Cambridge AS & A Level · Further Mathematics 9231

Further Mathematics 9231 formulas and definitions

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1 Further Pure Mathematics 1

1.1 Roots of polynomial equations

Quadratic $ax^2 + bx + c = 0$: $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

Cubic $ax^3 + bx^2 + cx + d = 0$: $\sum \alpha = -\frac{b}{a}$, $\sum \alpha\beta = \frac{c}{a}$, $\alpha\beta\gamma = -\frac{d}{a}$

Quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$: $\sum \alpha = -\frac{b}{a}$, $\sum \alpha\beta = \frac{c}{a}$, $\sum \alpha\beta\gamma = -\frac{d}{a}$, $\alpha\beta\gamma\delta = \frac{e}{a}$

$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$ (any degree)

$\sum \alpha^3 = \left(\sum \alpha\right)^3 - 3\sum \alpha \sum \alpha\beta + 3\sum \alpha\beta\gamma$ (cubic or quartic)

Cubic: $\sum \frac{1}{\alpha} = \dfrac{\sum \alpha\beta}{\alpha\beta\gamma}$; quartic: $\sum \frac{1}{\alpha} = \dfrac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta}$

Cubic: $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = \left(\sum \alpha\beta\right)^2 - 2\alpha\beta\gamma \sum \alpha$

Cubic: $\sum \frac{1}{\alpha^2} = \dfrac{\left(\sum \alpha\beta\right)^2 - 2\alpha\beta\gamma \sum \alpha}{(\alpha\beta\gamma)^2}$

Power sums from the equation: $aS_{n+3} + bS_{n+2} + cS_{n+1} + dS_n = 0$ for a cubic (one more term for a quartic); S_0 = number of roots

New roots $y = \mathrm{g}(\alpha)$: write x in terms of y and substitute into the equation

1.2 Rational functions and graphs

Vertical asymptotes: solve (denominator) $= 0$

Degree of top < degree of bottom: asymptote $y = 0$

Same degree: $y = \dfrac{\text{leading coefficient of top}}{\text{leading coefficient of bottom}}$

Quadratic over linear: $\dfrac{ax^2 + bx + c}{px + q} = mx + k + \dfrac{r}{px + q}$, oblique asymptote $y = mx + k$

Stationary points: $\dfrac{dy}{dx} = 0$, that is $vu' - uv' = 0$

$ax^2 + bx + c = 0$ has real roots $\iff b^2 - 4ac \geqslant 0$

Values taken: rewrite $y = f(x)$ as a quadratic in x and use discriminant $\geqslant 0$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f(x)^2}$$

$|f(x)| < k \iff -k < f(x) < k$

1.3 Summation of series

$\sum_{r=1}^{n} c = cn$ for a constant c

$\sum(a\,u_r + b\,v_r) = a\sum u_r + b\sum v_r$

$$\sum_{r=m}^{n} u_r = S_n - S_{m-1}$$

$$\sum_{r=1}^{n} \big(f(r) - f(r+1)\big) = f(1) - f(n+1)$$

$$\sum_{r=1}^{n} \big(f(r) - f(r+2)\big) = f(1) + f(2) - f(n+1) - f(n+2)$$

$$\frac{1}{(dr+a)(dr+a+d)} = \frac{1}{d}\left(\frac{1}{dr+a} - \frac{1}{dr+a+d}\right)$$

Convergent means S_n tends to a finite limit; $S_\infty = \lim_{n\to\infty} S_n$

$x^n \to 0$ for $-1 < x < 1$; $\dfrac{1}{n^k} \to 0$ for $k > 0$

$\ln a - \ln b = \ln\dfrac{a}{b}$, $\ln(ab) = \ln a + \ln b$, $x^{\log_x a} = a$

1.4 Matrices

$\mathbf{AB}$: row of $\mathbf{A}$ times column of $\mathbf{B}$; $(m \times n)(n \times p) = m \times p$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

3×3 determinant: expand along a row with signs $+\ -\ +$

Singular $\Leftrightarrow \det\mathbf{M} = 0$; non-singular $\Leftrightarrow \det\mathbf{M} \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

