

Hyperbolic functions A2

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What you need to know

By the end of this topic you should be able to:

1. **Define the six hyperbolic functions using e^x .** $\sinh x$, $\cosh x$ and $\tanh x$ are built from e^x and e^{-x} ; $\operatorname{sech} x$, $\operatorname{cosech} x$ and $\operatorname{coth} x$ are their reciprocals.
2. **Sketch the graph of each hyperbolic function**, showing where it crosses the axes, its maximum or minimum point, its symmetry and its asymptotes.
3. **Prove and use hyperbolic identities**, such as $\cosh^2 x - \sinh^2 x \equiv 1$, $\sinh 2x \equiv 2 \sinh x \cosh x$, and the other results that match the standard trigonometric identities. "Prove" means starting from the exponential definitions.
4. **Understand the inverse hyperbolic functions** $\sinh^{-1} x$, $\cosh^{-1} x$ and $\tanh^{-1} x$ (including their domains), **derive their logarithmic forms** and use them, for example to give exact answers to equations.

Differentiating and integrating hyperbolic functions belong to the Differentiation and Integration topics. Exam questions often start with an identity from this topic and then use it in a calculus part. This note covers choosing and using the identity; the calculus itself is in those topics' notes.

Understand it

Definitions from e^x

Every function can be split into an even part and an odd part. For e^x these parts are the two main hyperbolic functions:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Say them "cosh" and "shine" (or "sinch"). Adding and subtracting them gives two facts that make many proofs short:

$$\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x}.$$

The others are defined from these two, just as in trigonometry:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}, \quad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},$$

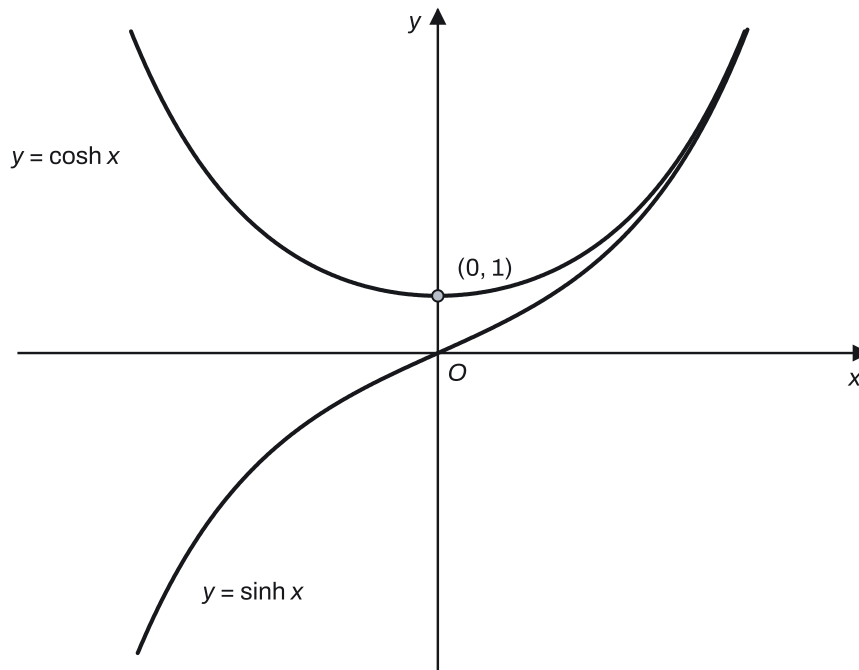
$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

Values at a logarithm come out exactly, because $e^{\ln a} = a$ and $e^{-\ln a} = \frac{1}{a}$. For $x = \ln 5$:

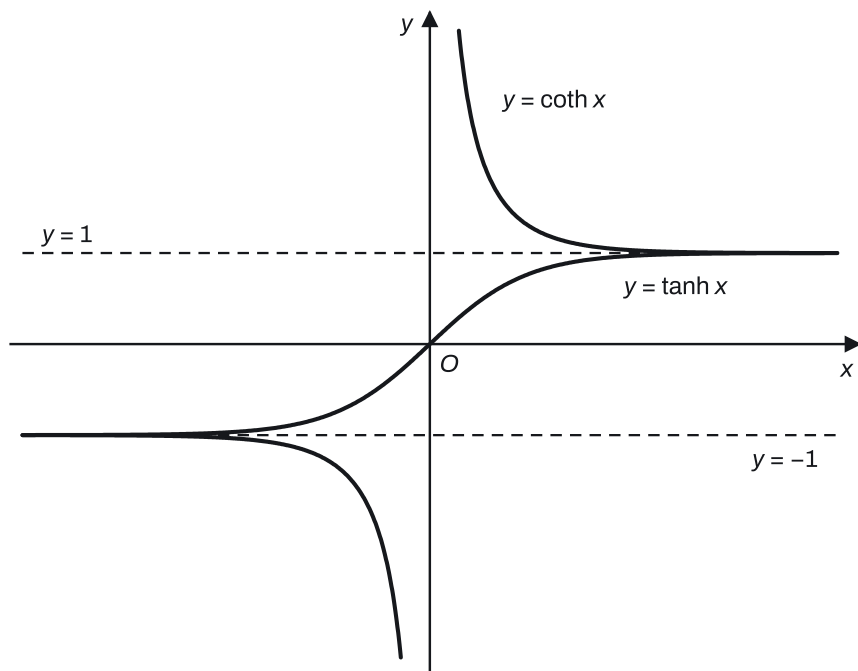
$$\sinh(\ln 5) = \frac{5 - \frac{1}{5}}{2} = \frac{12}{5}, \quad \cosh(\ln 5) = \frac{5 + \frac{1}{5}}{2} = \frac{13}{5}, \quad \tanh(\ln 5) = \frac{12}{13}.$$

