

# Differentiation A2

Read online at [pastopic.com/cambridge/further-mathematics-9231/notes/differentiation](https://pastopic.com/cambridge/further-mathematics-9231/notes/differentiation)

Last checked 30 September 2026

Read first: [Hyperbolic functions](#)

© 2026 Pastopic. Free for personal study. Please share the link, not the file.

## What you need to know

By the end of this topic you should be able to:

1. **Differentiate the hyperbolic functions** ( $\sinh$ ,  $\cosh$ ,  $\tanh$  and their reciprocals  $\operatorname{sech}$ ,  $\operatorname{cosech}$ ,  $\coth$ ), and **differentiate the inverse functions**  $\sin^{-1} x$ ,  $\cos^{-1} x$ ,  $\sinh^{-1} x$ ,  $\cosh^{-1} x$  and  $\tanh^{-1} x$ , including with the chain, product and quotient rules.
2. **Find the second derivative**  $\frac{d^2y}{dx^2}$  **when  $x$  and  $y$  are linked implicitly** (by one equation, such as  $y^3 + 2xy - x^2 = 1$ ) **or parametrically** (both given in terms of a third variable  $t$ ).
3. **Derive and use the first few terms of the Maclaurin's series of a function**: by differentiating and evaluating at  $x = 0$ , by combining the standard series in MF19, or by differentiating an equation over and over (for example  $y = \tanh x$  satisfies  $y' = 1 - y^2$ ). You never need the general term.

The definitions, identities and logarithmic forms of the hyperbolic functions are in the Hyperbolic functions note; this note uses them. The first-derivative rules (chain, product, quotient, implicit and parametric  $\frac{dy}{dx}$ ) are from A Level Mathematics and are recapped here only where they are needed.

## Understand it

### Derivatives of the hyperbolic functions

Straight from the definitions, because  $\frac{d}{dx}e^{-x} = -e^{-x}$ :

$$\frac{d}{dx} \sinh x = \frac{e^x + e^{-x}}{2} = \cosh x, \quad \frac{d}{dx} \cosh x = \frac{e^x - e^{-x}}{2} = \sinh x.$$

Notice there is **no minus sign** for  $\cosh$ , unlike  $\frac{d}{dx} \cos x = -\sin x$ . The quotient rule then gives  $\tanh$ :

$$\frac{d}{dx} \tanh x = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$

The reciprocals come from the chain rule, for example  $\operatorname{sech} x = (\cosh x)^{-1}$  gives  $-(\cosh x)^{-2} \sinh x = -\operatorname{sech} x \tanh x$ :

| $f(x)$                    | $f'(x)$                            |
|---------------------------|------------------------------------|
| $\operatorname{sech} x$   | $-\operatorname{sech} x \tanh x$   |
| $\operatorname{cosech} x$ | $-\operatorname{cosech} x \coth x$ |
| $\coth x$                 | $-\operatorname{cosech}^2 x$       |

With the chain rule, the inside's derivative multiplies the result:  $\frac{d}{dx} \cosh 5x = 5 \sinh 5x$ , and  $\frac{d}{dx} \tanh(x^2) = 2x \operatorname{sech}^2(x^2)$ .

## Derivatives of the inverse functions

The trick is to **undo the inverse and differentiate implicitly**. Let  $y = \sinh^{-1} x$ . Then  $\sinh y = x$ , and differentiating with respect to  $x$ :

$$\cosh y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.$$

The same three steps work for every inverse function. The only care needed is the **sign of the square root**:

- $y = \sin^{-1} x$ :  $\cos y \frac{dy}{dx} = 1$ . Here  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ , so  $\cos y \geq 0$  and  $\cos y = +\sqrt{1 - x^2}$ . So  $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$ . For  $\cos^{-1} x$  the same working gives  $-\frac{1}{\sqrt{1 - x^2}}$ .
- $y = \cosh^{-1} x$ :  $\sinh y \frac{dy}{dx} = 1$ , and  $y \geq 0$  so  $\sinh y = +\sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$ . So  $\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$ .
- $y = \tanh^{-1} x$ :  $\operatorname{sech}^2 y \frac{dy}{dx} = 1$ , and  $\operatorname{sech}^2 y = 1 - \tanh^2 y = 1 - x^2$ . So  $\frac{dy}{dx} = \frac{1}{1 - x^2}$ .

