

Equilibrium of a rigid body

A2

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What you need to know

By the end of this topic you should be able to:

1. **Find the moment of a force about a point:** the force times its perpendicular distance from the point, with a turning sense (clockwise or anticlockwise). All forces lie in one plane, so moments are just numbers with a sign; you never need them as vectors.
2. **Replace gravity by one force at the centre of mass.** The weight of a rigid body acts as if it all pulled at a single point, its centre of mass G . For a uniform body, G lies on every line or plane of symmetry, which often fixes it straight away (the middle of a rod, the centre of a rectangle or disc).
3. **Use given centres of mass** for a triangular lamina and the other shapes in the MF19 list (hemisphere, hemispherical shell, arc, sector, cone). You don't have to prove them.
4. **Find the centre of mass of a composite body** by treating each piece as a particle at its own centre of mass: shapes joined together, shapes with a piece cut out, and bodies with particles attached. Only simple cases, such as an L-shaped lamina or a cone joined to a hemisphere.
5. **Use the two conditions for equilibrium:** a rigid body acted on by forces in one plane is in equilibrium exactly when the forces add to zero (resolve in two directions) **and** their moments about any one point add to zero. It works both ways: if both hold, the body is in equilibrium.
6. **Solve equilibrium problems for one rigid body**, including deciding whether it **topples** (turns over about an edge) or **slides** (friction gives way).

Understand it

Moments

A force can push a body along, and it can also turn it. How strongly it turns the body about a point P is its **moment** about P :

$$\text{moment} = \text{force} \times \text{perpendicular distance from } P \text{ to the line of the force.}$$

A 12 N force whose line passes 0.5 m from P has moment 6 N m about P ; so does a 20 N force whose line passes 0.3 m away. Moments have a sense: choose clockwise or anticlockwise as positive and stick to it. A force whose line passes **through** P has zero moment about P , and that is the key to choosing where to take moments.

When the force isn't at right angles to the distance you know, either find the perpendicular distance with trigonometry, or split the force into two components and add their moments.

Centre of mass and symmetry

The weight of a rigid body is spread over every bit of it, but for moments and resolving it acts exactly like a single force W at the **centre of mass** G . For a **uniform** body (the same density everywhere) G lies on every axis of symmetry: a uniform rod has G at its midpoint, a rectangle or square at the crossing of its diagonals, a disc or cylinder at its centre.

For shapes without enough symmetry, MF19 gives the result:

- **Triangular lamina:** $\frac{2}{3}$ of the way along a median from a vertex. Equivalently, its coordinates are the **mean of the three vertices**: the triangle $(0, 0)$, $(6, 0)$, $(0, 9)$ has G at $(2, 3)$. This version is much quicker in exams.
- **Solid hemisphere** of radius r : $\frac{3}{8}r$ from the centre, on the axis. **Hemispherical shell:** $\frac{1}{2}r$ from the centre.
- **Solid cone or pyramid** of height h : $\frac{3}{4}h$ from the vertex, so $\frac{1}{4}h$ from the base.
- **Circular arc** of radius r and angle 2α : $\frac{r \sin \alpha}{\alpha}$ from the centre. **Sector** of radius r and angle 2α : $\frac{2r \sin \alpha}{3\alpha}$ from the centre. Both lie on the line of symmetry, and α is in radians. A **quarter circle** has $2\alpha = \frac{\pi}{2}$, so its centre of mass is $\frac{4\sqrt{2}r}{3\pi}$ from the centre along the bisector, which is $\frac{4r}{3\pi}$ from each straight edge.

Composite bodies: an equivalent system of particles

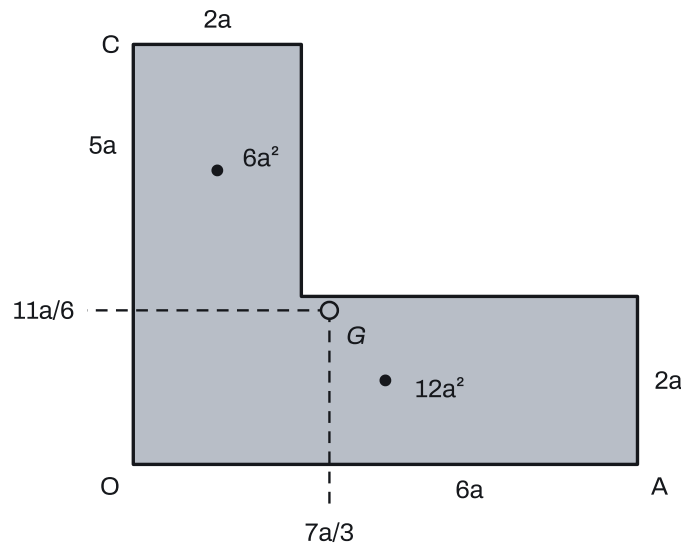
To find G for a body made of simpler pieces, replace each piece by a particle of the same mass at that piece's centre of mass. The moment of the whole equals the sum of the moments of the pieces, so

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}, \quad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}.$$

For a uniform lamina, mass is proportional to **area**; for a uniform solid, to **volume**; for a shell or hollow body made of thin uniform material, to **surface area**. So you can use areas or volumes in place of masses. Mixing a lamina with a particle needs real masses (or the ratio of masses).

