

Differential equations A2

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Read first: [Integration](#), [Hyperbolic functions](#), [Differentiation](#)

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What you need to know

By the end of this topic you should be able to:

1. **Solve a first order linear differential equation with an integrating factor:** write it as $\frac{dy}{dx} + P(x)y = Q(x)$, find the integrating factor $e^{\int P dx}$, and use it to find the general solution.
2. **Know what "complementary function" and "particular integral" mean** for a linear differential equation, and that the general solution is the complementary function plus a particular integral.
3. **Find the complementary function** of a first or second order linear equation with constant coefficients, from the auxiliary equation. For second order equations this covers two different real roots, a repeated real root, and a pair of complex conjugate roots.
4. **Find a particular integral** when the right-hand side is a polynomial, ae^{bx} , or $a \cos px + b \sin px$, knowing the form to try. In other simple cases you are given the form (for example kxe^{bx}) and find the coefficient.
5. **Use a given substitution** to turn a differential equation into a first or second order linear equation with constant coefficients, or into a first order equation whose variables separate. Examples are $x = e^t$, $y = vx$ and $z = x + y$.
6. **Use initial conditions to find a particular solution**, and **interpret a solution** in terms of the problem it models, such as what happens for large values of x or t .

The integrals you need along the way (standard forms, partial fractions, integration by parts, hyperbolic functions) are taught in the Integration note, and the hyperbolic identities and logarithmic forms in the Hyperbolic functions note. This note uses them.

Understand it

What a differential equation is

A **differential equation** links an unknown function y with its derivatives. Its **order** is the highest derivative in it:

$\frac{dy}{dx} + 3y = x$ is first order, $\frac{d^2y}{dx^2} - y = 0$ is second order. It is **linear** when y and its derivatives appear only to the power 1 and are never multiplied together, although their coefficients can be any functions of x .

To **solve** it means to find y as a function of x . The **general solution** contains arbitrary constants: one for a first order equation, two for a second order one, because solving it undoes one differentiation per order. Extra information, such as " $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$ " (the **initial conditions**), fixes the constants and gives a **particular solution**.

First order: the integrating factor

Every first order linear equation can be written in the **standard form**

$$\frac{dy}{dx} + P(x)y = Q(x),$$

so first **divide by the coefficient of $\frac{dy}{dx}$** . The left side is almost the derivative of a product. Multiply the whole equation by the **integrating factor**

$$I = e^{\int P dx}.$$

Since $\frac{dI}{dx} = PI$, the product rule gives $\frac{d}{dx}(Iy) = I\frac{dy}{dx} + PIy$, which is exactly the new left side. So the equation becomes

$$\frac{d}{dx}(Iy) = IQ \Rightarrow Iy = \int IQ dx.$$

For example, $\frac{dy}{dx} + \frac{2}{x}y = 10x^2$ has $I = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$. Then $\frac{d}{dx}(x^2y) = 10x^4$, so $x^2y = 2x^5 + C$ and $y = 2x^3 + \frac{C}{x^2}$. The constant goes in **when you integrate**, before you divide by I , so it is divided too.

$e^{\ln f(x)} = f(x)$ is what makes integrating factors simple. You don't need a constant in $\int P dx$ (it would only multiply I by a number), and when $f(x) > 0$ you can drop the modulus. Common ones:

$P(x)$	$\int P dx$	Integrating factor
$\frac{k}{x}$	$k \ln x$	x^k
$\frac{f'(x)}{f(x)}$ or $-\frac{f'(x)}{f(x)}$	$\pm \ln f(x)$	$f(x)$ or $\frac{1}{f(x)}$
$\tan x$	$\ln \sec x$	$\sec x$
$\tanh x$ or $\coth x$	$\ln \cosh x$ or $\ln \sinh x$	$\cosh x$ or $\sinh x$
$\frac{1}{\sqrt{x^2 + a^2}}$	$\sinh^{-1} \frac{x}{a} = \ln \frac{x + \sqrt{x^2 + a^2}}{a}$	$x + \sqrt{x^2 + a^2}$ (the $\frac{1}{a}$ is a constant, so drop it)

A multiple changes the power: $P = \frac{f'}{2f}$ gives $\int P dx = \frac{1}{2} \ln f$, so $I = \sqrt{f(x)}$.