A second route uses the logarithmic forms. For example,  $\tanh^{-1} x = \frac{1}{2} \ln(1 + x) - \frac{1}{2} \ln(1 - x)$ , so its derivative is  $\frac{1}{2(1+x)} + \frac{1}{2(1-x)} = \frac{(1-x)+(1+x)}{2(1-x^2)} = \frac{1}{1-x^2}$ . Either route earns the marks when a question says "show that".

With an inside function, use the chain rule:  $\frac{d}{dx} \sinh^{-1}(2x) = \frac{2}{\sqrt{1 + 4x^2}}$ , and  $\frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{a}$ .

$$\frac{1}{\sqrt{1 - x^2/a^2}} = \frac{1}{\sqrt{a^2 - x^2}} \text{ for } a > 0.$$

## The second derivative from an implicit equation

When  $y$  is a function of  $x$ , any expression in  $y$  is differentiated with the chain rule, so a factor  $\frac{dy}{dx}$  (written  $y'$ ) appears:  $\frac{d}{dx}(y^3) = 3y^2 y'$ , and a product needs the product rule,  $\frac{d}{dx}(xy) = xy' + y$ .

To get  $y''$ , **differentiate the differentiated equation again**. Each  $y'$  is now a function of  $x$  too, so, for example,

$$\frac{d}{dx}(yy') = y' \cdot y' + y y'' = (y')^2 + y y''.$$

For the circle  $x^2 + y^2 = 25$  at  $(3, 4)$ : once gives  $2x + 2yy' = 0$ , so  $y' = -\frac{3}{4}$ . Again gives  $2 + 2(y')^2 + 2yy'' = 0$ , that is  $2 + \frac{9}{8} + 8y'' = 0$ , so  $y'' = -\frac{25}{64}$ . (Check: the top half of the circle is  $y = \sqrt{25 - x^2}$ , whose second derivative  $-25(25 - x^2)^{-3/2}$  is  $-\frac{25}{64}$  at  $x = 3$ .)

You never need to make  $y''$  the subject in general. Substitute the numbers ( $x$ ,  $y$  and  $y'$ ) as soon as you have differentiated twice, then solve a simple linear equation.

## The second derivative of a parametric curve

For  $x = f(t)$ ,  $y = g(t)$ , the first derivative is  $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ , which is a function of  $t$ . To differentiate **that** with respect to  $x$ , use the chain rule once more:

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right) \times \frac{dt}{dx} = \frac{d}{dt} \left( \frac{dy}{dx} \right) \div \frac{dx}{dt}.$$

So: **differentiate  $\frac{dy}{dx}$  with respect to  $t$ , then divide by  $\frac{dx}{dt}$** . It is **not**  $\frac{d^2y}{dt^2} \div \frac{d^2x}{dt^2}$ .

For  $x = t^2$ ,  $y = t^3$  ( $t > 0$ ):  $\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3}{2}t$ , so  $\frac{d^2y}{dx^2} = \frac{3}{2} \div 2t = \frac{3}{4t}$ . (Check:  $y = x^{3/2}$  has  $y'' = \frac{3}{4}x^{-1/2} = \frac{3}{4t}$ .)

## Maclaurin's series

A Maclaurin's series is a polynomial that copies a function near  $x = 0$ : it has the same value, the same gradient, the same second derivative, and so on, at  $x = 0$ . If  $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ , then putting  $x = 0$  into  $f$ ,  $f'$ ,  $f''$ ,  $f'''$  gives  $a_0 = f(0)$ ,  $a_1 = f'(0)$ ,  $2a_2 = f''(0)$ ,  $6a_3 = f'''(0)$ , which is the formula in MF19:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

There are three ways to find one.