Why "hyperbolic"? For any t , the point $(\cosh t, \sinh t)$ lies on the hyperbola $x^2 - y^2 = 1$, just as $(\cos t, \sin t)$ lies on the circle $x^2 + y^2 = 1$.

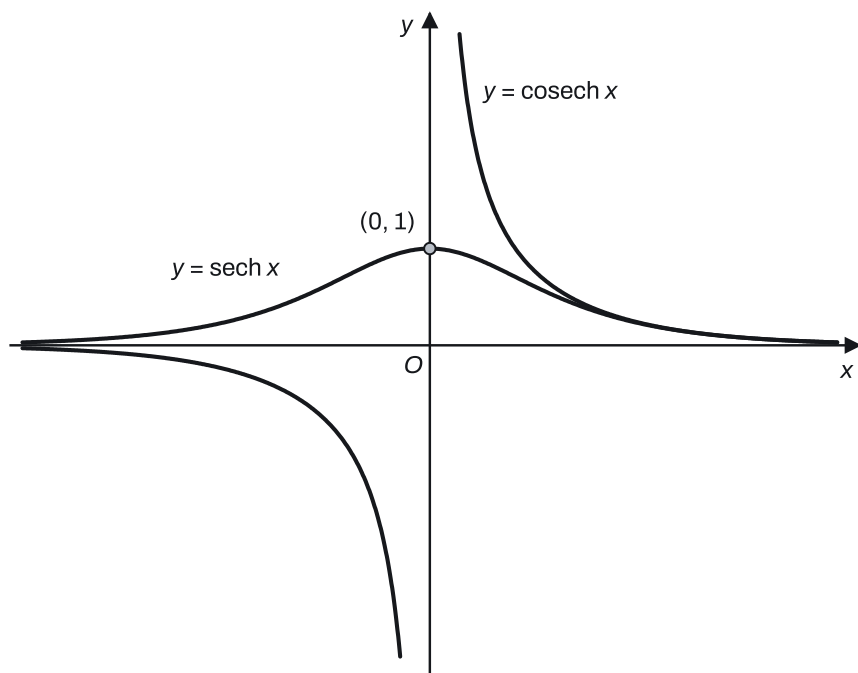
Graphs



- $\cosh x$ is **even** (symmetrical about the y -axis), with minimum point $(0, 1)$, so $\cosh x \geq 1$ for every x .
- $\sinh x$ is **odd** (rotational symmetry about O), passes through the origin and takes every real value. It is always below $\cosh x$, and the gap $\cosh x - \sinh x = e^{-x}$ shrinks to 0 as x grows.
- For large positive x , both behave like $\frac{1}{2}e^x$, because e^{-x} is tiny.



- $\tanh x$ is odd, passes through O and stays strictly between -1 and 1 : the asymptotes are $y = 1$ and $y = -1$. So an equation such as $\tanh x = \frac{5}{3}$ has no solution.
- $\coth x$ is odd, is not defined at $x = 0$ (the y -axis is an asymptote) and has $|\coth x| > 1$, with the same horizontal asymptotes.



- $\operatorname{sech} x$ is even, has maximum point $(0, 1)$, satisfies $0 < \operatorname{sech} x \leq 1$, and has asymptote $y = 0$. It is smooth (rounded) at its maximum, not pointed.
- $\operatorname{cosech} x$ is odd, is not defined at $x = 0$, and has both axes as asymptotes.

A useful way to sketch a reciprocal: where $\cosh x$ is large, $\operatorname{sech} x$ is close to 0 ; where $\sinh x$ passes through 0 , $\operatorname{cosech} x$ has an asymptote.

Proving identities from the definitions

When a question says "starting from the definitions ... in terms of exponentials", write every hyperbolic function as exponentials, then simplify **one side** until it becomes the other. For example, to prove $\cosh(A + B) \equiv \cosh A \cosh B + \sinh A \sinh B$, start with the right-hand side:

$$\begin{aligned}\cosh A \cosh B + \sinh A \sinh B &= \frac{1}{4}(e^A + e^{-A})(e^B + e^{-B}) + \frac{1}{4}(e^A - e^{-A})(e^B - e^{-B}) \\ &= \frac{1}{4}(e^{A+B} + e^{A-B} + e^{-A+B} + e^{-A-B}) + \frac{1}{4}(e^{A+B} - e^{A-B} - e^{-A+B} + e^{-A-B}) \\ &= \frac{1}{2}(e^{A+B} + e^{-(A+B)}) = \cosh(A + B).\end{aligned}$$

The most important identity comes almost for free from the two facts above:

$$\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x) = e^x \times e^{-x} = 1.$$

Dividing it by $\cosh^2 x$ gives $1 - \tanh^2 x \equiv \operatorname{sech}^2 x$; dividing by $\sinh^2 x$ gives $\coth^2 x - 1 \equiv \operatorname{cosech}^2 x$. Putting $B = A$ in the addition formulas gives the double-angle formulas, and $\cosh^2 x - \sinh^2 x = 1$ turns $\cosh 2x = \cosh^2 x + \sinh^2 x$ into $2 \cosh^2 x - 1$ or $1 + 2 \sinh^2 x$. In an exam, though, a "starting from the definitions" proof must go back to exponentials.