Separating the variables

After a substitution (see the last section), a first order equation often becomes $\frac{dv}{dx} = f(x)g(v)$. Put everything with v on one side and everything with x on the other, then integrate both sides:

$$\int \frac{1}{g(v)} dv = \int f(x) dx.$$

One constant, on one side, is enough.

Constant coefficients: complementary function plus particular integral

Now look at equations such as

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x),$$

where a, b, c are numbers. The **complementary function** (CF) is the general solution of the same equation with 0 on the right. It carries the arbitrary constants. A **particular integral** (PI) is any one function that gives exactly $f(x)$ on the right, with no arbitrary constants.

Because the left side is linear, putting in $y = \text{CF} + \text{PI}$ gives $0 + f(x) = f(x)$. So

general solution = complementary function + particular integral.

Finding the complementary function

Try $y = e^{mx}$ in $ay'' + by' + cy = 0$. Every term has a factor e^{mx} , which is never zero, so it works when m solves the **auxiliary equation**

$$am^2 + bm + c = 0.$$

Its roots decide the form of the CF:

Roots of the auxiliary equation	Complementary function	Example
Two different real roots m_1, m_2	$y = Ae^{m_1 x} + Be^{m_2 x}$	$y'' - 5y' + 6y = 0$: $m = 2, 3$, so $y = Ae^{2x} + Be^{3x}$
One repeated root m	$y = (A + Bx)e^{mx}$	$y'' + 6y' + 9y = 0$: $m = -3$, so $y = (A + Bx)e^{-3x}$
Complex roots $\alpha \pm \beta i$	$y = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$	$y'' + 2y' + 10y = 0$: $m = -1 \pm 3i$, so $y = e^{-x}(A \cos 3x + B \sin 3x)$

For a repeated root, $Ae^{mx} + Be^{mx}$ is really only one constant, so the extra factor x provides the second solution. For complex roots, $e^{(\alpha+\beta i)x} = e^{\alpha x}(\cos \beta x + i \sin \beta x)$, and combining it with its conjugate leaves the real functions $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$. Note that β , not βi , goes inside the cosine and sine.

A **first order** equation $\frac{dy}{dx} + ky = f(x)$ with constant k works the same way: the auxiliary equation $m + k = 0$ gives the CF $y = Ae^{-kx}$. (The integrating factor e^{kx} also solves it.)

Finding a particular integral

Try a function of the **same kind as the right-hand side**, with unknown coefficients:

Right-hand side $f(x)$	Try
A polynomial of degree n , such as $x^2 - 3$	A general polynomial of degree n : $px^2 + qx + r$ (all terms, even if f has some missing)
ae^{bx}	ke^{bx}
$a \cos px$, $b \sin px$, or $a \cos px + b \sin px$	$\lambda \cos px + \mu \sin px$ (both terms, even if f has only one)

Differentiate your trial function twice, substitute into the left side, and **equate coefficients** of each power of x (or of $\cos px$ and $\sin px$) to get simultaneous equations for the unknowns.

When the trial function is already part of the CF it gives 0 on the left, so it can't work. The question then gives you the form to use, usually the ordinary form times x (or times x^2 if the root is repeated). For example, in $y'' + 16y = 8 \cos 4x$ the CF is $A \cos 4x + B \sin 4x$, so try $y = kx \sin 4x$:

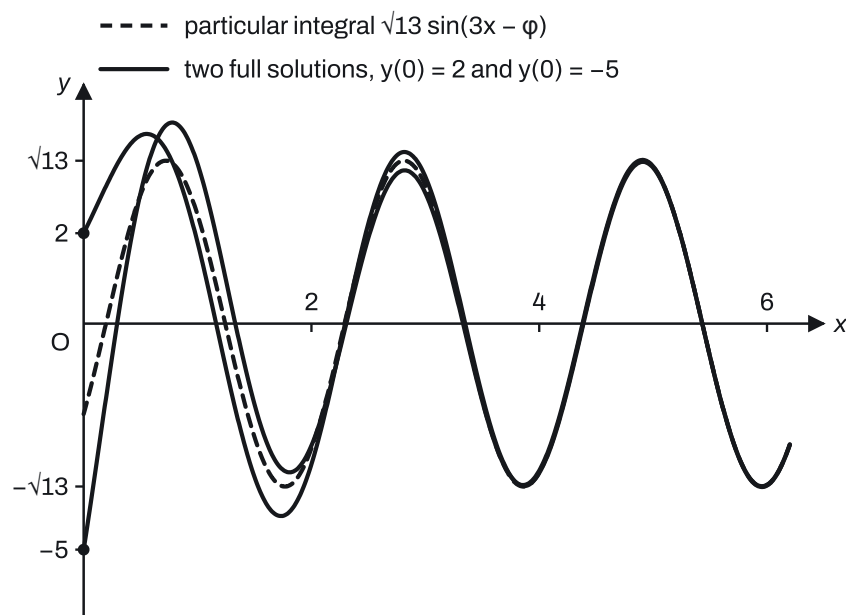
$$y'' = 8k \cos 4x - 16kx \sin 4x, \quad \text{so} \quad y'' + 16y = 8k \cos 4x = 8 \cos 4x \Rightarrow k = 1.$$