- **Differentiate and evaluate.** Find  $f'(x)$ ,  $f''(x)$ , ... as far as needed, put  $x = 0$  into each, then into the formula. For example  $f(x) = \cosh 2x$ :  $f(0) = 1$ ,  $f'(0) = 2 \sinh 0 = 0$ ,  $f''(0) = 4 \cosh 0 = 4$ , so  $\cosh 2x \approx 1 + 2x^2$ .
- **Combine standard series.** MF19 gives the series for  $e^x$ ,  $\ln(1+x)$ ,  $\sin x$ ,  $\cos x$ ,  $\tan^{-1} x$ ,  $\sinh x$ ,  $\cosh x$  and  $\tanh^{-1} x$ . Replace  $x$  by something else, add, or multiply, keeping only the powers you need. For example,  $e^x \sin x \approx \left(1 + x + \frac{x^2}{2}\right) \left(x - \frac{x^3}{6}\right) = x + x^2 + \frac{1}{3}x^3$ .
- **Differentiate an equation repeatedly.** For  $y = \tanh x$ :  $y' = \operatorname{sech}^2 x = 1 - y^2$ . Then  $y'' = -2yy'$  and  $y''' = -2(y')^2 - 2yy''$ . At  $x = 0$ :  $y = 0$ ,  $y' = 1$ ,  $y'' = 0$ ,  $y''' = -2$ . So  $\tanh x \approx x - \frac{2}{3!}x^3 = x - \frac{1}{3}x^3$ . This avoids messy quotient-rule derivatives.

A function needs a value at  $x = 0$  to have a Maclaurin's series:  $\ln x$  and  $\frac{1}{x}$  have none. The standard series only hold for the  $x$  shown in MF19 (for example  $-1 < x \leq 1$  for  $\ln(1+x)$ ).

## Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

| Formula                                                                                                                | Status |
|------------------------------------------------------------------------------------------------------------------------|--------|
| $\frac{d}{dx} \sinh x = \cosh x$ , $\frac{d}{dx} \cosh x = \sinh x$ , $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$ | Given  |

| Formula                                                                                                                                                                                                 | Status   |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------|
| $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$ , $\frac{d}{dx} \operatorname{cosech} x = -\operatorname{cosech} x \coth x$ , $\frac{d}{dx} \coth x = -\operatorname{cosech}^2 x$ | Learn it |
| $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ , $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$ , $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$                                               | Given    |
| $\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$ , $\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$ , $\frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2}$                                             | Given    |
| Derivatives of $e^x$ , $\ln x$ , $\sin x$ , $\cos x$ , $\tan x$ , $\sec x$ , $\operatorname{cosec} x$ , $\cot x$                                                                                        | Given    |
| Product rule $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$ ; quotient rule $\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$                           | Given    |
| Chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$                                                                                                                                         | Learn it |
| Implicit: $\frac{d}{dx} f(y) = f'(y) \frac{dy}{dx}$ ; $\frac{d}{dx} \left( y \frac{dy}{dx} \right) = \left( \frac{dy}{dx} \right)^2 + y \frac{d^2 y}{dx^2}$                                             | Learn it |
| Parametric: $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$                                                                                                                                          | Given    |
| Parametric: $\frac{d^2 y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) \div \frac{dx}{dt}$                                                                                                         | Learn it |
| Maclaurin's series $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^r}{r!} f^{(r)}(0) + \dots$                                                                                          | Given    |
| Series for $e^x$ , $\ln(1+x)$ , $\sin x$ , $\cos x$ , $\tan^{-1} x$ , $\sinh x$ , $\cosh x$ , $\tanh^{-1} x$ , with their ranges of validity                                                            | Given    |
| Coefficient of $x^r$ is $\frac{f^{(r)}(0)}{r!}$ , so $f^{(r)}(0) = r! \times \text{coefficient}$                                                                                                        | Learn it |
| $\cosh^2 x - \sinh^2 x \equiv 1$ (to simplify derivatives)                                                                                                                                              | Given    |
| $1 - \tanh^2 x \equiv \operatorname{sech}^2 x$                                                                                                                                                          | Learn it |

## How to do it

In the Paper 2 questions we have tagged for this topic (52 questions, 2021–2025; 15 of them appear in two papers of the same session, so there are 37 different questions), the types came up this often. 20 of the 52 questions mix two types, so the counts add up to more than 52. Some also have parts on integration or differential equations; those parts belong to their own topics, and only the differentiation is counted here.

| Question type                                  | Questions |
|------------------------------------------------|-----------|
| Maclaurin's series                             | 23        |
| Differentiate hyperbolic and inverse functions | 22        |
| Second derivative of a parametric curve        | 17        |
| Second derivative from an implicit equation    | 10        |

## 1. Maclaurin's series (23 questions)

1. **Choose the route.** If the function is built from MF19 series (a sum, a product, or  $x$  replaced by  $2x$ ,  $-x^2$ ,  $1 + x \dots$ ), combine the series. If the question says "from first principles", or the function is a composite such as  $e^{f(x)}$  or  $\ln f(x)$ , differentiate.
2. **Differentiating:** find  $f'(x)$  and  $f''(x)$  (and  $f'''(x)$  for a term in  $x^3$ ). Tidy each derivative before the next one, for example with  $\cosh^2 x - \sinh^2 x = 1$ .
3. **Evaluate**  $f(0)$ ,  $f'(0)$ ,  $f''(0)$ , ... exactly.
4. **Substitute** into  $f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{6}f'''(0)$  and simplify the fractions. Stop at the power asked for.