Example: a uniform rod AB of mass 4 kg and length 2 m, with a 1 kg particle at A and a 3 kg particle at B . Taking moments about A : $8\bar{x} = 4(1) + 1(0) + 3(2) = 10$, so $\bar{x} = \frac{5}{4}$ m from A .

A **table** keeps the work tidy: one row per piece with its mass (or area) and the distance of its centre of mass from each reference line, then a row for the whole body.



For this L-shaped lamina (Example 1 works it in full), taking moments about OC gives $18a^2 \bar{x} = 12a^2(3a) + 6a^2(a)$, so $\bar{x} = \frac{7a}{3}$.

Removing a piece. A lamina with a hole or a corner cut off is "the whole shape **minus** the removed piece". Give the removed piece a **negative** mass (area) in the table:

$$(M_{\text{whole}} - m_{\text{cut}}) \bar{x} = M_{\text{whole}} x_{\text{whole}} - m_{\text{cut}} x_{\text{cut}}.$$

Equilibrium of a rigid body

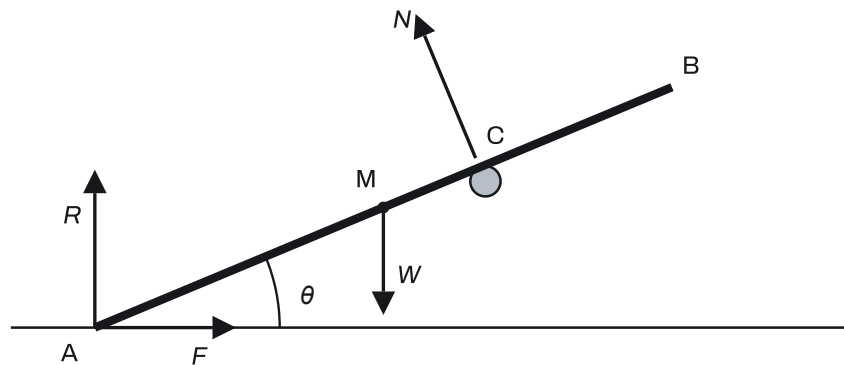
A particle is in equilibrium when the forces on it balance. A rigid body also must not start to turn, so it needs **two** conditions:

1. the **resultant force is zero**: resolving in any two directions gives zero;
2. the **total moment is zero** about **any** point you choose.

That gives three independent equations (two resolutions and one moments equation, or one resolution and two moments equations), so you can find up to three unknowns. Taking moments about a point where unknown forces act removes those forces from the equation, so choose that point with care.

Contact forces:

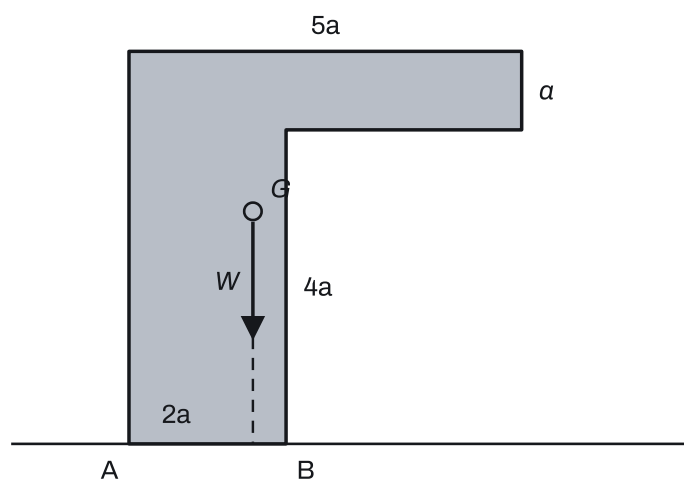
- A **smooth** surface, peg or cylinder gives only a **normal reaction**, at right angles to the surface. A rod resting on a smooth peg or cylinder gets a reaction at right angles to the **rod** (for a cylinder, it also passes through the cylinder's axis).
- A **rough** surface also gives **friction** F along the surface, with $F \leq \mu R$. Friction acts to oppose the way the body is about to move. "About to slip", "on the point of slipping" or "limiting equilibrium" means $F = \mu R$.
- A **light inextensible string** pulls along its own line with tension T .



The force diagram above is Example 4. The peg pushes the rod up and back towards A , so the ground's friction at A must push towards the peg to balance it.