Using initial conditions

The initial conditions go into the **general solution** (CF + PI), never into the CF alone. For a second order equation, differentiate the general solution (with the product rule when the CF is $e^{\alpha x}(\dots)$), then put in $x = 0$ with both conditions. That gives two equations for A and B .

What happens for large x

If every root of the auxiliary equation is negative (or has a negative real part), every CF term contains a decaying factor such as e^{-x} or e^{-2x} . It tends to 0 as $x \rightarrow \infty$ **whatever A and B are**, so for large x every solution is close to the particular integral. In a model where t is time, the CF is the part that dies out and the PI is the long-term behaviour.



The diagram shows the solutions of Example 3. When the PI is $\lambda \cos px + \mu \sin px$, a question may ask you to write it as a single sine wave $R \sin(px \pm \phi)$, using the compound angle formula (in MF19) and $R = \sqrt{\lambda^2 + \mu^2}$.

Using a given substitution

Some equations are not linear with constant coefficients as they stand, but a substitution given in the question turns them into one of the types above. The work is always the same: **express the old derivatives in terms of the new ones**, substitute, and simplify.

- $y = vx$, where v is a function of x : the product rule gives $\frac{dy}{dx} = v + x \frac{dv}{dx}$. This suits equations in which x and y appear as $\frac{y}{x}$ or with the same total power in every term; after substituting, the x 's cancel and the variables separate.
- $z = x + y$ (or another combination): $\frac{dz}{dx} = 1 + \frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{dz}{dx} - 1$.
- $x = e^t$, for equations with terms $x^2 \frac{d^2y}{dx^2}$, $x \frac{dy}{dx}$ and y . Since $\frac{dx}{dt} = e^t = x$, the chain rule gives $x \frac{dy}{dx} = \frac{dy}{dt}$. Differentiating again, $x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. The equation then has constant coefficients in t (Example 4).
- **A new dependent variable**, such as $v = y^n$, $y = x^2w$ or $y = \cosh u$. Differentiate the relation twice with the chain and product rules. For $v = y^n$: $\frac{dv}{dx} = ny^{n-1} \frac{dy}{dx}$ and $\frac{d^2v}{dx^2} = ny^{n-1} \frac{d^2y}{dx^2} + n(n-1)y^{n-2} \left(\frac{dy}{dx}\right)^2$. Then replace each group of terms in the given equation.

After solving the new equation, **go back to the original variables**: $v = \frac{y}{x}$, $t = \ln x$ (so $e^{2t} = x^2$), $w = \frac{y}{x^2}$, and so on.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it. MF19 has no formulas for differential equations themselves, so all of the methods must be learnt.

Formula	Status
$\frac{dy}{dx} + P(x)y = Q(x)$: integrating factor $I = e^{\int P dx}$, then $Iy = \int IQ dx$	Learn it
General solution = complementary function + particular integral	Learn it
Auxiliary equation of $ay'' + by' + cy = 0$: $am^2 + bm + c = 0$	Learn it
CF: $Ae^{m_1x} + Be^{m_2x}$ (real, different); $(A + Bx)e^{mx}$ (repeated); $e^{\alpha x}(A \cos \beta x + B \sin \beta x)$ (roots $\alpha \pm \beta i$)	Learn it
CF of $y' + ky = 0$: Ae^{-kx}	Learn it