### Variations you will meet:

- **Standard series with a shifted argument.** Rewrite first so that MF19 applies:  $e^{x-3} = e^{-3} \cdot e^x$ , and  $\ln(4 - x) = \ln 4 + \ln\left(1 - \frac{x}{4}\right)$ . Expand each part, then collect like powers.
- **Series inside a series.** For  $\ln(2 - \cos x)$ , write  $2 - \cos x = 1 + u$  with  $u = \frac{x^2}{2} - \frac{x^4}{24} + \dots$ , then use  $\ln(1 + u) = u - \frac{u^2}{2} + \dots$ , keeping only the powers you need:  $\ln(2 - \cos x) \approx \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{8}x^4 = \frac{1}{2}x^2 - \frac{1}{6}x^4$ .
- **An answer in the form  $\ln a + bx + cx^2$ .**  $f(0)$  is often an inverse hyperbolic value: use the logarithmic form, for example  $\tanh^{-1} \frac{1}{3} = \frac{1}{2} \ln 2 = \ln \sqrt{2}$ , or  $\cosh^{-1} \frac{5}{4} = \ln 2$ .
- **$y$  given implicitly** (with  $y = y_0$  at  $x = 0$ ). Differentiate the equation twice, substitute  $x = 0$ ,  $y = y_0$  to get  $y'(0)$ , then  $y''(0)$ , then use the formula.
- **$y$  in terms of  $x$  for a parametric curve.** Find the value of  $t$  where  $x = 0$ . Then  $y(0)$ ,  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at  $x = 0$  are the values of  $y$  and the parametric derivatives at that  $t$ . It is often not  $t = 0$ .
- **"Deduce the value of  $f^{(4)}(0)$ ":** it is  $4! = 24$  times the coefficient of  $x^4$ .
- **Approximate an integral:** integrate the polynomial between the (small) limits, giving a fraction if asked.

## 2. Differentiate hyperbolic and inverse functions (22 questions)

1. **Spot the outer function** and write its derivative from MF19 (or the table of reciprocals above).
2. **Multiply by the derivative of the inside** (chain rule), and use the product or quotient rule where two functions are multiplied or divided.
3. **Simplify towards the printed answer:** square roots often collapse with an identity or by factorising. For example  $\sqrt{(x^2 + 1)^2 - 1} = \sqrt{x^2(x^2 + 2)} = x\sqrt{x^2 + 2}$  for  $x > 0$ .

### Variations you will meet:

- **"Show that  $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$ "** (or another MF19 result). Quoting MF19 is not enough: either write  $\tanh y = x$  and differentiate implicitly, or differentiate the logarithmic form (both shown in "Understand it").
- **An inverse reciprocal function**, such as  $u = \operatorname{cosech}^{-1} x$  for  $x > 0$ : write  $\sinh u = \frac{1}{x}$ , differentiate implicitly to get  $\cosh u \frac{du}{dx} = -\frac{1}{x^2}$ , then write  $\cosh u$  in terms of  $x$  with an identity, taking the positive root:  $\cosh u = \sqrt{1 + \frac{1}{x^2}} = \frac{\sqrt{x^2 + 1}}{x}$ , so  $\frac{du}{dx} = -\frac{1}{x\sqrt{x^2 + 1}}$ .

- **Logarithms of hyperbolic functions:**  $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$ , for example  $\frac{d}{dx} \ln(\sinh x) = \cosh x$ . Then simplify with  $\sinh 2x = 2 \sinh x \cosh x$  if the answer has a double angle.
- **Stationary points:** set the derivative to 0, for example  $y = \tanh x - \frac{1}{2}x$  has  $\frac{dy}{dx} = \operatorname{sech}^2 x - \frac{1}{2} = 0$ , so  $\cosh^2 x = 2$  and  $x = \pm \cosh^{-1} \sqrt{2} = \pm \ln(\sqrt{2} + 1)$ . If the equation can't be solved exactly, show a root lies in an interval by a change of sign.
- **Inside a parametric or implicit question:** the same derivatives appear as  $\frac{dx}{dt}$ ,  $\frac{dy}{dt}$  or in an implicit equation. Keep the chain rule factor, for example  $\frac{d}{dx} \tan^{-1}(xy) = \frac{xy' + y}{1 + x^2y^2}$ .