Toppling on a horizontal surface

A body standing on a flat base stays up while the vertical line through G passes through its base. The ground's reaction can then act anywhere under the base, including directly below G , so the weight and reaction balance with no turning. If the vertical through G falls **outside** the base, the weight turns the body about the nearest edge, and nothing can stop it: it **topples**.

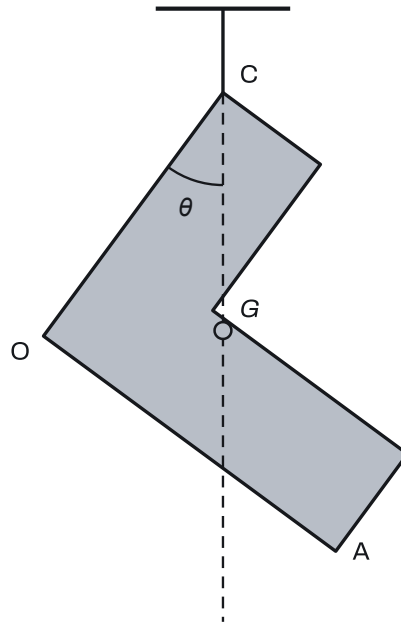


On the point of toppling about an edge B , the whole reaction is at B and G is **directly above** B . So the condition for the lamina above to stand on AB is $0 \leq \bar{x} \leq 2a$; here $\bar{x} = \frac{41a}{26} \approx 1.58a$, so it stands.

Freely suspended bodies

A body hanging freely from a point A (by a string, or on a smooth pin) settles with G **directly below** A .

Otherwise the weight would have a moment about A and swing it round. So the line AG is vertical, and every angle question comes from the right-angled triangle made by A , G and the edges through A .



Hanging from $C(0, 5a)$: G is $\frac{7a}{3}$ across from CO and $5a - \frac{11a}{6} = \frac{19a}{6}$ down from C along CO , so the angle between CO and the vertical satisfies $\tan \theta = \frac{7a/3}{19a/6} = \frac{14}{19}$.

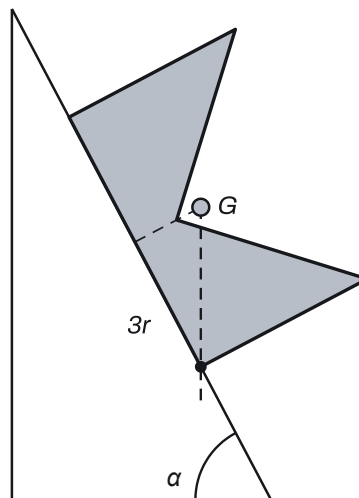
On an inclined plane: toppling or sliding

A body standing on a rough slope at angle α can fail in two ways:

- it **slides** when the slope is steeper than friction can hold: for a body with no other forces, this happens when $\tan \alpha > \mu$;
- it **topples** about the lowest point of its base when the vertical through G passes outside the base. On the point of toppling, G is vertically above that lowest point.

For a solid standing on its base (radius or half-width r , with G at height $\bar{y}$ above the base, on the axis), the right-angled triangle made by G , the foot of the axis and the lowest point of the base gives

$$\text{on the point of toppling: } \tan \alpha = \frac{r}{\bar{y}}, \quad \text{it stays up while } \tan \alpha \leq \frac{r}{\bar{y}}.$$



As the slope gets steeper, whichever of $\tan \alpha = \mu$ and $\tan \alpha = \frac{r}{y}$ is reached first decides what happens (Example 3).

Resting on a curved surface. A solid hemisphere (or an object with a hemispherical end) resting with its curved surface on a plane has its reaction and friction both acting at the contact point P . The only other force is the weight, so for no turning about P the weight must pass through P : G is **vertically above** P . Also, the radius to P is perpendicular to the plane, so the centre O of the curved surface is at a horizontal distance $r \sin \alpha$ from P .

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
Moment about P = force $\times$ perpendicular distance from P	Learn it
Composite body: $\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$, $\bar{y} = \frac{\sum m_i y_i}{\sum m_i}$; a removed piece counts with a minus sign	Learn it
Triangular lamina: $\frac{2}{3}$ along the median from a vertex	Given
Triangle's centre of mass = mean of its vertices, $(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3})$	Learn it
Solid hemisphere, radius r : $\frac{3}{8}r$ from the centre	Given
Hemispherical shell, radius r : $\frac{1}{2}r$ from the centre	Given
Circular arc, radius r , angle 2α : $\frac{r \sin \alpha}{\alpha}$ from the centre	Given