From trigonometry to hyperbolic identities (Osborn's rule)

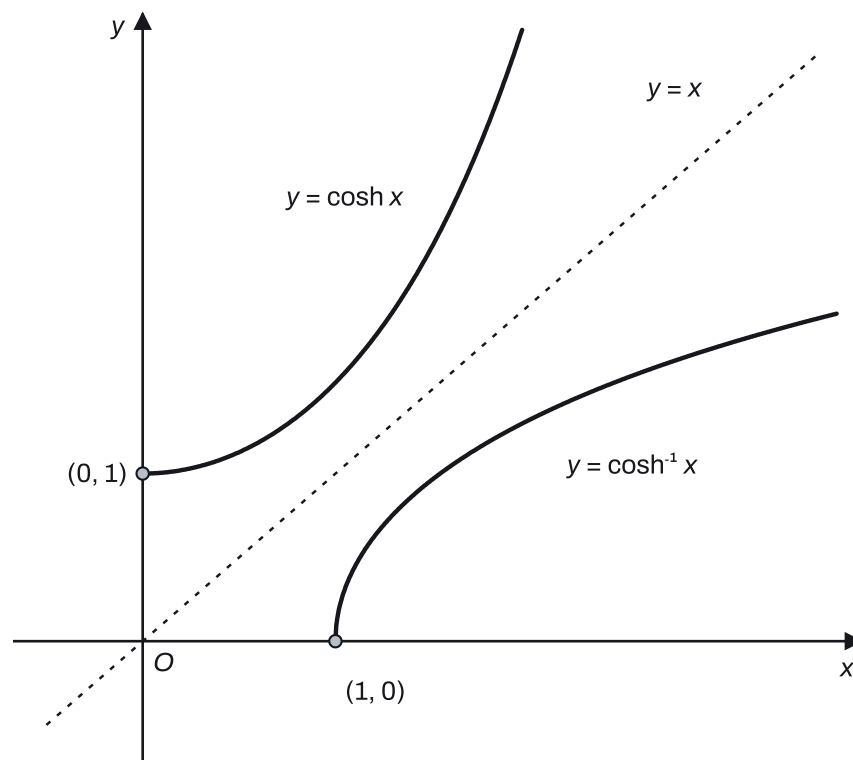
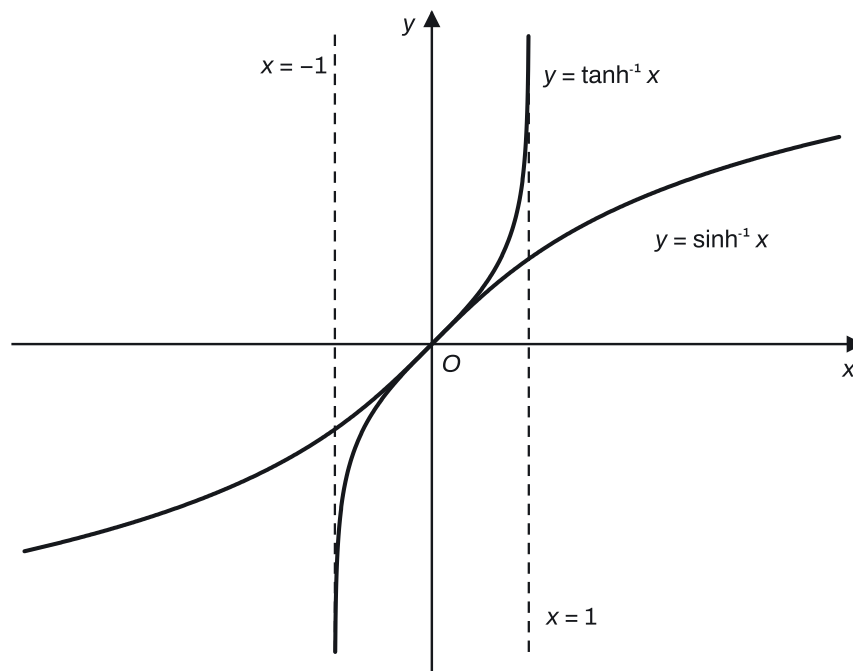
Every trigonometric identity has a hyperbolic partner. To find it, change $\cos$ to $\cosh$ and $\sin$ to $\sinh$, and **change the sign of every term that contains a product of two sines** (this includes $\tan^2$, which hides $\sin^2$). For example:

Trigonometry	Hyperbolic
$\cos^2 A + \sin^2 A \equiv 1$	$\cosh^2 A - \sinh^2 A \equiv 1$
$\cos 2A \equiv 1 - 2 \sin^2 A$	$\cosh 2A \equiv 1 + 2 \sinh^2 A$
$\sin 3A \equiv 3 \sin A - 4 \sin^3 A$	$\sinh 3A \equiv 3 \sinh A + 4 \sinh^3 A$
$1 + \tan^2 A \equiv \sec^2 A$	$1 - \tanh^2 A \equiv \operatorname{sech}^2 A$

($\sin^3 A = \sin A \times \sin^2 A$ contains a product of two sines, so its sign changes.) The rule is a way to **check or recall** an identity. It is not a proof.