Formula	Status
PI to try: polynomial of the same degree; ke^{bx} ; $\lambda \cos px + \mu \sin px$	Learn it
$y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$; $x = e^t \Rightarrow x \frac{dy}{dx} = \frac{dy}{dt}$, $x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$	Learn it
$\lambda \cos px + \mu \sin px = R \sin(px + \phi)$ with $R = \sqrt{\lambda^2 + \mu^2}$, $R \cos \phi = \mu$, $R \sin \phi = \lambda$	Learn it
$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$	Given
$\int \frac{f'(x)}{f(x)} dx = \ln f(x) $; integration by parts	Given
$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$, $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$	Given
$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$, $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a}$, $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$	Given
$\int \tan x dx = \ln \sec x $, $\int \tanh x dx = \ln \cosh x$, $\int \coth x dx = \ln \sinh x $	Learn it

How to do it

In the Paper 2 questions we have tagged for this topic (64 questions, 2021–2025; 20 of them repeat a question from another paper of the same session, so there are 44 different questions), the types came up this often. 3 of the 64 questions mix two types (a substitution, then solving the new second order equation), so the counts add up to more than 64. Some also have a part on hyperbolic identities, differentiation or integration; those parts are taught in their own topics, and only the differential equation is counted here.

Question type	Questions
First order linear: integrating factor	28
Second order: particular solution from initial conditions	23
Second order: general solution and behaviour for large x	9
Reducing an equation with a given substitution	7

1. First order linear: integrating factor (28 questions)

- Put it in standard form:** divide by the coefficient of $\frac{dy}{dx}$ so the equation reads $\frac{dy}{dx} + P(x)y = Q(x)$. Take care with the sign of P .
- Find the integrating factor** $I = e^{\int P dx}$ and simplify it to a function without e and $\ln$, using the table in "Understand it".
- Write the left side as** $\frac{d}{dx}(Iy)$ and the right side as IQ , simplified. It usually simplifies a lot (Example 1).
- Integrate the right side** and include $+C$.

5. **Use the initial condition** to find C , then **divide by I** to give $y = f(x)$ if the question asks for that form.

Variations you will meet:

- **A hyperbolic or trigonometric P** , such as $\coth x$, $\tanh x$, $\frac{2}{\sinh 2x}$ or $\tan x$. Recognise $\int P dx$ as a logarithm (often with a hint from part (a), for example the derivative of $\ln \tanh x$), then simplify $e^{\ln(\dots)}$.
- **A rational P** : if the top is a multiple of the derivative of the bottom, $\int P dx$ is a multiple of $\ln(\text{bottom})$, so I is a power of the bottom, such as $\frac{1}{x^2+1}$ (from $P = -\frac{2x}{x^2+1}$) or $(4-x^2)^{-1/2}$ (from $P = \frac{x}{4-x^2}$). Otherwise use partial fractions.
- **"Show that an appropriate integrating factor is ..."**: show the integral (for example a $\sinh^{-1}$ or $\cosh^{-1}$ form from MF19), turn it into a logarithm, and simplify $e^{\ln(\dots)}$ to the printed form.
- **Integrating the right side** needs a technique from the Integration note: a standard form after completing the square ($\sin^{-1}$, $\sinh^{-1}$), partial fractions, integration by parts (for example $\int \tan^{-1} x dx$), or a result proved in part (a) ("hence").
- **An equation starting $x \frac{dy}{dx} - y = \dots$** has $P = -\frac{1}{x}$ after dividing by x , so $I = \frac{1}{x}$.
- **Constant coefficients**, such as $\frac{dy}{dx} + ky = \sin x$: either use $I = e^{kx}$ (two rounds of integration by parts) or, usually quicker, CF Ae^{-kx} plus a PI $\lambda \cos x + \mu \sin x$.
- **Other letters**: the variables may be x and t , or y and θ . Keep the letters the question uses.

2. Second order: particular solution from initial conditions (23 questions)

1. **Auxiliary equation**: write $am^2 + bm + c = 0$ and solve it. Decide which case the roots are.
2. **Complementary function** in the right form for that case, in terms of the right variable (x or t).
3. **Particular integral**: write the trial function, its first and second derivatives, substitute and **equate coefficients**. Solve the simultaneous equations carefully and check by substituting back.
4. **General solution**: $y = \text{CF} + \text{PI}$, written as an equation starting " $y =$ ".
5. **Differentiate the general solution**, using the product rule on $e^{\alpha x}(A \cos \beta x + B \sin \beta x)$.
6. **Put in the initial conditions** ($x = 0$ usually), solve for A and B , and write the final answer as " $y = \dots$ ".