3×3 inverse $= \dfrac{1}{\det\mathbf{A}} \times$ (transpose of the matrix of cofactors)

$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$; $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$

$\det(\mathbf{AB}) = \det\mathbf{A}\det\mathbf{B}$; $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det\mathbf{A}}$

Columns of $\mathbf{M}$ = images of $(1,0)$ and $(0,1)$

Rotation θ anticlockwise about O: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Stretch, enlargement, shear and reflection matrices (table above)
$\mathbf{AB}$ = "$\mathbf{B}$ first, then $\mathbf{A}$"
Area of image $= \lvert\det \mathbf{M}\rvert \times$ area of object
Invariant point: $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
Invariant line $y = mx$: image (X, Y) of (x, mx) satisfies $Y = mX$

1.5 Polar coordinates

$x = r\cos\theta, y = r\sin\theta$
$r^2 = x^2 + y^2, \tan\theta = \dfrac{y}{x}$ (check the quadrant)
Convention: $r \geqslant 0$; points are written (r, θ)
Area $= \dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\, d\theta$
Sector of a circle: area $\frac{1}{2}r^2\theta$; triangle: area $\frac{1}{2}ab\sin C$
So $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ and $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$
$\int \tan x\, dx = \ln\lvert\sec x\rvert = -\ln\lvert\cos x\rvert$
Furthest from the pole: $\dfrac{dr}{d\theta} = 0$
Furthest from the initial line: $\dfrac{d}{d\theta}(r\sin\theta) = 0$; from $\theta = \frac{1}{2}\pi$: $\dfrac{d}{d\theta}(r\cos\theta) = 0$

1.6 Vectors

$\mathbf{a} \times \mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta\,\hat{\mathbf{n}}$
$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$
$\mathbf{b} \times \mathbf{a} = -\,\mathbf{a} \times \mathbf{b}$; $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ for parallel vectors
Line: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$
Plane: $ax + by + cz = d$, $\mathbf{r} \cdot \mathbf{n} = p$, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$
Normal of $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$: $\mathbf{n} = \mathbf{b} \times \mathbf{c}$, and $d = \mathbf{a} \cdot \mathbf{n}$
Distance from (x_1, y_1, z_1) to $ax + by + cz = d$: $\dfrac{\lvert ax_1 + by_1 + cz_1 - d\rvert}{\sqrt{a^2 + b^2 + c^2}}$
Distance of $ax + by + cz = d$ from O: $\dfrac{\lvert d\rvert}{\sqrt{a^2 + b^2 + c^2}}$
Angle between line and plane: $\sin\theta = \dfrac{\lvert\mathbf{b} \cdot \mathbf{n}\rvert}{\lvert\mathbf{b}\rvert\lvert\mathbf{n}\rvert}$

Angle between planes: $\cos\theta = \dfrac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$

Line of intersection of two planes has direction $\mathbf{n}_1 \times \mathbf{n}_2$

Shortest distance between skew lines: $\dfrac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$

Common perpendicular: $\overrightarrow{PQ} \cdot \mathbf{b}_1 = 0$ and $\overrightarrow{PQ} \cdot \mathbf{b}_2 = 0$

1.7 Proof by induction

The four parts: base case, hypothesis "for some k", step to $k+1$, conclusion "for all n"

Sum to $k+1$ terms = sum to k terms + the $(k+1)$th term

Chain rule: $\dfrac{d}{dx}(ax+b)^m = am(ax+b)^{m-1}$

$a^{k+1} = a \times a^k$, $(k+1)! = (k+1) \times k!$, $(-1)^{k+1} = -(-1)^k$

$\cos\theta = \sin\left(\theta + \frac{1}{2}\pi\right)$ and $-\sin\theta = \cos\left(\theta + \frac{1}{2}\pi\right)$ (from the given $\sin(A+B)$ and $\cos(A+B)$)

$a^{2n} - 1 = (a^n - 1)(a^n + 1)$ (difference of two squares)

Inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; $\det(\mathbf{M}^n) = (\det\mathbf{M})^n$