Formula	Status
Circular sector, radius r , angle 2α : $\frac{2r \sin \alpha}{3\alpha}$ from the centre	Given
Solid cone or pyramid, height h : $\frac{3}{4}h$ from the vertex ($\frac{1}{4}h$ from the base)	Given
Volume of a sphere $\frac{4}{3}\pi r^3$ (so a solid hemisphere $\frac{2}{3}\pi r^3$); volume of a cone $\frac{1}{3}\pi r^2 h$	Given
Area of a sector of radius r and angle θ (radians): $\frac{1}{2}r^2\theta$	Given
Surface area of a sphere $4\pi r^2$ (so a hemispherical shell $2\pi r^2$)	Given
Cylinder: volume $\pi r^2 h$, curved surface $2\pi r h$; disc (end of a closed cylinder) πr^2	Learn it
Equilibrium: resultant force zero (resolve two ways) and total moment zero about any point	Learn it
Friction: $F \leq \mu R$; $F = \mu R$ when about to slip	Learn it
Freely suspended from A : G is vertically below A	Learn it
On a horizontal surface: stands if the vertical through G meets the base; on the point of toppling, G is above the edge	Learn it
On a slope: slides first if $\mu < \tan \alpha$; topples when $\tan \alpha > \frac{r}{y}$	Learn it
Uniform rods, rectangles, discs, cylinders and spheres have G at their centre by symmetry; nothing to learn beyond that.	

How to do it

In the Paper 3 questions we have tagged for this topic (33 questions, 2021–2025; 10 of them repeat a question from another paper of the same session, so there are 23 different questions), the types came up this often. Most questions find a centre of mass first and then use it, so 23 of the 33 mix two types and the counts add up to more than 33.

Question type	Questions
Centre of mass of a composite lamina or solid	25
Toppling on a horizontal surface	9
Rigid body held by friction, strings and contact forces	8
On a rough inclined plane: toppling or equilibrium	8
Freely suspended body	6

1. Centre of mass of a composite lamina or solid (25 questions)

1. **Choose axes or reference lines**, usually the two edges the question measures from (for example "the distance from AD and from AB "). Put the origin at a corner so that most distances are positive.
2. **Make a table**: each piece, its area (lamina), volume (solid) or surface area (shell), and the distance of its centre of mass from each reference line. Use MF19 for triangles, hemispheres, cones and sectors, and symmetry for the rest. Removed pieces go in with a minus sign.
3. **Take moments** about each reference line: (total) $\bar{x} = \sum(\text{piece} \times \text{distance})$.
4. **Simplify** to the form asked for. For a "show that", keep every line of algebra.

Variations you will meet:

- **Triangles cut from a square or rectangle**, with a corner point given by a letter such as h . Use the mean of the three vertices for each triangle's centre of mass: the triangle with vertices $(4a, 0)$, $(4a, 4a)$ and $(4a - h, 0)$ has its centre of mass $4a - \frac{h}{3}$ from the y -axis. Splitting the remaining shape into different pieces (two triangles, or a rectangle and a triangle) is just as good; pick whatever gives the fewest terms.
- **A lamina given by coordinates** of its vertices (for example a trapezium): split it into rectangles and triangles, or subtract triangles from a rectangle, then take moments about the y -axis.
- **A quarter circle or other sector** joined to a rectangle: use $\frac{2r \sin \alpha}{3\alpha}$ with 2α the sector's angle. For a quarter circle it is $\frac{4r}{3\pi}$ from each straight edge, which you need when measuring from one of those edges.
- **A cone given by its semi-vertical angle θ** : its height is $\frac{\text{radius}}{\tan \theta}$; find it before working out the volume.
- **Solids joined or removed**: a cone on a cylinder, a hemisphere removed from a cone, a cylinder removed from a hemisphere or another cylinder. Measure everything from one plane face. Watch which way each piece's centre of mass lies from the joining face: a cone attached to the top of a cylinder of height h has its centre of mass at $h + \frac{1}{4}(\text{cone height})$ above the base.
- **Pieces not on the same axis** (for example a small cylinder attached off-centre to a hemisphere): you need both $\bar{x}$ and $\bar{y}$, one moments equation for each. A hemisphere's centre of mass is $\frac{3}{8}r$ **below** its plane face when it hangs underneath the axes, so give that distance a minus sign.
- **Hollow bodies** made from the same thin material (a hemispherical shell joined to a closed hollow cylinder): mass is proportional to surface area. A closed cylinder has its curved surface $2\pi ah$ **and** two ends πa^2 each, one at each end; don't forget them.
- **A particle attached** to a lamina: work with masses, not areas. If the lamina has mass m and the particle km , the particle is one more row with mass km .
- **Using symmetry**: if the shape is symmetric about the line $y = x$, then $\bar{y} = \bar{x}$ with no more work; a single line of reasoning earns the mark.

2. Toppling on a horizontal surface (9 questions)

1. **Find $\bar{x}$** (the horizontal position of G measured along the base), usually from the earlier part.
2. **Identify the edge it would topple about**: the end of the base nearer to G 's overhang.
3. **"On the point of toppling"**: set $\bar{x}$ equal to that edge's position and solve.

4. **"The set of values for which it remains in equilibrium"**: require $\bar{x}$ to be between the two ends of the base, write the inequality, clear the denominator and solve. A quadratic inequality needs its roots and a sketch; then keep only values that make physical sense (a length can't be negative or bigger than the shape allows).