Variations you will meet:

- **All three cases of roots** appear: distinct real (for example $10m^2 - 7m + 1 = 0$ gives $m = \frac{1}{2}, \frac{1}{5}$), repeated, and complex with fractions such as $m = -\frac{1}{4} \pm \frac{3}{4}i$, which give $e^{-x/4} (A \cos \frac{3x}{4} + B \sin \frac{3x}{4})$.
- **Right-hand sides**: a quadratic (try $px^2 + qx + r$), a constant (try $y = k$; a report praised this quick choice), an exponential (try ke^{bx}), or a sine or cosine.
- **A right-hand side in disguise**: $\cosh x + \sinh x = e^x$, so a power of $\cosh x + \sinh x$ is an exponential. Part (a) often proves this.
- **"Find k such that kxe^{bx} is a particular integral"**: the given form clashes with the CF. Differentiate it twice with the product rule, substitute and match the coefficients of e^{bx} ; the terms in xe^{bx} cancel.
- **Initial values that are not zero**, such as $x = 5$ and $\frac{dx}{dt} = -2$ when $t = 0$: the PI's value and gradient at 0 still count, so include them when you form the equations for A and B .
- **A follow-up**, such as "find the exact value of x for which $y = 0$ ": set the particular solution to 0 and solve, for example as a $\sinh$ or logarithm equation.

3. Second order: general solution and behaviour for large x (9 questions)

1. Find the **general solution** as in type 2, steps 1–4.
2. **Say why the CF disappears:** each term has a factor such as e^{-x} or $e^{-t/2}$, so it tends to 0 as $x \rightarrow \infty$ for any initial conditions.
3. **State the approximation** $y \approx \text{PI}$ as an equation (not with an arrow).
4. **If asked for** $R \sin(px \pm \phi)$, expand it with the compound angle formula, match coefficients with the PI, and find $R = \sqrt{\lambda^2 + \mu^2}$ and ϕ .

Variations you will meet:

- **A polynomial PI:** "state an approximate solution for large x " means, for example, $y \approx x - \frac{3}{2}$.
- **Other quantities:** "deduce an approximate value of $\frac{dx}{dt}$ for large t " means differentiate the PI.
- **Choosing the quadrant of ϕ :** with " $R \sin(px - \phi)$ ", where R and ϕ are positive", both $R \cos \phi$ and $R \sin \phi$ come out with signs. If the PI is $-\frac{4}{25} \cos 2x - \frac{3}{25} \sin 2x$, then $R \cos \phi = -\frac{3}{25}$ and $R \sin \phi = \frac{4}{25}$, so $R = \frac{1}{5}$ and ϕ is obtuse: $\phi = \pi - \tan^{-1} \frac{4}{3} \approx 2.21$. Always check both signs.
- **A substitution first:** the second order equation may come from part (a) of a substitution question (type 4).

4. Reducing an equation with a given substitution (7 questions)

1. **Differentiate the substitution** to express the old derivatives in terms of the new ones (see the last section of "Understand it").
2. **Substitute everything** into the equation and simplify. If a result is printed, show enough lines to reach it exactly.
3. **Solve the new equation:** separate the variables (first order), or CF + PI (second order).
4. **Change back** to the original variables, and convert the initial conditions too if they are given for the original ones.
5. **Give the answer in the form asked**, for example " $a \ln(x + y) + b(x - y) + c = 0$ " or "a polynomial in x ".

Variations you will meet:

- $y = vx$ **with a square root**, such as $\sqrt{x^2 + y^2}$: x comes out of the root, the variables separate, and the v integral is a $\sinh^{-1}$ or $\cosh^{-1}$ form from MF19. Turn it into a logarithm, and square to remove the root if a polynomial answer is wanted.
- $z = x + y$ (or $z = ax + by$): the right side becomes a function of z only, then separate. A fraction such as $\frac{z+1}{z-1}$ needs dividing first.
- $v = y^n$, $y = x^2 w$, $y = \cosh u$: part (a) is a "show that" with the chain and product rules; part (b) solves the constant-coefficient equation and converts back. With $y = \cosh u$, the initial conditions for u become conditions for y : $y(0) = \cosh u(0)$ and $\frac{dy}{dx} = \sinh u \frac{du}{dx}$. At the end, $u = \cosh^{-1} y$ written as a logarithm.
- $x = e^t$ (in the syllabus, not yet in our tagged questions): the equation in t has constant coefficients, and the answer goes back with $e^t = x$ (Example 4).