Inverse functions and their logarithmic forms

$\sinh x$ and $\tanh x$ are one-to-one, so they have inverses straight away: $\sinh^{-1} x$ is defined for every x , and $\tanh^{-1} x$ for $-1 < x < 1$. $\cosh x$ is not one-to-one ($\cosh(-x) = \cosh x$), so we keep only the part with $x \geq 0$. Then $\cosh^{-1} x$ is defined for $x \geq 1$ and its value is never negative.



Because the hyperbolic functions are made of exponentials, their inverses are logarithms. **Deriving $\sinh^{-1} x$:** let $y = \sinh^{-1} x$, so

$$x = \sinh y = \frac{e^y - e^{-y}}{2} \Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow (e^y)^2 - 2x(e^y) - 1 = 0.$$

This is a quadratic in e^y , so $e^y = x \pm \sqrt{x^2 + 1}$. Since $\sqrt{x^2 + 1} > |x|$, the minus sign makes e^y negative, which is impossible. So

$$\sinh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right).$$

$\cosh^{-1} x$ works the same way: $x = \cosh y$ gives $(e^y)^2 - 2x(e^y) + 1 = 0$, so $e^y = x \pm \sqrt{x^2 - 1}$. Both roots are positive, but their product is 1, so one is at least 1 and the other at most 1. We need $y \geq 0$, that is $e^y \geq 1$, so we take the plus sign: $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ for $x \geq 1$. The other root gives the negative solution, so **the equation** $\cosh x = c$ (with $c > 1$) **has two solutions**, $x = \pm \ln(c + \sqrt{c^2 - 1})$.

$\tanh^{-1} x$ is derived in Example 3: $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ for $|x| < 1$.

For the reciprocal functions, turn them upside down first: $\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$ (for $0 < x \leq 1$), $\operatorname{cosech}^{-1} x = \sinh^{-1} \frac{1}{x}$ and $\operatorname{coth}^{-1} x = \tanh^{-1} \frac{1}{x}$ (for $|x| > 1$). For example, $\sinh^{-1} \frac{3}{4} = \ln\left(\frac{3}{4} + \frac{5}{4}\right) = \ln 2$.

Solving hyperbolic equations

There are two routes.

- **Use an identity** to get a single function, usually a quadratic in $\sinh x$ or $\cosh x$. Solve it, throw out impossible values ($\cosh x < 1$, $|\tanh x| \geq 1$), then use the logarithmic forms. For example, $\sinh x = \frac{3}{4}$ gives $x = \ln 2$ only, while $\cosh x = \frac{5}{4}$ gives $x = \pm \ln\left(\frac{5}{4} + \frac{3}{4}\right) = \pm \ln 2$.
- **Use exponentials** when the equation mixes $\sinh x$ and $\cosh x$ in a linear way, such as $a \cosh x + b \sinh x = c$. Replace each function by its definition, multiply through by e^x , and solve the quadratic in e^x (Example 1). Reject any negative value of e^x .

Identities that later parts need

Most exam questions on this topic prove an identity in part (a), then use it in a later part, usually on differentiation, integration or a differential equation. The skill that belongs to this topic is spotting which identity turns an awkward expression into a simple one:

- **A square becomes a double angle:** $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$, $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$, $\tanh^2 x = 1 - \operatorname{sech}^2 x$.
- **A square root disappears:** $\sqrt{1 + \sinh^2 x} = \cosh x$, $\sqrt{\cosh^2 x - 1} = \sinh x$ (for $x \geq 0$) and $\sqrt{1 - \tanh^2 x} = \operatorname{sech} x$.
- **A fraction collapses:** for example $\frac{2 \sinh x \cosh x}{1 + \sinh^2 x} = \frac{2 \sinh x \cosh x}{\cosh^2 x} = 2 \tanh x$, and $\frac{\cosh^2 t - 1}{\sinh t \cosh t} = \frac{\sinh^2 t}{\sinh t \cosh t} = \tanh t$.
- **An inverse function is written the other way round:** if $y = \cosh^{-1} x$ then $\cosh y = x$, and $\sinh y = +\sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$ because $y \geq 0$.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$	Learn it
$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$; $\operatorname{sech} x = \frac{1}{\cosh x}$, $\operatorname{cosech} x = \frac{1}{\sinh x}$, $\operatorname{coth} x = \frac{1}{\tanh x}$	Learn it