For example, the Γ -shaped lamina in the diagram, with its arm of general length h (where $h \geq 2a$), has $\bar{x} = \frac{16a^2 + h^2}{2(8a + h)}$. It stands on AB when $\bar{x} \leq 2a$: $16a^2 + h^2 \leq 32a^2 + 4ah$, so $h^2 - 4ah - 16a^2 \leq 0$, which gives $h \leq 2(1 + \sqrt{5})a \approx 6.47a$.

Variations you will meet:

- **The standing edge is part of a cut**: after a corner is removed, the base may be shorter than the original side (for example it runs from A to a point E), so compare $\bar{x}$ with the position of E , not the old corner.
- **A particle attached** that makes it tip: take moments about the toppling vertex, with the reaction there contributing nothing. The lamina's weight on one side balances the particle's weight on the other: $mg \times (\text{lamina's } G \text{ to the vertex}) = kmg \times (\text{particle to the vertex})$.
- **A lamina on a smooth surface**: smoothness doesn't matter for toppling, since only the vertical reaction and the weight act.
- **A letter that also sets a corner**, such as k in a length ka : the answer is a condition on k , or on h in terms of a and k .

3. Rigid body held by friction, strings and contact forces (8 questions)

1. **Draw a force diagram**: weight at G (and any particle's weight at its point), a normal reaction at each contact at right angles to the surface (or to the rod, at a smooth peg or cylinder), friction along each rough surface opposing the motion that is about to happen, tension along each string.
2. **Take moments about the point where most unknowns act**, usually the contact point with friction and a reaction. State the point you use.
3. **Resolve** in two directions, horizontally and vertically, or along and perpendicular to a slope or rod, whichever makes fewer components.
4. **Use** $F = \mu R$ for a limiting contact, then solve the equations together.

Variations you will meet:

- **A rod resting on a smooth peg or cylinder**, with its lower end on rough ground (Example 4). Moments about the lower end give the reaction N at once; resolving gives R and F .
- **A rod against a rough vertical wall, held by a string** from a point on the rod to the wall. When the string, rod and wall make an isosceles triangle, the string makes the same angle with the wall as the rod does; its perpendicular distance from the rod's end is then (length of that side) $\times \sin(\text{angle between string and rod})$. Friction at the wall acts **up** if the rod is about to slip down, and **down** if it's about to slip up.
- **A lamina hanging from a string with a vertex touching a rough wall**: the same recipe; moments about the vertex on the wall remove the friction and the reaction.

- **A ring or disc on a slope held by a string:** moments about the centre remove the weight and the reaction, so they relate friction to tension; moments about the contact point remove friction and reaction together.
- **Two rods rigidly joined and resting on a rough cylinder** at two contact points: there are two normal reactions and two frictional forces. If both contacts are about to slip, both frictions are $\mu \times$ their reactions. Moments about each contact point in turn is often the quickest way to relate the reactions.
- **A particle of weight kW attached** somewhere on the rod: one more force in every equation, and the answer may be the value of k that makes it about to slip.

4. On a rough inclined plane: toppling or equilibrium (8 questions)

1. **Find $\bar{y}$** , the height of G above the face in contact with the plane (from the earlier part).
2. **Draw the cross-section on the slope** and mark the lowest point of the base. On the point of toppling, G is vertically above it, so $\tan \alpha = \frac{r}{\bar{y}}$ (with r the distance from the axis to that lowest point).
3. **Solve** for the angle, or for the unknown length when the angle is given.

For example, a uniform solid cylinder of radius 2 cm and height 10 cm standing on a rough slope (rough enough to stop sliding) topples when $\tan \alpha = \frac{2}{5}$, that is at $\alpha = 21.8^\circ$.

Variations you will meet:

- **"The set of values for which it is in equilibrium":** $\tan \alpha \leq \frac{r}{\bar{y}}$, then substitute $\bar{y}$ in terms of the unknown and solve the inequality (a quadratic inequality when $\bar{y}$ is a fraction with the unknown on top and bottom).
- **A hollow object standing on its rim:** the same condition, with r the radius of the rim in contact with the plane.
- **Topple or slide first?** It slides when $\tan \alpha = \mu$ and topples when $\tan \alpha = \frac{r}{\bar{y}}$; the smaller angle happens first. "The plane is rough enough to prevent sliding" means you ignore sliding.
- **Resting on a curved (hemispherical) surface:** G is vertically above the contact point P , and the centre O of the curved face is $r \sin \alpha$ horizontally from P . So the horizontal distance between O and G equals $r \sin \alpha$. For a solid hemisphere of radius r on a slope of 15° , with G on the axis $\frac{3}{8}r$ from O : $\frac{3}{8}r \sin \phi = r \sin 15^\circ$, where ϕ is the angle between the axis and the vertical, so $\sin \phi = \frac{8}{3} \sin 15^\circ = 0.690$ and $\phi = 43.6^\circ$. When something else is fixed instead (for example a flat face stays horizontal), read the horizontal distance between O and G straight from the coordinates of G .