2 Further Pure Mathematics 2

2.1 Hyperbolic functions

$\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$; $\operatorname{sech} x = \dfrac{1}{\cosh x}$, $\operatorname{cosech} x = \dfrac{1}{\sinh x}$, $\coth x = \dfrac{1}{\tanh x}$

$\cosh x + \sinh x = e^x$, $\cosh x - \sinh x = e^{-x}$

$\cosh 2x \equiv 2\cosh^2 x - 1 \equiv 1 + 2\sinh^2 x$

$1 - \tanh^2 x \equiv \operatorname{sech}^2 x$, $\coth^2 x - 1 \equiv \operatorname{cosech}^2 x$

$\sinh(A \pm B) \equiv \sinh A\cosh B \pm \cosh A\sinh B$, $\cosh(A \pm B) \equiv \cosh A\cosh B \pm \sinh A\sinh B$

Osborn's rule (to check a hyperbolic identity from its trigonometric partner)

$\cosh x = c\,(c > 1) \Rightarrow x = \pm\ln\left(c + \sqrt{c^2 - 1}\right)$

$\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}$, $\operatorname{cosech}^{-1} x = \sinh^{-1}\frac{1}{x}$, $\coth^{-1} x = \tanh^{-1}\frac{1}{x}$

Ranges: $\cosh x \geqslant 1$; $-1 < \tanh x < 1$; $0 < \operatorname{sech} x \leqslant 1$; $|\coth x| > 1$

2.2 Matrices

Three equations $\Leftrightarrow \mathbf{Ax} = \mathbf{b}$; if $\det\mathbf{A} \neq 0$, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ (a unique point)

$\det \mathbf{A} = 0$: no unique solution; consistent (sheaf, a common line) or inconsistent (triangular prism, or parallel planes)

$\mathbf{Ae} = \lambda\mathbf{e}$, with $\mathbf{e} \neq \mathbf{0}$

Characteristic equation: $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

2×2: $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$

Triangular matrix: eigenvalues are the diagonal entries

Eigenvector of a 3×3: vector product of two non-parallel rows of $\mathbf{A} - \lambda\mathbf{I}$

$\mathbf{A}^n\mathbf{e} = \lambda^n\mathbf{e}$; $(\mathbf{A} + k\mathbf{I})\mathbf{e} = (\lambda + k)\mathbf{e}$; $\mathbf{A}^{-1}\mathbf{e} = \frac{1}{\lambda}\mathbf{e}$

$\mathbf{A} = \mathbf{QDQ}^{-1}$: columns of $\mathbf{Q}$ are eigenvectors, $\mathbf{D}$ the matching eigenvalues

$\mathbf{A}^n = \mathbf{QD}^n\mathbf{Q}^{-1}$; $\mathbf{D}^n$ has diagonal $\lambda_1^n, \lambda_2^n, \lambda_3^n$

Cayley–Hamilton: $\mathbf{A}$ satisfies its characteristic equation ($\lambda \to \mathbf{A}$, number $c \to c\mathbf{I}$)

Vector product $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$

2.3 Differentiation

$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$, $\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x$, $\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x$

Chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Implicit: $\frac{d}{dx}f(y) = f'(y)\frac{dy}{dx}$; $\frac{d}{dx}\left(y\frac{dy}{dx}\right) = \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}$

Parametric: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \div \dfrac{dx}{dt}$

Coefficient of x^r is $\dfrac{f^{(r)}(0)}{r!}$, so $f^{(r)}(0) = r! \times$ coefficient

$1 - \tanh^2 x \equiv \operatorname{sech}^2 x$

2.4 Integration

$\int \tanh x\,dx = \ln\cosh x$, $\int \coth x\,dx = \ln|\sinh x|$

$\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x$, $\int \operatorname{cosech}^2 x\,dx = -\coth x$

$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$, $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$

Substitutions $x = a\sin\theta$, $x = a\sinh u$, $x = a\cosh u$ for $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a^2}$

Increasing f: $\frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) < \int_0^1 f(x)\,dx < \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$ (reversed if decreasing)

Arc length $s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$, $\int \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$, $\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

