \cos \theta)^2 = -64 \sin^2 \theta \cos^4 \theta$.

Expanding:

$$\begin{aligned} (z^2 + 2 + z^{-2}) (z^4 - 2 + z^{-4}) &= z^6 + 2z^4 + z^2 - 2z^2 - 4 - 2z^{-2} + z^{-2} + 2z^{-4} + z^{-6} \\ &= (z^6 + z^{-6}) + 2(z^4 + z^{-4}) - (z^2 + z^{-2}) - 4. \end{aligned}$$

M1 A1 for expanding and grouping.

$$-64 \sin^2 \theta \cos^4 \theta = 2 \cos 6\theta + 4 \cos 4\theta - 2 \cos 2\theta - 4.$$

M1 for $z^n + z^{-n} = 2 \cos n\theta$. Dividing by -64 :

$$\sin^2 \theta \cos^4 \theta = \frac{1}{32} (2 + \cos 2\theta - 2 \cos 4\theta - \cos 6\theta).$$

A1

(b)

$$\int_0^{\pi/2} \sin^2 \theta \cos^4 \theta d\theta = \frac{1}{32} \left[2\theta + \frac{1}{2} \sin 2\theta - \frac{1}{2} \sin 4\theta - \frac{1}{6} \sin 6\theta \right]_0^{\pi/2}.$$

M1 for using (a), **A1** for the integral. Every sine is 0 at both limits ($\sin \pi = \sin 2\pi = \sin 3\pi = 0$), so the value is $\frac{1}{32} \times \pi = \frac{\pi}{32}$. **A1**

Example 4

(a) Show that $1 - e^{2i\theta} = -2i \sin \theta e^{i\theta}$. [2]

(b) By considering $(1 - e^{2i\theta})^5$, use de Moivre's theorem to show that

$$\sum_{r=0}^5 (-1)^r \binom{5}{r} \cos 2r\theta = 32 \sin^5 \theta \sin 5\theta. \quad [4]$$

(c) Find the exact value of $\sum_{r=0}^5 (-1)^r \binom{5}{r} \sin 2r\theta$ when $\theta = \frac{\pi}{6}$. [2]

(a) Take out $e^{i\theta}$: $1 - e^{2i\theta} = e^{i\theta} (e^{-i\theta} - e^{i\theta})$. **M1** Since $e^{i\theta} - e^{-i\theta} = 2i \sin \theta$, this is $-2i \sin \theta e^{i\theta}$. **A1**

(b) By the binomial theorem and de Moivre, $(e^{2i\theta})^r = \cos 2r\theta + i \sin 2r\theta$, so

$$(1 - e^{2i\theta})^5 = \sum_{r=0}^5 \binom{5}{r} (-e^{2i\theta})^r = \sum_{r=0}^5 (-1)^r \binom{5}{r} \cos 2r\theta + i \sum_{r=0}^5 (-1)^r \binom{5}{r} \sin 2r\theta.$$

M1 for writing the series as $C + iS$ from the binomial expansion.

From (a), $(1 - e^{2i\theta})^5 = (-2i)^5 \sin^5 \theta e^{5i\theta}$, and $(-2i)^5 = -32i^5 = -32i$. **M1** So

$$(1 - e^{2i\theta})^5 = -32i \sin^5 \theta (\cos 5\theta + i \sin 5\theta) = 32 \sin^5 \theta \sin 5\theta - 32i \sin^5 \theta \cos 5\theta.$$

A1. Taking real parts, $\sum_{r=0}^5 (-1)^r \binom{5}{r} \cos 2r\theta = 32 \sin^5 \theta \sin 5\theta$. **A1**

(c) The imaginary parts give $\sum_{r=0}^5 (-1)^r \binom{5}{r} \sin 2r\theta = -32 \sin^5 \theta \cos 5\theta$. **M1** At $\theta = \frac{\pi}{6}$: $-32 \times \left(\frac{1}{2}\right)^5 \times \cos \frac{5\pi}{6} = -1 \times \left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$. **A1**