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

Find the solution of the differential equation

$$\frac{dy}{dx} + y \tan x = \sin 2x \cos x, \quad 0 \leq x < \frac{\pi}{2},$$

for which $y = 1$ when $x = 0$. Give your answer in the form $y = f(x)$. **[7]**

The equation is already in standard form, with $P = \tan x$.

$$I = e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x.$$

M1 A1 for the integrating factor.

$$\frac{d}{dx}(y \sec x) = \sec x \sin 2x \cos x = \sin 2x.$$

M1 for the left side as $\frac{d}{dx}(Iy)$ and the right side multiplied by I .

$$y \sec x = -\frac{1}{2} \cos 2x + C.$$

A1

At $x = 0, y = 1: 1 = -\frac{1}{2} + C$, so $C = \frac{3}{2}$. **M1**

$$y = \cos x \left(\frac{3}{2} - \frac{1}{2} \cos 2x \right) = \cos x (1 + \sin^2 x).$$

M1 for multiplying by $\cos x$, **A1**. (Using $\cos 2x = 1 - 2\sin^2 x$; the answer $2 \cos x - \cos^3 x$ is the same.)

Example 2

Find the particular solution of the differential equation

$$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 10y = 10x^2 + 4x + 2,$$

given that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$. **[10]**

Auxiliary equation $m^2 + 2m + 10 = 0$, so $m = -1 \pm 3i$. **M1**

CF: $y = e^{-x}(A \cos 3x + B \sin 3x)$. **A1**

PI: try $y = px^2 + qx + r$, so $y' = 2px + q$ and $y'' = 2p$. **B1**

$$2p + 2(2px + q) + 10(px^2 + qx + r) = 10px^2 + (4p + 10q)x + (2p + 2q + 10r).$$

Equating coefficients: $10p = 10$, $4p + 10q = 4$, $2p + 2q + 10r = 2$, so $p = 1$, $q = 0$, $r = 0$. **M1 A1**

General solution: $y = e^{-x}(A \cos 3x + B \sin 3x) + x^2$. **A1**

Differentiate with the product rule:

$$\frac{dy}{dx} = e^{-x}(-3A \sin 3x + 3B \cos 3x) - e^{-x}(A \cos 3x + B \sin 3x) + 2x.$$

M1

At $x = 0$: $y = A = 1$, and $\frac{dy}{dx} = 3B - A = 0$, so $B = \frac{1}{3}$. **M1 A1**

$$y = e^{-x} \left(\cos 3x + \frac{1}{3} \sin 3x \right) + x^2.$$

A1

Example 3

(a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 26 \cos 3x.$$

[7]

(b) Show that, for large positive values of x and for any initial conditions, $y \approx R \sin(3x - \phi)$, where R and ϕ are positive constants to be found. **[3]**

(a) $m^2 + 2m + 5 = 0$, so $m = -1 \pm 2i$. **M1**

CF: $y = e^{-x}(A \cos 2x + B \sin 2x)$. **A1**

PI: try $y = \lambda \cos 3x + \mu \sin 3x$, so $y' = -3\lambda \sin 3x + 3\mu \cos 3x$ and $y'' = -9\lambda \cos 3x - 9\mu \sin 3x$. **M1 A1**

Substituting and collecting:

$$\cos 3x : -9\lambda + 6\mu + 5\lambda = 26 \Rightarrow -4\lambda + 6\mu = 26, \quad \sin 3x : -9\mu - 6\lambda + 5\mu = 0 \Rightarrow -6\lambda - 4\mu = 0.$$

M1

From the second, $\mu = -\frac{3}{2}\lambda$; then $-4\lambda - 9\lambda = 26$, so $\lambda = -2$ and $\mu = 3$. **A1**

$$y = e^{-x}(A \cos 2x + B \sin 2x) - 2 \cos 3x + 3 \sin 3x.$$

A1

(b) As $x \rightarrow \infty$, $e^{-x} \rightarrow 0$, so the CF tends to 0 whatever A and B are, and $y \approx -2 \cos 3x + 3 \sin 3x$. **M1** (with setting up $R \sin(3x - \phi) = R \sin 3x \cos \phi - R \cos 3x \sin \phi$)

Matching with $\lambda = -2$, $\mu = 3$: $R \cos \phi = \mu = 3$ and $R \sin \phi = -\lambda = 2$, so $R = \sqrt{9+4} = \sqrt{13}$ and $\tan \phi = \frac{2}{3}$, $\phi = 0.588$ (3 s.f.). **A1 A1**

$$y \approx \sqrt{13} \sin(3x - 0.588).$$

The diagram in "Understand it" shows two solutions of this equation settling onto this curve.