5. Freely suspended body (6 questions)

1. **Find G** relative to two lines through the suspension point A (usually the two edges meeting there).
2. **Draw A with G directly below it.** The angle θ between an edge through A and the vertical satisfies $\tan \theta = \frac{\text{distance of } G \text{ from that edge}}{\text{distance of } G \text{ from } A \text{ measured along that edge}}$.
3. **Substitute and solve.** When G involves an unknown, this gives an equation (often a quadratic) for it.

For example, a uniform solid cone of height 12 cm and base radius 4 cm hung from a point on the rim of its base: G is on the axis 3 cm from the base, so the axis makes an angle with the vertical of $\tan^{-1} \frac{4}{3} = 53.1^\circ$.

Variations you will meet:

- **Measure from the right point.** If G is given as distances from AB and AD but the body hangs from D , the distance along DA from D is $AD - \bar{y}$, not $\bar{y}$.
- **Two answers.** A quadratic may give two lengths; keep both if both fit the shape (the question often says "possible values"), and reject any that would make a length negative or larger than the shape.
- **The angle with a different edge,** such as the angle between AB and the vertical, or with the upward vertical: redraw the right-angled triangle for that edge rather than adjusting angles afterwards.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
-

Example 1

A uniform lamina $OABDEC$ is L-shaped. O is the origin, $OA = 6a$ lies along the x -axis and $OC = 5a$ along the y -axis. The lamina is made of a rectangle $6a$ by $2a$ along OA and a rectangle $2a$ by $3a$ on top of it along OC (see the first diagram in "Understand it").

(a) Find the distances of the centre of mass of the lamina from OC and from OA . **[4]**

(b) The lamina is freely suspended from C . Find the angle between CO and the vertical. **[3]**

(a) Areas and centres of mass: bottom rectangle $12a^2$ at $(3a, a)$; upright rectangle $6a^2$ at $(a, 3.5a)$; whole lamina $18a^2$. **B1**

Moments about OC : $18a^2 \bar{x} = 12a^2(3a) + 6a^2(a) = 42a^3$, so $\bar{x} = \frac{7a}{3}$. **M1 A1**

Moments about OA : $18a^2 \bar{y} = 12a^2(a) + 6a^2(3.5a) = 33a^3$, so $\bar{y} = \frac{11a}{6}$. **A1**

(b) G is vertically below C . Along CO , G is $5a - \frac{11a}{6} = \frac{19a}{6}$ from C ; across, it is $\frac{7a}{3}$ from CO . **M1** for the right-angled triangle with both distances from C .

$$\tan \theta = \frac{7a/3}{19a/6} = \frac{14}{19} \Rightarrow \theta = 36.4^\circ.$$

A1 A1

Example 2

A uniform square lamina $ABCD$ has side $4a$. The point E on AB is such that $EB = h$, where $0 < h < 4a$. The triangle BCE is removed, leaving the lamina $AECD$.

(a) Show that the distance of the centre of mass of $AECD$ from AD is $\frac{48a^2 - 12ah + h^2}{3(8a - h)}$. [4]

(b) The lamina $AECD$ is placed in a vertical plane with its edge AE on a horizontal surface. Find the set of values of h for which it does not topple. [3]

(a) Take A as the origin with AB along the x -axis, so $B(4a, 0)$, $C(4a, 4a)$ and $E(4a - h, 0)$.

Piece	Area	Distance of centre of mass from AD
Square $ABCD$	$16a^2$	$2a$
Triangle BCE	$2ah$	$\frac{1}{3}(4a + 4a + (4a - h)) = 4a - \frac{h}{3}$
Lamina $AECD$	$16a^2 - 2ah$	$\bar{x}$

B1 for the triangle's area and distance.

$$(16a^2 - 2ah)\bar{x} = 16a^2(2a) - 2ah\left(4a - \frac{h}{3}\right) = 32a^3 - 8a^2h + \frac{2ah^2}{3}.$$

M1 A1 for the moments equation with all three terms.

Dividing by $2a(8a - h)$ and multiplying top and bottom by 3:

$$\bar{x} = \frac{96a^2 - 24ah + 2h^2}{6(8a - h)} = \frac{48a^2 - 12ah + h^2}{3(8a - h)}.$$

A1, the given answer with the working shown.

(b) The base runs from A to E , so it does not topple (about E) when $\bar{x} \leq 4a - h$. **B1**

$$48a^2 - 12ah + h^2 \leq 3(8a - h)(4a - h) = 96a^2 - 36ah + 3h^2 \Rightarrow h^2 - 12ah + 24a^2 \geq 0.$$

M1 for a three-term quadratic inequality ($8a - h > 0$, so multiplying through keeps the sign).