Common mistakes

- **Not grouping before substituting.** After expanding $(z \pm \frac{1}{z})^n$, examiners want to see the terms paired into brackets like $(z^4 + z^{-4})$ before you write $2 \cos 4\theta$. Jumping straight to the answer loses marks on a "show that".
- **Dropping a 2 or an i .** $z^n - z^{-n} = 2i \sin n\theta$, not $i \sin n\theta$ or $2 \sin n\theta$. For a power of $\sin \theta$, show the i on the left cancelling the i on the right.
- **Using only one of the two identities.** In a $\tan^n \theta$ or $\cot^n \theta$ question you need both $z + \frac{1}{z} = 2 \cos \theta$ and $z - \frac{1}{z} = 2i \sin \theta$; write both down so the examiner sees them.
- **Getting stuck turning $\sin$ and $\cos$ into $\sec$, cosec or $\cot$.** Get the polynomial in one function first, take the reciprocal, then divide the top and bottom by the right power. $\cot^2 \theta = \operatorname{cosec}^2 \theta - 1$ and $\tan^2 \theta = \sec^2 \theta - 1$ help.
- **Skipping the expansion of $(1 - c^2)^2$ or $(1 - s^2)^3$.** On a given answer, each bracket must be expanded in full on the page, or the last mark goes.

- **Missing or repeated roots in a "hence".** Degree n means up to n distinct roots: find them all, and don't list $\cos \frac{\pi}{9}$ and $\cos(-\frac{\pi}{9})$ as two roots. Remember $\pm$ when a square root is taken, and leave out $\sin \theta = 0$ if you divided by $\sin \theta$.
- **Not linking the polynomial to part (a).** Say what x is and which simple equation, such as $\cos 5\theta = \frac{1}{2}$, the polynomial becomes.
- **Wrong argument for a root.** Start from the number in exponential form with the argument in the correct quadrant: $-1 + i$ has argument $\frac{3\pi}{4}$, not $-\frac{\pi}{4}$. Then put every root's argument in the range asked for.
- **Forgetting to root the modulus,** or writing e^θ instead of $e^{i\theta}$. Each costs a mark.
- **Series: extra letters in a sum to infinity.** For $\sum_{r=1}^{\infty} z^r$ just write $\frac{z}{1-z}$; there is no n . Simplify the numerator and denominator (multiply by the conjugate) before taking the real or imaginary part.

How to get the marks

- **"Use de Moivre's theorem"** means write $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ (or $z^n + z^{-n} = 2 \cos n\theta$) and actually use it. A route that only uses double-angle formulae scores little.
- **"Show that" answers are given,** so every step must be visible: the binomial expansion, which part you take (real or imaginary), each substitution of $s^2 = 1 - c^2$ expanded, and the final collected line exactly as printed.
- **"Hence"** means the roots must come from the identity you have just shown. Write the substitution ($x = \cos \theta$), the resulting equation in $n\theta$, its general solution, then the roots. Listing values of q without saying $x = \cos(q\pi)$ does not earn full marks.
- **Give roots in the form asked.** For $re^{i\theta}$ state $r > 0$ and keep θ in the stated interval; mark schemes often give one mark for the first root and one for the rest, so get the pattern right. Numbers such as $2^{1/3}$ can stay as they are unless told otherwise.
- **Exact answers:** leave π in arguments and surds in moduli. Don't give decimals unless asked.
- **"State" or "write down"** is one mark with no working needed, such as the sum of a geometric series or the fourth roots of unity. Get the formula exactly right, because the next parts build on it.
- **"You do not need to simplify"** means stop once the expression is correct; don't risk errors tidying it.
- **Method marks follow through.** A slip in a modulus usually still earns the marks for the other roots, so keep going.