Surface area about the x-axis $S = 2\pi\int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ or $2\pi\int y\sqrt{\dot{x}^2 + \dot{y}^2}\,dt$ (about the y-axis, x in place of y)

2.5 Complex numbers

$z_1 z_2$: multiply the moduli, add the arguments; $\dfrac{z_1}{z_2}$: divide the moduli, subtract the arguments

De Moivre: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for any integer n

$e^{i\theta} = \cos\theta + i\sin\theta$, so $\left(re^{i\theta}\right)^n = r^n e^{in\theta}$

$(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

With $z = \cos\theta + i\sin\theta$: $z^n + z^{-n} = 2\cos n\theta$, $z^n - z^{-n} = 2i\sin n\theta$

nth roots of unity: $e^{2k\pi i/n}$, $k = 0, 1, \ldots, n-1$; they add up to 0

Roots of $z^n = Re^{i\alpha}$: $R^{1/n}e^{i(\alpha+2k\pi)/n}$, $k = 0, 1, \ldots, n-1$

$1 + e^{i\theta} = 2\cos\frac{\theta}{2}\,e^{i\theta/2}$ and $e^{i\theta} - 1 = 2i\sin\frac{\theta}{2}\,e^{i\theta/2}$ (half-angle trick)

2.6 Differential equations

$\dfrac{dy}{dx} + P(x)y = Q(x)$: integrating factor $I = e^{\int P\,dx}$, then $Iy = \int IQ\,dx$

General solution = complementary function + particular integral

Auxiliary equation of $ay'' + by' + cy = 0$: $am^2 + bm + c = 0$

CF: $Ae^{m_1 x} + Be^{m_2 x}$ (real, different); $(A + Bx)e^{mx}$ (repeated); $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$ (roots $\alpha \pm \beta i$)

CF of $y' + ky = 0$: Ae^{-kx}

PI to try: polynomial of the same degree; ke^{bx}; $\lambda\cos px + \mu\sin px$

$y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$; $x = e^t \Rightarrow x\dfrac{dy}{dx} = \dfrac{dy}{dt}$, $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$

$\lambda\cos px + \mu\sin px = R\sin(px + \phi)$ with $R = \sqrt{\lambda^2 + \mu^2}$, $R\cos\phi = \mu$, $R\sin\phi = \lambda$

$\int \tan x\,dx = \ln|\sec x|$, $\int \tanh x\,dx = \ln\cosh x$, $\int \coth x\,dx = \ln|\sinh x|$

3 Further Mechanics

3.1 Motion of a projectile

Components at time t: $v_x = u\cos\theta$, $v_y = u\sin\theta - gt$

Displacements: $x = (u\cos\theta)t$, $y = (u\sin\theta)t - \frac{1}{2}gt^2$

Trajectory in $\tan\theta$: $y = x\tan\theta - \dfrac{gx^2}{2u^2}(1 + \tan^2\theta)$

Speed $= \sqrt{v_x^2 + v_y^2}$; direction $\tan\alpha = \dfrac{v_y}{v_x}$

Gradient of the path $\dfrac{dy}{dx} = \dfrac{v_y}{v_x}$

Time to greatest height $\frac{u\sin\theta}{g}$; greatest height $H = \frac{u^2\sin^2\theta}{2g}$
Time of flight on level ground $\frac{2u\sin\theta}{g}$; range $R = \frac{u^2\sin 2\theta}{g}$
Two directions are perpendicular when $v_{x1}v_{x2} + v_{y1}v_{y2} = 0$ (gradients multiply to -1)

3.2 Equilibrium of a rigid body

Moment about P = force $\times$ perpendicular distance from P
Composite body: $\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$, $\bar{y} = \frac{\sum m_i y_i}{\sum m_i}$; a removed piece counts with a minus sign
Triangle's centre of mass = mean of its vertices, $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$
Cylinder: volume $\pi r^2 h$, curved surface $2\pi r h$; disc (end of a closed cylinder) πr^2
Equilibrium: resultant force zero (resolve two ways) **and** total moment zero about any point
Friction: $F \leqslant \mu R$; $F = \mu R$ when about to slip
Freely suspended from A: G is vertically below A
On a horizontal surface: stands if the vertical through G meets the base; on the point of toppling, G is above the edge
On a slope: slides first if $\mu < \tan\alpha$; topples when $\tan\alpha > \frac{r}{\bar{y}}$