The roots are $h = (6 \pm 2\sqrt{3})a$, and the quadratic is ≥ 0 outside them. Since $h < 4a < (6 + 2\sqrt{3})a$, the lamina stands for $0 < h \leq (6 - 2\sqrt{3})a$, about $2.54a$. **A1**

Example 3

A uniform solid cylinder has radius $3r$ and height $4r$. A solid cone of radius $3r$ and height $3r$ is removed from it, the base of the cone being the top face of the cylinder and the axes coinciding. The remaining object has a full circular face at the bottom.

(a) Find the height of the centre of mass of the object above its full circular face. [4]

(b) The object stands with its full circular face on a rough plane inclined at α to the horizontal. The plane is rough enough to prevent sliding. Find the value of α for which the object is on the point of toppling. [2]

(c) Instead, the coefficient of friction is 0.75 and α is increased slowly from 0. Determine whether the object slides or topples first. [2]

(a)

Piece	Volume	Height of centre of mass
Cylinder	$\pi(3r)^2(4r) = 36\pi r^3$	$2r$
Cone (removed)	$\frac{1}{3}\pi(3r)^2(3r) = 9\pi r^3$	$4r - \frac{1}{4}(3r) = \frac{13r}{4}$
Object	$27\pi r^3$	$\bar{y}$

B1 for the volumes, **B1** for the cone's centre of mass, measured down from the top face.

$$27\pi r^3 \bar{y} = 36\pi r^3(2r) - 9\pi r^3 \left(\frac{13r}{4} \right) = 72\pi r^4 - \frac{117}{4}\pi r^4 = \frac{171}{4}\pi r^4 \Rightarrow \bar{y} = \frac{19r}{12}.$$

M1 A1

(b) On the point of toppling, G is vertically above the lowest point of the base, so (see the fifth diagram in "Understand it")

$$\tan \alpha = \frac{3r}{19r/12} = \frac{36}{19} \Rightarrow \alpha = 62.2^\circ.$$

M1 A1

(c) It would slide when $\tan \alpha = 0.75$, that is at $\alpha = 36.9^\circ$. **M1** This is reached before 62.2° (since $0.75 < \frac{36}{19}$), so the object **slides** first. **A1**

Example 4

A uniform rod AB of length $6a$ and weight W has its end A on rough horizontal ground. The rod rests on a smooth peg at C , where $AC = 4a$, and makes an angle θ with the ground, where $\tan \theta = \frac{5}{12}$. The rod is in a vertical plane perpendicular to the peg and is about to slip.

Find the coefficient of friction between the rod and the ground. [7]

The forces are drawn in the second diagram in "Understand it". $\sin \theta = \frac{5}{13}$, $\cos \theta = \frac{12}{13}$. The peg's reaction N is at right angles to the rod; at A there is a normal reaction R and friction F towards the peg.

Moments about A : $N(4a) = W(3a \cos \theta) = W \cdot 3a \cdot \frac{12}{13}$, so $N = \frac{9W}{13}$. **M1 A1**

Resolving vertically: $R + N \cos \theta = W$, so $R = W - \frac{9}{13} \cdot \frac{12}{13} W = \frac{61W}{169}$. **M1 A1**

Resolving horizontally: $F = N \sin \theta = \frac{9}{13} \cdot \frac{5}{13} W = \frac{45W}{169}$. **B1**

About to slip: $F = \mu R$, so **M1**

$$\mu = \frac{45W/169}{61W/169} = \frac{45}{61} = 0.738 \text{ (3 s.f.)}.$$

A1

Common mistakes

- **Getting a piece's centre of mass the wrong way round.** Examiners often see $\frac{2}{3}$ used where $\frac{1}{3}$ belongs for a triangle, or a cone's centre of mass measured from the wrong end. For a triangle, use the mean of its three vertices; for a cone on top of another piece, add $\frac{1}{4}$ of its height to the height of its base.
- **Equations that are not dimensionally consistent.** Every term of a moments equation must have the same dimensions (for a lamina, a^3 ; for a solid, a^4). Reports stress that a quick check here catches missing factors, and without a consistent equation you can't go further.
- **Forgetting parts of a hollow body**, such as the ends of a closed cylinder, or using volume for a shell. A shell's mass is proportional to its surface area.
- **Measuring from the wrong line**, for example taking moments about A and then forgetting to subtract the distance from A to the line you actually want. Say which line or point you take moments about, and convert at the end.
- **Missing a symmetry.** When a lamina is symmetric about $y = x$, the second coordinate is the same as the first; some candidates did the whole calculation again (and made slips).
- **Missing forces or wrong directions.** Draw the force diagram first. A smooth peg or cylinder pushes at right angles to the rod, not vertically; friction opposes the motion that is about to happen.
- **Taking moments about an awkward point.** Reports note that candidates who took moments about the contact point (where friction and a reaction act) usually succeeded, and those who chose other points often left out terms. Pick the point that removes the most unknowns and say what it is.
- **Using the wrong inequality or keeping impossible answers.** The set of values for equilibrium needs the correct end of the base, and the answers must fit the shape (lengths between 0 and the side length).
- **Giving an angle with the wrong edge or the wrong vertical.** Draw the suspended body with G below the point of suspension, and read off the angle the question asks for.