### 3. Second derivative of a parametric curve (17 questions)

1. Find  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$ .
2.  $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ , simplified. A "show that" usually needs an identity here.
3. Differentiate  $\frac{dy}{dx}$  with respect to  $t$  (quotient or product rule).
4. Divide by  $\frac{dx}{dt}$  and simplify to the form asked for.

#### Variations you will meet:

- **Find the value at a given  $t$ ,** sometimes  $t = \ln a$ : use exact values such as  $\sinh(\ln 2) = \frac{3}{4}$ ,  $\cosh(\ln 2) = \frac{5}{4}$ .
- **Find the point where  $\frac{d^2y}{dx^2}$  takes a given value:** set your expression equal to it, solve for  $t$ , then find  $x$  and  $y$  at that  $t$ .
- **"Find the set of values of  $t$  for which  $\frac{d^2y}{dx^2} < 0$ ":** factorise the expression and look at the sign of each factor over the given range of  $t$ .
- **"In the form  $at^m(1 - t^2)^n \dots$ ":** take out the lowest power of each bracket as a common factor, for example  $(1 - t^2)^{-3/2} + t^2(1 - t^2)^{-5/2} = (1 - t^2)^{-5/2} [(1 - t^2) + t^2]$ .

### 4. Second derivative from an implicit equation (10 questions)

1. **Differentiate every term with respect to  $x$ :** a term in  $y$  gets a factor  $y'$ ; a product of  $x$  and  $y$  needs the product rule. The right-hand side constant gives 0.
2. **Substitute the point** to find  $y'$  (usually a "show that": show the substitution line).
3. **Differentiate the equation from step 1 again.** Every  $y'$  term is a product or a power, so it produces a  $y''$  and a  $(y')^2$  term. Keep brackets such as  $(x + y)^2$  factorised; don't expand them first.
4. **Substitute  $x, y$  and  $y'$ ,** then solve the linear equation for  $y''$ .

#### Variations you will meet:

- **Logarithms and inverse functions of a product:** split a logarithm first, for example  $\ln(x^2y) = 2 \ln x + \ln y$  differentiates to  $\frac{2}{x} + \frac{y'}{y}$ , which is easier to differentiate again. An inverse function of a product needs the chain rule with the product rule inside:  $\frac{d}{dx} \tan^{-1}(xy) = \frac{xy' + y}{1 + x^2y^2}$ .
- **Quotients in the first derivative:** you can differentiate the quotient again, or first multiply out the denominator and differentiate the product. Either is fine; substitute numbers as early as possible.

- Followed by a Maclaurin's series for  $y$  (type 1).

## Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

### Example 1

(a) Show that, for  $x > 0$ ,  $\frac{d}{dx}(\cosh^{-1}(x^2 + 1)) = \frac{2}{\sqrt{x^2 + 2}}$ . [3]

(b) Find the exact value of  $\frac{d}{dx}(\operatorname{sech} 2x)$  when  $x = \ln 2$ . [3]

(a) By MF19 and the chain rule, with  $u = x^2 + 1$ :

$$\frac{d}{dx} \cosh^{-1}(x^2 + 1) = \frac{1}{\sqrt{(x^2 + 1)^2 - 1}} \times 2x.$$

**B1** for  $\frac{1}{\sqrt{u^2 - 1}}$ , **M1** for  $\times 2x$ .

$(x^2 + 1)^2 - 1 = x^4 + 2x^2 = x^2(x^2 + 2)$ , and  $\sqrt{x^2} = x$  because  $x > 0$ , so the derivative is  $\frac{2x}{x\sqrt{x^2 + 2}} = \frac{2}{\sqrt{x^2 + 2}}$ . **A1**, the given answer with the factorising shown.