3.3 Circular motion

$v = r\omega$
One revolution $= 2\pi$ rad; n rpm $\Rightarrow \omega = \frac{2\pi n}{60}$; period $\frac{2\pi}{\omega}$
Newton's second law towards the centre: resultant $= mr\omega^2 = \frac{mv^2}{r}$
Conical pendulum (string length l, angle θ to vertical): $r = l\sin\theta, T\cos\theta = mg, T = ml\omega^2$
Energy: $\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgh$ (KE $\frac{1}{2}mv^2$, GPE mgh)
Vertical circle, θ from downward vertical: $v^2 = u^2 - 2ga(1-\cos\theta), T - mg\cos\theta = \frac{mv^2}{a}$
Vertical circle, α from upward vertical: $v^2 = u^2 - 2ga(1+\cos\alpha), T + mg\cos\alpha = \frac{mv^2}{a}$
Slack string or leaving the surface: $T = 0$ or $R = 0$ (speed is **not** zero)
Complete circles: string or inside a surface $u^2 \geqslant 5ga$; rod or bead on a wire $u^2 > 4ga$
Outside a sphere: $mg\cos\alpha - R = \frac{mv^2}{a}$; from the top with speed u, leaves where $\cos\alpha = \frac{u^2+2ga}{3ga}$
Limiting friction $F = \mu R$ (and $F \leqslant \mu R$ always)

3.4 Hooke's law

Work done stretching from x_1 to x_2: $\dfrac{\lambda(x_2^2 - x_1^2)}{2l}$

Equilibrium extension of a hanging particle: $\dfrac{\lambda e}{l} = mg$

KE $= \frac{1}{2}mv^2$, GPE $= mgh$

Energy: KE + GPE + EPE is constant (no friction); otherwise the loss equals the work done against friction

Work done against friction $= F \times d$, with $F = \mu R$ while sliding

Newton's second law: resultant force $= ma$

Half of a string of natural length l has natural length $\frac{l}{2}$ and the same λ

3.5 Linear motion under a variable force

Newton's second law: resultant force $= ma$

$v = \dfrac{dx}{dt}, \quad a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

$a = v\dfrac{dv}{dx}$ (use when the question links v and x)

Weight $= mg$, with $g = 10$ m s^{-2}

Terminal (limiting) speed: the value of v that makes the acceleration zero

$\displaystyle\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

$\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c\,(n \neq -1), \quad \int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$

3.6 Momentum

Momentum $= mv$ (a vector: signs show direction)

Conservation of momentum: $m_Au_A + m_Bu_B = m_Av_A + m_Bv_B$

Newton's experimental law: $v_B - v_A = e(u_A - u_B)$ (separation $= e\times$ approach)

$0 \leqslant e \leqslant 1$; $e = 1$ perfectly elastic, $e = 0$ inelastic

Smooth spheres: velocity components **perpendicular** to the line of centres are unchanged

Smooth fixed surface: parallel component unchanged, perpendicular component $\times e$ and reversed

Wall, angles measured from the wall: $\tan\beta = e\tan\alpha$

Kinetic energy $= \frac{1}{2}mv^2$, with v^2 = sum of the squares of the components

4 Further Probability & Statistics

4.1 Continuous random variables

$\int_{\text{all } x} f(x)\,dx = 1$ and $f(x) \geqslant 0$

$\mathrm{P}(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx = F(b) - F(a)$

$F(x) = \mathrm{P}(X \leqslant x) = \int_{-\infty}^{x} f(t)\,dt$, and $f(x) = F'(x)$

$F = 0$ below the range and $F = 1$ above it; F has no jumps

$\mathrm{E}\big(g(X)\big) = \int g(x)\,f(x)\,dx$

$\mathrm{Var}\big(g(X)\big) = \mathrm{E}\big(g(X)^2\big) - \big(\mathrm{E}(g(X))\big)^2$