How to get the marks

- **Show your table** of masses (areas, volumes) and distances, even when you then write the moments equation straight away. The first mark is usually for correct areas and distances, and it can be earned from the table.
- **Moments equations earn method marks** when they have the right number of terms and are dimensionally correct, even with a sign slip. Write the equation in full before simplifying.
- **"Show that":** the answer is printed, so show enough algebra that each step is clear, and end with exactly the printed form.

- **State the condition in words or symbols** before you use it: "on the point of toppling, G is vertically above E , so $\bar{x} = 4a - h$ ", or " $F = \mu R$ as the rod is about to slip". A mark is often for the condition itself.
- **Set of values:** a strict or non-strict inequality is usually accepted, but give the final set clearly (such as $0 < h \leq (6 - 2\sqrt{3})a$), and don't include values the shape can't have.
- **Two possible values:** if the question says "possible values", expect two, and give both if both are valid.
- **Accuracy:** give exact answers (fractions, surds) when the data are exact; angles to 1 decimal place in degrees, other non-exact answers to 3 significant figures.
- **Follow-through:** later parts usually use your answer from part (a) with follow-through, so carry on with it even if you doubt it.

Quick check

1. A uniform rod AB of mass 3 kg and length 2 m has particles of mass 1 kg at A and 2 kg at B . Find the distance of the centre of mass of the system from A .
2. A uniform lamina is made of a square $ABCD$ of side $2a$ and a triangle DCE joined along DC . With A as origin, $B(2a, 0)$, $C(2a, 2a)$, $D(0, 2a)$ and $E(ka, 3a)$, where $k > 0$. The lamina stands in a vertical plane with AB on a horizontal surface. Find the set of values of k for which it does not topple.
3. A uniform solid hemisphere is freely suspended from a point on the rim of its plane face. Find the angle between its plane face and the vertical.
4. A uniform rod AB of length $2a$ and weight W has its end A on rough horizontal ground. It is held at 60° to the horizontal by a light string attached at B and at right angles to the rod, in the vertical plane containing the rod. Find the tension, and the least possible value of the coefficient of friction.
5. A uniform solid cube stands on a rough plane whose angle to the horizontal is slowly increased. The coefficient of friction is 0.8. Does the cube slide or topple first, and at what angle?

Answers

1. Moments about A : $6\bar{x} = 3(1) + 1(0) + 2(2) = 7$, so $\bar{x} = \frac{7}{6}$ m.
2. Square: area $4a^2$, centre of mass a from AD . Triangle: area $\frac{1}{2}(2a)(a) = a^2$, centre of mass $\frac{1}{3}(0 + 2a + ka) = \frac{(2+k)a}{3}$ from AD . So $5a^2 \bar{x} = 4a^3 + \frac{(2+k)a^3}{3}$, giving $\bar{x} = \frac{(14+k)a}{15}$. It stands while $\bar{x} \leq 2a$: $14 + k \leq 30$, so $0 < k \leq 16$.
3. G is on the axis $\frac{3}{8}r$ from the centre O of the plane face. Hanging from A on the rim, $AO = r$ along the face and $OG = \frac{3}{8}r$ at right angles to it, so $\tan \theta = \frac{3r/8}{r} = \frac{3}{8}$ and $\theta = 20.6^\circ$.
4. Moments about A : $T(2a) = W(a \cos 60^\circ)$, so $T = \frac{W}{4}$. The string pulls at right angles to the rod, at 60° to the vertical. Vertically: $R = W - T \cos 60^\circ = \frac{7W}{8}$. Horizontally: $F = T \sin 60^\circ = \frac{\sqrt{3}W}{8}$. Since $F \leq \mu R$, $\mu \geq \frac{\sqrt{3}}{7} = 0.247$.
5. It would topple when $\tan \alpha = \frac{\text{half-side}}{\text{half-side}} = 1$, at 45° . It slides when $\tan \alpha = 0.8$, at 38.7° . That comes first, so it **slides** at 38.7° .

Past questions to try

- **9231/33/M/J/25 Q4**: a lamina with a particle attached, on the point of toppling about a vertex.
- **9231/32/O/N/22 Q2**: a triangle with a piece removed, and the set of values for which it stands.
- **9231/33/O/N/23 Q3**: a square lamina hanging from a string with a vertex on a rough wall.
- **9231/32/M/J/23 Q4**: a hemisphere with a cylinder attached, resting on its curved surface on a rough slope.