Formula	Status
$\cosh x + \sinh x = e^x, \cosh x - \sinh x = e^{-x}$	Learn it
$\cosh^2 x - \sinh^2 x \equiv 1$	Given
$\sinh 2x \equiv 2 \sinh x \cosh x$	Given
$\cosh 2x \equiv \cosh^2 x + \sinh^2 x$	Given
$\cosh 2x \equiv 2 \cosh^2 x - 1 \equiv 1 + 2 \sinh^2 x$	Learn it
$1 - \tanh^2 x \equiv \operatorname{sech}^2 x, \coth^2 x - 1 \equiv \operatorname{cosech}^2 x$	Learn it
$\sinh(A \pm B) \equiv \sinh A \cosh B \pm \cosh A \sinh B, \cosh(A \pm B) \equiv \cosh A \cosh B \pm \sinh A \sinh B$	Learn it
Osborn's rule (to check a hyperbolic identity from its trigonometric partner)	Learn it
$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$	Given
$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), x \geq 1$	Given
$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), x < 1$	Given
$\cosh x = c (c > 1) \Rightarrow x = \pm \ln(c + \sqrt{c^2 - 1})$	Learn it
$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}, \operatorname{cosech}^{-1} x = \sinh^{-1} \frac{1}{x}, \coth^{-1} x = \tanh^{-1} \frac{1}{x}$	Learn it
Ranges: $\cosh x \geq 1; -1 < \tanh x < 1; 0 < \operatorname{sech} x \leq 1; \coth x > 1$	Learn it

How to do it

In the Paper 2 questions we have tagged for this topic (23 questions, 2021–2025; 8 of them appear in two papers of the same session, so there are 15 different questions), the types came up this often. Every one of the 23 questions mixes two or more types, so the counts add up to more than 23. Most also have parts on differentiation, integration or differential equations; those parts are taught in their own topics, and only the identity step is counted here.

Question type	Questions
Use an identity to rewrite an expression in a later part	20
Prove an identity from the exponential definitions	19
Exact values at logarithms, and answers in terms of e	8
Solve a hyperbolic equation	5
Inverse hyperbolic functions and their logarithmic forms	5
Sketch hyperbolic graphs	4

1. Use an identity to rewrite an expression in a later part (20 questions)

The later part is usually calculus, and its method belongs to another topic. The marks for this topic come from choosing the right identity and using it correctly.

1. **Use the result of part (a)** if there is one: it is there because you need it. "Hence" means you must.
2. **Look for a square, a square root or a fraction.** A square of $\cosh$, $\sinh$ or $\tanh$ usually needs a double-angle identity or $1 - \tanh^2 x = \operatorname{sech}^2 x$. A root of a sum of squares usually becomes a single hyperbolic function. A fraction often cancels once its top or bottom is rewritten.
3. **Rewrite, then simplify** until only one function (or one power of it) is left, and check the sign of any square root you take.

Variations you will meet:

- **Powers of $\cosh x$ in terms of $\sinh x$:** even powers of $\cosh x$ turn into powers of $1 + \sinh^2 x$, for example $\cosh^4 x = (1 + \sinh^2 x)^2$.
- **Sums of two squares:** expand, collect, and look for $\sinh^2 + 1$ or $\cosh^2 - 1$, for example $\sinh^2 u + \cosh^2 u = 2 \sinh^2 u + 1 = \cosh 2u$.
- **Inverse functions:** write $y = \sinh^{-1}(\dots)$ as $\sinh y = \dots$, then use $\cosh^2 y = 1 + \sinh^2 y$ to write $\cosh y$ in terms of x (positive root, since $\cosh y \geq 1$).
- **Exponentials in disguise:** $e^{\ln \cosh x} = \cosh x$, and $\cosh x \pm \sinh x = e^{\pm x}$.

2. Prove an identity from the exponential definitions (19 questions)

This is almost always part (a), worth 3 marks, and the identity is used later in the question.

1. **Write each function you need in exponentials**, for example $\sinh x = \frac{1}{2}(e^x - e^{-x})$. This line usually earns the first mark on its own, so write it out.
2. **Start from one side** (the more complicated one) and substitute.
3. **Expand carefully.** Squares are where the errors happen: $(e^x - e^{-x})^2 = e^{2x} - 2 + e^{-2x}$, because $e^x \times e^{-x} = 1$.
4. **Collect terms** and recognise the other side, for example $\frac{1}{2}(e^{2x} + e^{-2x}) = \cosh 2x$. End with a line that says the two sides are equal.

Variations you will meet:

- **Identities with $\tanh$ and sech .** Write $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ and $\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$, then put everything over the common denominator $(e^x + e^{-x})^2$. The numerator simplifies because $(e^x + e^{-x})^2 - (e^x - e^{-x})^2 = 4$.
- **Double-angle identities** ($\sinh 2x$, the three forms of $\cosh 2x$, or $2 \sinh^2 x$ and $2 \cosh^2 x$ in terms of $\cosh 2x$): expand the squares or the product, then match to $\frac{1}{2}(e^{2x} \pm e^{-2x})$.
- **A product of sums**, such as $(\cosh x + \sinh x)(\cosh 3x - \sinh 3x)$: writing both brackets in exponentials shows they are e^x and e^{-3x} , so the product is e^{-2x} .
- **Triple angles** such as $\cosh 3x$ in terms of $\cosh x$: expand $(e^x + e^{-x})^3$ with the binomial theorem (Example 2).
- **Letters.** If the question uses A or t , use the same letter all the way through; mixing x and A in one line has cost the accuracy mark.