(b)  $\frac{d}{dx}(\operatorname{sech} 2x) = -2 \operatorname{sech} 2x \tanh 2x$ . **B1**

At  $x = \ln 2$ ,  $2x = \ln 4$ :  $\cosh(\ln 4) = \frac{4 + \frac{1}{4}}{2} = \frac{17}{8}$  and  $\sinh(\ln 4) = \frac{15}{8}$ , so  $\operatorname{sech} 2x = \frac{8}{17}$  and  $\tanh 2x = \frac{15}{17}$ . **M1** for exact values from exponentials.

$$-2 \times \frac{8}{17} \times \frac{15}{17} = -\frac{240}{289}.$$

**A1**

### Example 2

The curve  $C$  has equation  $y^3 + 2xy - x^2 = 1$ .

(a) Show that, at the point  $(2, 1)$  on  $C$ ,  $\frac{dy}{dx} = \frac{2}{7}$ . [3]

(b) Find the value of  $\frac{d^2y}{dx^2}$  at the point (2, 1). [5]

(a) Differentiating with respect to  $x$ :

$$3y^2y' + 2xy' + 2y - 2x = 0.$$

**B1** for  $3y^2y'$ , **B1** for  $2xy' + 2y$ .

At (2, 1):  $3y' + 4y' + 2 - 4 = 0$ , so  $7y' = 2$  and  $y' = \frac{2}{7}$ . **B1**, with the substitution shown.

(b) Differentiate again, term by term:

$$\underbrace{6y(y')^2 + 3y^2y''}_{\text{from } 3y^2y'} + \underbrace{2y' + 2xy''}_{\text{from } 2xy'} + 2y' - 2 = 0.$$

**B1** for the first bracket, **B1** for the second, **B1** for  $2y' - 2$ .

At (2, 1) with  $y' = \frac{2}{7}$ :  $6 \times \frac{4}{49} + 3y'' + \frac{4}{7} + 4y'' + \frac{4}{7} - 2 = 0$ . **M1** for substituting all three values.

$$7y'' = 2 - \frac{8}{7} - \frac{24}{49} = \frac{18}{49} \Rightarrow y'' = \frac{18}{343}.$$

**A1**

### Example 3

A curve has parametric equations  $x = t - \tanh t$ ,  $y = \operatorname{sech} t$ , for  $t > 0$ .

(a) Show that  $\frac{dy}{dx} = -\operatorname{cosech} t$ . [3]

(b) Show that  $\frac{d^2y}{dx^2} = \frac{\cosh^3 t}{\sinh^4 t}$ . [4]

(c) Find the exact value of  $\frac{d^2y}{dx^2}$  when  $t = \ln 2$ . [2]

(a)  $\frac{dx}{dt} = 1 - \operatorname{sech}^2 t = \tanh^2 t$  and  $\frac{dy}{dt} = -\operatorname{sech} t \tanh t$ . **B1** for both.

$$\frac{dy}{dx} = \frac{-\operatorname{sech} t \tanh t}{\tanh^2 t} = -\frac{\operatorname{sech} t}{\tanh t} = -\frac{1}{\cosh t} \times \frac{\cosh t}{\sinh t} = -\operatorname{cosech} t.$$

**M1** for dividing the right way round, **A1** for the given answer.

(b)  $\frac{d}{dt}(-\operatorname{cosech} t) = \operatorname{cosech} t \coth t = \frac{\cosh t}{\sinh^2 t}$ . **M1 A1**

Now divide by  $\frac{dx}{dt} = \tanh^2 t = \frac{\sinh^2 t}{\cosh^2 t}$ : **M1**

$$\frac{d^2y}{dx^2} = \frac{\cosh t}{\sinh^2 t} \times \frac{\cosh^2 t}{\sinh^2 t} = \frac{\cosh^3 t}{\sinh^4 t}.$$

**A1**

(c)  $\sinh(\ln 2) = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$  and  $\cosh(\ln 2) = \frac{5}{4}$ . **M1**

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{5}{4}\right)^3}{\left(\frac{3}{4}\right)^4} = \frac{125}{64} \times \frac{256}{81} = \frac{500}{81}.$$

**A1****Example 4**

(a) Find the Maclaurin's series for  $\ln(1 + \sinh x)$  up to and including the term in  $x^3$ . **[6]**

(b) Use your answer to find an approximation to  $\int_0^{0.2} \ln(1 + \sinh x) dx$ , giving your answer as a fraction in its lowest terms. **[2]**