Quick check

1. Let $w = \sqrt{2} e^{i\pi/4}$. Describe the effect on the Argand diagram of multiplying a complex number by w , and find w^{-6} in the form $a + bi$.
2. Find the cube roots of -8 , giving your answers in the form $re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leq \pi$.
3. Use de Moivre's theorem to show that $\frac{\sin 4\theta}{\sin \theta} = 8 \cos^3 \theta - 4 \cos \theta$.
4. By considering $(z - \frac{1}{z})^3 (z + \frac{1}{z})^2$, where $z = \cos \theta + i \sin \theta$, express $\sin^3 \theta \cos^2 \theta$ in the form $a \sin \theta + b \sin 3\theta + c \sin 5\theta$.

5. By considering a geometric series in $z = \cos \theta + i \sin \theta$, show that $\sum_{r=1}^n \cos(2r-1)\theta = \frac{\sin 2n\theta}{2 \sin \theta}$, where $\sin \theta \neq 0$.

Answers

- Multiplying by w enlarges by scale factor $|w| = \sqrt{2}$ about the origin and rotates anticlockwise through $\frac{\pi}{4}$.
By de Moivre, $w^{-6} = (\sqrt{2})^{-6} e^{-6i\pi/4} = \frac{1}{8} e^{-3i\pi/2} = \frac{1}{8} (\cos \frac{3\pi}{2} - i \sin \frac{3\pi}{2}) = \frac{1}{8} i$, so $a = 0$, $b = \frac{1}{8}$.
- $-8 = 8e^{i\pi}$, so $z = 2e^{i(\pi+2k\pi)/3}$: $2e^{i\pi/3}$, $2e^{i\pi}$ and $2e^{5i\pi/3} = 2e^{-i\pi/3}$.
- $\sin 4\theta = \text{Im}(c + is)^4 = 4c^3s - 4cs^3$. Dividing by s : $4c^3 - 4cs^2 = 4c^3 - 4c(1 - c^2) = 8c^3 - 4c$.
- The left side is $(2i \sin \theta)^3 (2 \cos \theta)^2 = -32i \sin^3 \theta \cos^2 \theta$. Writing it as $(z - \frac{1}{z})(z^2 - \frac{1}{z^2})^2$ and expanding:
 $z^5 - z^3 - 2z + 2z^{-1} + z^{-3} - z^{-5} = (z^5 - z^{-5}) - (z^3 - z^{-3}) - 2(z - z^{-1}) = 2i \sin 5\theta - 2i \sin 3\theta - 4i \sin \theta$. Dividing by $-32i$: $\sin^3 \theta \cos^2 \theta = \frac{1}{8} \sin \theta + \frac{1}{16} \sin 3\theta - \frac{1}{16} \sin 5\theta$.
- $\sum_{r=1}^n z^{2r-1} = \frac{z(z^{2n} - 1)}{z^2 - 1} = \frac{z^{2n} - 1}{z - z^{-1}} = \frac{\cos 2n\theta - 1 + i \sin 2n\theta}{2i \sin \theta}$. Dividing by i swaps the parts: this is $\frac{\sin 2n\theta}{2 \sin \theta} + i \frac{1 - \cos 2n\theta}{2 \sin \theta}$. The real part is $\sum \cos(2r - 1)\theta = \frac{\sin 2n\theta}{2 \sin \theta}$. (The imaginary part gives $\sum \sin(2r - 1)\theta = \frac{\sin^2 n\theta}{\sin \theta}$.)

Past questions to try

- 9231/22/O/N/24 Q4**: $\cot 6\theta$ in powers of $\cot \theta$, then the roots of a sextic as $\cot(q\pi)$.
- 9231/23/O/N/23 Q2**: cube roots of a shifted equation, $(z + 5i)^3 = \dots$, in a special form.
- 9231/23/M/J/24 Q6**: a geometric series in z^4 used to sum $\sin 4r\theta$.
- 9231/23/M/J/23 Q3**: $\cot^4 \theta$ in multiple angles, from both binomial expansions.