Example 4

(a) Show that the substitution $x = e^t$ transforms the differential equation

$$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 6x^3, \quad x > 0,$$

into $\frac{d^2 y}{dt^2} - 3 \frac{dy}{dt} + 2y = 6e^{3t}$. **[4]**

(b) Hence find the general solution for y in terms of x . **[5]**

(a) $\frac{dx}{dt} = e^t = x$, so $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = x \frac{dy}{dx}$. **B1**

Differentiate $\frac{dy}{dt} = x \frac{dy}{dx}$ with respect to t , using the product rule:

$$\frac{d^2 y}{dt^2} = \frac{dx}{dt} \frac{dy}{dx} + x \frac{d^2 y}{dx^2} \frac{dx}{dt} = x \frac{dy}{dx} + x^2 \frac{d^2 y}{dx^2}.$$

M1 A1

So $x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt}$, and the equation becomes

$$\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) - 2 \frac{dy}{dt} + 2y = 6e^{3t} \Rightarrow \frac{d^2 y}{dt^2} - 3 \frac{dy}{dt} + 2y = 6e^{3t}.$$

A1, the given answer.

(b) $m^2 - 3m + 2 = 0$, so $m = 1, 2$, and the CF is $y = Ae^t + Be^{2t}$. **M1 A1**

PI: try $y = ke^{3t}$: $9k - 9k + 2k = 6$, so $k = 3$. **M1 A1**

$y = Ae^t + Be^{2t} + 3e^{3t}$, and with $e^t = x$:

$$y = Ax + Bx^2 + 3x^3.$$

A1. (Check: $y = 3x^3$ gives $x^2(18x) - 2x(9x^2) + 2(3x^3) = 6x^3$.)

Common mistakes

- **Finding the integrating factor before dividing through.** The coefficient of $\frac{dy}{dx}$ must be 1 first. For $x \frac{dy}{dx} - y = \dots$, $P = -\frac{1}{x}$ and $I = \frac{1}{x}$, not $e^{-\int 1 dx}$. Mark schemes give a mark just for dividing through.
- **Sign slips in the integrating factor.** $e^{-\ln x} = \frac{1}{x}$, not $-x$; $e^{-\frac{1}{2} \ln f} = f^{-1/2}$.
- **Not multiplying the right side by I , or not simplifying it.** Reports say the best answers simplified IQ fully first; then the integral was often easy to see.

- **Adding $+C$ after dividing by I .** It must be added when you integrate: $y = \frac{1}{I} \left(\int IQ \, dx + C \right)$.
- **Errors solving for the particular integral.** Reports every year mention slips in the simultaneous equations for the coefficients. Write each equation separately and check the PI by substituting it back.
- **The wrong form of the CF.** A repeated root needs $(A + Bx)e^{mx}$; complex roots $\alpha \pm \beta i$ give $e^{\alpha x}(A \cos \beta x + B \sin \beta x)$, not $e^{\beta x}$ or $\cos(\beta i x)$.
- **Using initial conditions on the CF alone.** Put them into the whole general solution, and differentiate it with the product rule.
- **An expression instead of an equation.** Reports often say candidates wrote only the right-hand side. The general and particular solutions must start " $y =$ " (or " $x =$ "), or the final mark is lost.
- **The wrong variable.** If the equation is in x and t , the CF must use t : writing e^{-2x} for e^{-2t} loses the mark.
- **For large x , working with the wrong part or an arrow.** Say the CF tends to 0, then write $y \approx \text{PI}$ as an equation; reports mention arrows used instead of " $=$ ". Check the signs of $R \cos \phi$ and $R \sin \phi$ before choosing ϕ .
- **Stopping after the substitution.** Reports on a $z = x + y$ question say some candidates reached the separable equation but did not separate the variables, or did not give the answer in the required form. Finish the job, and read which form is wanted.