**(a)** Let  $f(x) = \ln(1 + \sinh x)$ . Then  $f'(x) = \frac{\cosh x}{1 + \sinh x}$ . **B1**

By the quotient rule, using  $\cosh^2 x = 1 + \sinh^2 x$ :

$$f''(x) = \frac{\sinh x(1 + \sinh x) - \cosh^2 x}{(1 + \sinh x)^2} = \frac{\sinh x + \sinh^2 x - 1 - \sinh^2 x}{(1 + \sinh x)^2} = \frac{\sinh x - 1}{(1 + \sinh x)^2}.$$

**B1**

Write  $f''(x) = (\sinh x - 1)(1 + \sinh x)^{-2}$ ; the product rule gives

$$f'''(x) = \cosh x (1 + \sinh x)^{-2} - 2(\sinh x - 1) \cosh x (1 + \sinh x)^{-3}.$$

**B1**

At  $x = 0$ :  $f(0) = \ln 1 = 0$ ,  $f'(0) = 1$ ,  $f''(0) = -1$ ,  $f'''(0) = 1 + 2 = 3$ . **M1** for evaluating the derivatives at 0.

$$\ln(1 + \sinh x) = 0 + x - \frac{x^2}{2!} + \frac{3x^3}{3!} + \dots = x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \dots$$

**M1 A1**

(Check with MF19:  $\sinh x = x + \frac{x^3}{6} + \dots$ , and  $\ln(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3}$  gives  $x + \frac{x^3}{6} - \frac{x^2}{2} + \frac{x^3}{3} = x - \frac{1}{2}x^2 + \frac{1}{2}x^3$ .)

**(b)**

$$\int_0^{0.2} \left( x - \frac{1}{2}x^2 + \frac{1}{2}x^3 \right) dx = \left[ \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{8} \right]_0^{0.2} = \frac{1}{50} - \frac{1}{750} + \frac{1}{5000} = \frac{283}{15000}.$$

**M1** for integrating the series, **A1**.

**Common mistakes**

- **Not dividing by  $\frac{dx}{dt}$  for the second derivative.** Examiners comment on this almost every year: after differentiating  $\frac{dy}{dx}$  with respect to  $t$ , you must divide by  $\frac{dx}{dt}$ . And  $\frac{d^2y}{dx^2}$  is never  $\frac{d^2y}{dt^2} \div \frac{d^2x}{dt^2}$ .

- **Losing a term in the second implicit differentiation.**  $3y^2y'$  is a product, so it gives  $6y(y')^2 + 3y^2y''$ ;  $xy'$  gives  $y' + xy''$ . Treat each  $y'$  as something that must itself be differentiated.
- **Expanding brackets first.** Multiplying out  $(x + y)^3$  or a long quotient before differentiating wastes time and invites errors; reports say factorised forms are more successful.
- **Wrong signs from hyperbolic derivatives.**  $\frac{d}{dx} \cosh x = +\sinh x$  (no minus sign), but  $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$ .
- **Missing the chain-rule factor**, for example writing  $\frac{d}{dx} \sinh^{-1}(2x) = \frac{1}{\sqrt{1+4x^2}}$  instead of  $\frac{2}{\sqrt{1+4x^2}}$ .
- **Losing a sign when substituting.** A final answer such as  $y''$  with the minus sign dropped is a common loss; write each substituted term in brackets.
- **Trusting a wrong  $f'(0)$ .** Reports mention candidates who carried a wrong  $f'(0)$  through a whole Maclaurin's series. Check it quickly with a standard series if you can.
- **Forgetting the factorials.** The coefficient of  $x^2$  is  $\frac{f''(0)}{2}$ , of  $x^3$  is  $\frac{f'''(0)}{6}$ . When deducing  $f^{(4)}(0)$  from a series, multiply the coefficient by  $4! = 24$ , not by 4 or 2.
- **Differentiating many times when a standard series is quicker.** Reports note that finding four derivatives led to more errors than combining MF19 series, unless the question says "from first principles".
- **Leaving  $f(0)$  unevaluated**, such as  $\tanh^{-1} \frac{1}{3}$  when the form is  $\ln a + bx + cx^2$ . Use the logarithmic form.