3. Exact values at logarithms, and answers in terms of e (8 questions)

1. **Write the function in exponentials** and use $e^{\ln a} = a$, $e^{-\ln a} = \frac{1}{a}$, $e^{k \ln a} = a^k$.

2. For a double angle, use $2 \ln a = \ln a^2$: for example $\sinh(2 \ln 2) = \sinh(\ln 4) = \frac{4 - \frac{1}{4}}{2} = \frac{15}{8}$.

Variations you will meet:

- **A value given as an inverse function**, such as $x = \sinh^{-1} \frac{3}{4}$: turn it into a logarithm first ($\ln 2$), then evaluate cosh, sech or tanh of it exactly.
- **"Give your answer in terms of e "** (or π and e): replace $\sinh 2$ by $\frac{1}{2}(e^2 - e^{-2})$, and so on. An answer left as $\sinh 2$ is not finished.
- **"In the form $a + \ln b$ "**: a term like $\ln(\tanh t)$ at a known t becomes $\ln$ of a fraction, for example $\ln \tanh(\ln 7) = \ln \frac{49-1}{49+1} = \ln \frac{24}{25}$.

4. Solve a hyperbolic equation (5 questions)

1. **Choose the route.** Squares or double angles: use an identity to get a quadratic in one function (for $\cosh 2x$ with $\sinh x$, use $\cosh 2x = 1 + 2 \sinh^2 x$; with $\cosh x$, use $2 \cosh^2 x - 1$). A linear mix of $\sinh x$ and $\cosh x$: use exponentials.
2. **Solve the quadratic and reject impossible values**, saying why: $\cosh x \geq 1$, $|\tanh x| < 1$, $e^x > 0$.
3. **Convert to logarithms** with the MF19 forms. Remember the $\pm$ for $\cosh x = c$.
4. Give the answer in the form asked for, usually $\ln(\dots)$ with the inside simplified.

Variations you will meet:

- **Exponential route instead:** after writing in exponentials, you may get a quadratic in e^{2x} (for example $e^{4x} - pe^{2x} + 1 = 0$). Solve for e^{2x} , then $x = \frac{1}{2} \ln(\dots)$. Both routes earn full marks.
- **How many roots?** For a quadratic in $s = \sinh x$, each real value of s gives exactly **one** x , because $\sinh$ is one-to-one. So "two distinct real roots" in x means the quadratic has two distinct real roots: discriminant $b^2 - 4ac > 0$. For a quadratic in $c = \cosh x$ it's different: $c > 1$ gives two values of x , $c = 1$ gives one ($x = 0$), $c < 1$ gives none.
- **Equations from an earlier part**, where an expression is set equal to a number: simplify with identities until you have one function, such as $\tanh^2 t = \dots$, then solve.

5. Inverse hyperbolic functions and their logarithmic forms (5 questions)

1. **To derive a logarithmic form:** let $y = \sinh^{-1} x$ (or $\cosh^{-1}$, $\tanh^{-1}$), write $x = \sinh y$ in exponentials, rearrange into a quadratic in e^y (or an equation in e^{2y} for $\tanh$), solve, choose the valid root with a reason, then take logs.
2. **To evaluate one exactly:** substitute into the MF19 form and simplify the surd, for example $\sinh^{-1} \frac{3}{4} = \ln 2$.
3. **For sech^{-1} , $\operatorname{cosech}^{-1}$, coth^{-1} :** write them as $\cosh^{-1} \frac{1}{x}$, $\sinh^{-1} \frac{1}{x}$, $\tanh^{-1} \frac{1}{x}$ first.

Variations you will meet:

- **"Express $\cosh y$ in terms of x "** when y is an inverse reciprocal function, such as $y = \operatorname{cosech}^{-1}(\dots)$: take cosech of both sides, turn it upside down to get $\sinh y$, then use $\cosh y = \sqrt{1 + \sinh^2 y}$.

6. Sketch hyperbolic graphs (4 questions)

1. **Know the six basic shapes** from "Understand it": the points $(0, 0)$ or $(0, 1)$, the symmetry and the asymptotes.
2. **Mark the key points and state the asymptotes** when asked: for $y = \tanh x$ they are $y = \pm 1$ (for $x \geq 0$ just $y = 1$), for $y = \operatorname{sech} x$ it is $y = 0$.
3. Draw smooth curves: rounded at a maximum or minimum, getting closer to an asymptote without touching or crossing it.

Variations you will meet:

- **Two curves on the same diagram**, such as $y = 3 \cosh x$ and $y = 5 \sinh x$: find each curve's value at $x = 0$ (3 and 0), its symmetry and how fast it grows (for large x , $3 \cosh x \approx \frac{3}{2}e^x$ and $5 \sinh x \approx \frac{5}{2}e^x$, so $5 \sinh x$ ends up above). Solve the equation to mark where they cross (here $\tanh x = \frac{3}{5}$, so $x = \ln 2$ only), and make the sketch agree with your solutions. Squares such as $\sinh^2 x$ are even and never negative, and grow like a multiple of e^{2x} .

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

Solve the equation $12 \cosh x + 8 \sinh x = 21$, giving your answers in exact logarithmic form. **[5]**

Write both functions in exponentials:

$$12 \times \frac{e^x + e^{-x}}{2} + 8 \times \frac{e^x - e^{-x}}{2} = 21 \Rightarrow 10e^x + 2e^{-x} = 21.$$

B1 for correct exponentials.