How to get the marks

- **Integrating factor questions** usually score **B1** for dividing through, **M1 A1** for the integrating factor, **M1** for $\frac{d}{dx}(Iy) = IQ$, **M1 A1** for integrating the right side, **M1** for using the initial condition, and **A1** for the answer. Show $\int P \, dx$ as well as I .
- **Second order questions** usually score **M1** for the auxiliary equation (with the right type of roots), **A1** for the CF, **M1 A1** for the PI's form and derivatives, **M1** for substituting and equating coefficients, **A1** for the general solution, then **M1** for differentiating, **M1 A1** for the initial conditions and **A1** for the final answer.
- **Method marks follow through.** A slip in the PI still leaves the marks for using the initial conditions, as long as your CF has the correct form.
- **"Show that"** (a substitution or an integrating factor): the answer is printed, so every step must be seen, including each derivative and the line before the result.
- **"Hence"** means use part (a): the derivative you proved in (a) is usually exactly the integral you need in (b).
- **"Exact form"**: keep logarithms, surds, π and inverse functions such as $\sinh^{-1} \frac{x-2}{3}$; an unsimplified but correct exact answer is accepted.
- **"Give your answer in the form $y = f(x)$ ":** divide through by the integrating factor at the end.
- **Large x :** the method mark needs a reason (the exponential factor tends to 0) as well as the PI; angles can be in radians or degrees, to 3 significant figures.

Quick check

1. Find the general solution of $\frac{dy}{dx} + 4y = 8x + 6$.

2. Write down the complementary function of (a) $y'' - y' - 12y = 0$, (b) $4y'' + 4y' + y = 0$, (c) $y'' - 4y' + 13y = 0$.
3. Find the value of k for which $y = kx^2e^{2x}$ is a particular integral of $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 6e^{2x}$, and write down the general solution.
4. Use the substitution $z = x + y$ to find the solution of $\frac{dy}{dx} = (x + y)^2 - 1$ for which $y = 1$ when $x = 0$.
Give your answer in the form $y = f(x)$.
5. Use the substitution $y = vx$ to find the solution of $x\frac{dy}{dx} = y + xe^{-y/x}$, for $x \geq 1$, for which $y = 0$ when $x = 1$.

Answers

1. CF: $m + 4 = 0$, so $y = Ae^{-4x}$. PI: try $y = px + q$: $p + 4(px + q) = 8x + 6$, so $4p = 8$ and $p + 4q = 6$, giving $p = 2$, $q = 1$. $y = Ae^{-4x} + 2x + 1$. (With $I = e^{4x}$ and integration by parts you get the same.)
2. (a) $m^2 - m - 12 = (m - 4)(m + 3) = 0$: $y = Ae^{4x} + Be^{-3x}$. (b) $4m^2 + 4m + 1 = (2m + 1)^2 = 0$, so $m = -\frac{1}{2}$ twice: $y = (A + Bx)e^{-x/2}$. (c) $m = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i$: $y = e^{2x}(A \cos 3x + B \sin 3x)$.
3. $y' = k(2x + 2x^2)e^{2x}$, $y'' = k(2 + 8x + 4x^2)e^{2x}$. Then $y'' - 4y' + 4y = ke^{2x}[(2 + 8x + 4x^2) - (8x + 8x^2) + 4x^2] = 2ke^{2x}$, so $2k = 6$ and $k = 3$. The auxiliary equation $(m - 2)^2 = 0$ gives the CF, so $y = (A + Bx)e^{2x} + 3x^2e^{2x}$.
4. $\frac{dy}{dx} = \frac{dz}{dx} - 1$, so $\frac{dz}{dx} - 1 = z^2 - 1$, that is $\frac{dz}{dx} = z^2$. Separating, $\int z^{-2} dz = \int dx$, so $-\frac{1}{z} = x + C$. At $x = 0$, $z = 1$, so $C = -1$ and $x + y = \frac{1}{1 - x}$. So $y = \frac{1}{1 - x} - x$.
5. $v + x \frac{dv}{dx} = v + e^{-v}$, so $x \frac{dv}{dx} = e^{-v}$. Separating, $\int e^v dv = \int \frac{dx}{x}$, so $e^v = \ln x + C$. At $x = 1$, $v = 0$: $C = 1$. So $v = \ln(1 + \ln x)$ and $y = x \ln(1 + \ln x)$.

Past questions to try

- **9231/22/O/N/24 Q7**: an integrating factor from the derivative of $\ln \tanh x$, proved in part (a).
- **9231/23/M/J/24 Q5**: a general solution with a repeated root, then the $R \sin(t - \phi)$ form for large t .
- **9231/22/O/N/23 Q8**: the substitution $v = y^4$, then a particular solution of the new equation.
- **9231/22/M/J/22 Q6**: the substitution $y = vx$ with a square root, and an answer as a polynomial.