## How to get the marks

- **"Show that"** with a given derivative: write the unsimplified derivative first (this line usually earns the method mark), then every simplification step. In an implicit "show that", write the line with the point's values substituted before the answer.
- **Differentiating a given MF19 result** ("show that  $\frac{d}{dx} \tanh^{-1} x = \dots$ ") needs a derivation, by implicit differentiation or from the logarithmic form. Quoting it earns nothing.
- **Second derivatives:** each correctly differentiated term usually earns its own **B1**, so show the full differentiated equation before you substitute. The **M1** for substituting needs a  $y''$  in your equation.
- **Parametric  $\frac{d^2y}{dx^2}$ :** marks are for  $\frac{d}{dt} \left( \frac{dy}{dx} \right)$  (product or quotient rule) and then for dividing by  $\frac{dx}{dt}$ ; say "chain rule" or write the formula so the method is clear.
- **Maclaurin's series:** a method mark needs your derivatives evaluated at 0 (with an attempt at the second derivative), then another for putting them into the formula. The final **A1** needs simplified coefficients (for example  $2!$  written as 2) and exact values, such as  $\ln 10$  rather than 2.303.
- **"Use standard results from MF19"** means combine the given series; differentiating instead may still be allowed ("or otherwise"), but "from first principles" means you must differentiate.
- **"Exact"** means fractions, surds, logarithms and  $e$ , no decimals.
- **Method marks follow through.** A wrong  $y'$  used correctly in the second derivative still earns the method marks, so keep going.

---

## Quick check

1. Differentiate (a)  $y = \sinh^{-1}(3x)$ , (b)  $y = e^x \operatorname{sech} x$ .
2. The curve  $xe^y + y = 2$  passes through  $(2, 0)$ . Find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at this point.
3. A curve has parametric equations  $x = 3t - t^3$ ,  $y = 3t^2$ . Find  $\frac{d^2y}{dx^2}$  in terms of  $t$ , and its value when  $t = 2$ .
4. Use standard results from MF19 to find the Maclaurin's series for  $e^x \cosh x$  up to and including the term in  $x^3$ .

## Answers

- (a)  $\frac{dy}{dx} = \frac{3}{\sqrt{1+9x^2}}$ . (b) By the product rule,  $\frac{dy}{dx} = e^x \operatorname{sech} x - e^x \operatorname{sech} x \tanh x = e^x \operatorname{sech} x (1 - \tanh x)$ .
- Differentiating:  $e^y + xe^y y' + y' = 0$ . At  $(2, 0)$ :  $1 + 2y' + y' = 0$ , so  $y' = -\frac{1}{3}$ . Again:  $e^y y' + e^y y' + xe^y (y')^2 + xe^y y'' + y'' = 0$ . At  $(2, 0)$ :  $-\frac{1}{3} - \frac{1}{3} + 2 \times \frac{1}{9} + 2y'' + y'' = 0$ , so  $3y'' = \frac{4}{9}$  and  $y'' = \frac{4}{27}$ .
- $\frac{dx}{dt} = 3 - 3t^2$ ,  $\frac{dy}{dt} = 6t$ , so  $\frac{dy}{dx} = \frac{2t}{1-t^2}$ . Then  $\frac{d}{dt} \left( \frac{2t}{1-t^2} \right) = \frac{2(1-t^2)+4t^2}{(1-t^2)^2} = \frac{2(1+t^2)}{(1-t^2)^2}$ , and dividing by  $3(1-t^2)$  gives  $\frac{d^2y}{dx^2} = \frac{2(1+t^2)}{3(1-t^2)^3}$ . At  $t = 2$ :  $\frac{2 \times 5}{3 \times (-27)} = -\frac{10}{81}$ .
- $\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) \left(1 + \frac{x^2}{2}\right) = 1 + x + \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(\frac{1}{6} + \frac{1}{2}\right)x^3 + \dots = 1 + x + x^2 + \frac{2}{3}x^3$ . (Check:  $e^x \cosh x = \frac{1}{2}(e^{2x} + 1)$ , whose  $x^3$  coefficient is  $\frac{1}{2} \times \frac{8}{6} = \frac{2}{3}$ .)

## Past questions to try

- 9231/22/M/J/22 Q5**: implicit differentiation twice, then the Maclaurin's series for  $y$ .
- 9231/22/M/J/25 Q6**: the derivative of an inverse  $\operatorname{sech}$ , then a parametric second derivative.
- 9231/23/O/N/23 Q3**: a Maclaurin's series with an inverse  $\tanh$ , in the form  $\ln a + bx + cx^2$ .
- 9231/22/O/N/24 Q2**: implicit second derivative with a logarithm of  $xy$ .