Multiply by e^x : $10e^{2x} - 21e^x + 2 = 0$. **M1** for getting a quadratic in e^x , **A1** for the correct quadratic.

$(10e^x - 1)(e^x - 2) = 0$, so $e^x = 2$ or $e^x = \frac{1}{10}$. **M1** for solving.

$x = \ln 2$ or $x = \ln \frac{1}{10} = -\ln 10$. **A1** for both.

Example 2

(a) Starting from the definition of $\cosh$ in terms of exponentials, prove that $\cosh 3x \equiv 4 \cosh^3 x - 3 \cosh x$. **[3]**

(b) Hence solve the equation $4 \cosh 3x = 13 \cosh x$, giving your answers in logarithmic form. **[4]**

(a) Start with the right-hand side and write $\cosh x = \frac{1}{2}(e^x + e^{-x})$. **B1**

$$4 \cosh^3 x - 3 \cosh x = \frac{4}{8} (e^{3x} + 3e^x + 3e^{-x} + e^{-3x}) - \frac{3}{2} (e^x + e^{-x}).$$

M1 for expanding the cube with the binomial theorem.

$$= \frac{1}{2}e^{3x} + \frac{3}{2}e^x + \frac{3}{2}e^{-x} + \frac{1}{2}e^{-3x} - \frac{3}{2}e^x - \frac{3}{2}e^{-x} = \frac{1}{2}(e^{3x} + e^{-3x}) = \cosh 3x.$$

A1, the given result reached from one side.

(b) Using (a): $4(4 \cosh^3 x - 3 \cosh x) = 13 \cosh x$, so $16 \cosh^3 x - 25 \cosh x = 0$. **M1** for using the identity.

$\cosh x (16 \cosh^2 x - 25) = 0$, so $\cosh x = 0$ or $\cosh x = \pm \frac{5}{4}$. Since $\cosh x \geq 1$, only $\cosh x = \frac{5}{4}$ is possible. **A1** for $\frac{5}{4}$ with the others rejected.

$$x = \pm \ln \left(\frac{5}{4} + \sqrt{\frac{25}{16} - 1} \right) = \pm \ln \left(\frac{5}{4} + \frac{3}{4} \right) = \pm \ln 2.$$

M1 for the logarithmic form, **A1** for both answers, including the negative one.

Example 3

(a) Starting from the definition of $\tanh$ in terms of exponentials, show that $\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$ for $-1 < x < 1$. **[4]**

(b) Find the exact value of x for which $\tanh^{-1} x = \sinh^{-1} \frac{3}{4}$. **[3]**

(a) Let $y = \tanh^{-1} x$, so $x = \tanh y = \frac{e^{2y} - 1}{e^{2y} + 1}$ (multiplying top and bottom of $\frac{e^y - e^{-y}}{e^y + e^{-y}}$ by e^y). **B1**

$$xe^{2y} + x = e^{2y} - 1 \Rightarrow e^{2y}(1 - x) = 1 + x \Rightarrow e^{2y} = \frac{1+x}{1-x}.$$

M1 for making e^{2y} the subject, **A1**.

The right-hand side is positive because $-1 < x < 1$, so we can take logs: $2y = \ln \left(\frac{1+x}{1-x} \right)$, giving $\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$. **A1**

(b) $\sinh^{-1} \frac{3}{4} = \ln \left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1} \right) = \ln \left(\frac{3}{4} + \frac{5}{4} \right) = \ln 2$. **B1**

$\frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) = \ln 2 \Rightarrow \frac{1+x}{1-x} = 4$. **M1** for equating and removing the logs.

$1+x = 4-4x \Rightarrow x = \frac{3}{5}$. **A1** (Check: $\tanh(\ln 2) = \frac{4-1}{4+1} = \frac{3}{5}$.)

Example 4

(a) Starting from the definitions of $\cosh$ and $\sinh$ in terms of exponentials, prove that $\cosh^4 x - \sinh^4 x \equiv \cosh 2x$. **[3]**

(b) Hence solve the equation $\cosh^4 x - \sinh^4 x = 4 \sinh x + 7$, giving your answers in logarithmic form. **[5]**

(a) $\cosh^2 x = \frac{1}{4}(e^{2x} + 2 + e^{-2x})$ and $\sinh^2 x = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$. **B1**

Factorise the left-hand side as a difference of two squares:

$$\cosh^4 x - \sinh^4 x = (\cosh^2 x - \sinh^2 x)(\cosh^2 x + \sinh^2 x) = \frac{4}{4} \times \frac{2e^{2x} + 2e^{-2x}}{4}.$$

M1 for both brackets in exponentials.

$$= 1 \times \frac{1}{2}(e^{2x} + e^{-2x}) = \cosh 2x.$$

A1

(b) By (a), the equation is $\cosh 2x = 4 \sinh x + 7$. Use $\cosh 2x = 1 + 2 \sinh^2 x$, the form that matches the $\sinh x$ on the right: **M1**

$$1 + 2 \sinh^2 x = 4 \sinh x + 7 \Rightarrow \sinh^2 x - 2 \sinh x - 3 = 0 \Rightarrow (\sinh x - 3)(\sinh x + 1) = 0.$$

A1 for the quadratic, **M1** for solving it: $\sinh x = 3$ or $\sinh x = -1$. Both are allowed, because $\sinh x$ takes every real value.

$$x = \ln(3 + \sqrt{10}) \quad \text{or} \quad x = \ln(-1 + \sqrt{2}).$$

M1 for the logarithmic form, **A1** for both answers. (Each value of $\sinh x$ gives exactly one x , so there are two solutions.)