Median m: $F(m) = \frac{1}{2}$; quartiles: $F(q_1) = \frac{1}{4}$, $F(q_3) = \frac{3}{4}$; pth percentile: $F(x) = \frac{p}{100}$

$Y = g(X)$ with g increasing: $G(y) = F\big(g^{-1}(y)\big)$, then $g_Y(y) = G'(y)$

4.2 Inference using normal and t-distributions

$\Sigma(x - \bar{x})^2 = (n-1)s^2$

One-sample t: $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$, $\nu = n - 1$

t interval for μ: $\bar{x} \pm t\dfrac{s}{\sqrt{n}}$, $\nu = n - 1$

Paired t: $t = \dfrac{\bar{d} - k}{s_d/\sqrt{n}}$, $\nu = n - 1$ (n pairs)

Two-sample t: $t = \dfrac{(\bar{x} - \bar{y}) - c}{\sqrt{s_p^2\left(\frac{1}{n_x} + \frac{1}{n_y}\right)}}$, $\nu = n_x + n_y - 2$

Pooled t interval: $(\bar{x} - \bar{y}) \pm t\sqrt{s_p^2\left(\frac{1}{n_x} + \frac{1}{n_y}\right)}$

Large samples: $z = \dfrac{(\bar{x} - \bar{y}) - c}{\sqrt{\frac{s_x^2}{n_x} + \frac{s_y^2}{n_y}}}$; interval $(\bar{x} - \bar{y}) \pm z\sqrt{\frac{s_x^2}{n_x} + \frac{s_y^2}{n_y}}$

Common z values: 90% or one-tailed 5%: 1.645; 95% or two-tailed 5%: 1.960; one-tailed 1%: 2.326; two-tailed 1%: 2.576

χ²-tests

$X^2 = \sum \dfrac{(O - E)^2}{E}$, or equivalently $X^2 = \sum \dfrac{O^2}{E} - N$

Goodness of fit:
$\nu = (\text{number of classes after combining}) - 1 - (\text{number of parameters estimated from the data})$

Contingency table: $E = \dfrac{\text{row total} \times \text{column total}}{\text{grand total}}$

Contingency table: $\nu = (r-1)(c-1)$, counting rows and columns after combining

Every expected frequency must be at least 5; combine classes until it is

Expected frequency $= N \times P(\text{class})$, and all the E add up to N

Normal: $P(a \leqslant X < b) = \Phi\left(\dfrac{b-\mu}{\sigma}\right) - \Phi\left(\dfrac{a-\mu}{\sigma}\right)$ (Φ is in the normal table)

Given pdf: $P(a \leqslant X < b) = \displaystyle\int_a^b f(x)\,dx$

4.4 Non-parametric tests

Sign test: number of $+$ signs $X \sim B(n, 0.5)$ under H_0

Sign test for large n: $X \approx N(0.5n,\ 0.25n)$, with a continuity correction

$P + Q = \frac{1}{2}n(n+1)$

Continuity correction: move the statistic 0.5 towards the mean before standardising

4.5 Probability generating functions

$\mathrm{G}_X(1) = 1; \mathrm{G}_X(0) = \mathrm{P}(X = 0)$

$\mathrm{P}(X = r)$ = coefficient of $t^r = \dfrac{\mathrm{G}_X^{(r)}(0)}{r!}$

X, Y independent: $\mathrm{G}_{X+Y}(t) = \mathrm{G}_X(t)\,\mathrm{G}_Y(t)$

Sum of n independent observations of X: $\big(\mathrm{G}_X(t)\big)^n$

Discrete uniform on $1, \ldots, n$: $\dfrac{1}{n}(t + t^2 + \cdots + t^n) = \dfrac{t(1-t^n)}{n(1-t)}$

$\mathrm{B}(n, p)$: $(q + pt)^n$

$\mathrm{Geo}(p)$: $\dfrac{pt}{1-qt}$

$\mathrm{Po}(\lambda)$: $e^{\lambda(t-1)}$

$\mathrm{Geo}(p)$ variance $\dfrac{q}{p^2}$