Common mistakes

- **Working on both sides at once.** Examiners keep reminding candidates that a proof of an identity must start from one side and reach the other. Writing the identity and then changing both sides until they match loses the final mark.
- **Skipping the expansion in a "prove" question.** The answer is given, so the marks are for the working. Write each function in exponentials and show the full expansion of every square or product.
- **Squaring wrongly.** $(e^x + e^{-x})^2 = e^{2x} + 2 + e^{-2x}$, not $e^{2x} + e^{-2x}$. The middle term is $2 \times e^x \times e^{-x} = 2$.
- **Mixing letters.** If the identity is in A (or t), don't write x halfway through a line. Mark schemes withhold the accuracy mark for this.
- **Losing a solution of** $\cosh x = c$. For $c > 1$ there are two answers, $\pm \ln(c + \sqrt{c^2 - 1})$.
- **Keeping impossible values.** $\cosh x = -2$, $\cosh x = \frac{1}{2}$, $\tanh x = 2$ and $e^x = -3$ have no real solutions. Say so and reject them.
- **Getting a sign wrong in Osborn's rule.** Only terms with a product of two sines change sign; $\sinh 2x = 2 \sinh x \cosh x$ has no sign change. And the rule is not a proof.
- **Sketches with the wrong features.** A pointed top on $y = \operatorname{sech} x$ (it must be smooth at $(0, 1)$), curves that cross their asymptotes, or a missing asymptote equation all lose marks.
- **Leaving answers unfinished.** If the question says "in terms of e ", $\sinh 2$ must become $\frac{1}{2}(e^2 - e^{-2})$; "exact" means logs and surds, not decimals.

How to get the marks

- **"Starting from the definitions"** means exponentials must appear. Quoting $\cosh^2 x - \sinh^2 x = 1$ from MF19 earns nothing in that part. The usual split is **B1** for the exponential forms, **M1** for expanding or putting over a common denominator, **A1** for reaching the printed result.
- **"Show that"** or a given answer: every step must be visible, and the last line must match the printed result exactly. Keep all earlier marks in mind: a given answer earns its final mark only if the working before it is complete.
- **"Hence"** means use the part before (usually the identity from part (a)). A different method may earn nothing.
- **"Exact" or "logarithmic form"** means an answer like $\ln 2$ or $\ln(3 + 2\sqrt{2})$, with the inside simplified. Decimals don't count.
- **Rejecting values:** write the reason, for example " $\cosh x \geq 1$, so $\cosh x = \frac{1}{2}$ is impossible".
- **Sketches:** correct shape and position earn one mark; key points and the asymptote (stated as an equation such as $y = 1$) earn the other. When two curves are asked for, their intersections must match your algebra.
- **Method marks follow through.** A slip in an early value still leaves method marks available later, so keep going.

Quick check

1. Find the exact values of $\sinh(\ln 4)$, $\cosh(\ln 4)$ and $\tanh(\ln 4)$.
2. Solve $\cosh 2x - 3 \cosh x = 8$, giving your answers in logarithmic form.
3. Show that the equation $3 \sinh x = 5 \cosh x$ has no real solutions.
4. Express $\operatorname{cosech}^{-1} \frac{5}{12}$ as a single logarithm.
5. Use Osborn's rule on $\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$ to write the matching identity for $\tanh 2A$. Check it with $A = \ln 2$.

Answers

1. $\sinh(\ln 4) = \frac{4 - \frac{1}{4}}{2} = \frac{15}{8}$, $\cosh(\ln 4) = \frac{4 + \frac{1}{4}}{2} = \frac{17}{8}$, $\tanh(\ln 4) = \frac{15}{17}$.
2. $\cosh 2x = 2 \cosh^2 x - 1$, so $2 \cosh^2 x - 3 \cosh x - 9 = 0$, that is $(2 \cosh x + 3)(\cosh x - 3) = 0$. $\cosh x = -\frac{3}{2}$ is impossible, so $\cosh x = 3$ and $x = \pm \ln(3 + \sqrt{8}) = \pm \ln(3 + 2\sqrt{2})$.
3. Dividing by $\cosh x$ (never 0) gives $\tanh x = \frac{5}{3}$. But $-1 < \tanh x < 1$ for every x , so there is no solution. (Or with exponentials: $3(e^x - e^{-x}) = 5(e^x + e^{-x})$ gives $e^{2x} = -4$, impossible.)
4. $\operatorname{cosech}^{-1} \frac{5}{12} = \sinh^{-1} \frac{12}{5} = \ln\left(\frac{12}{5} + \sqrt{\frac{144}{25} + 1}\right) = \ln\left(\frac{12}{5} + \frac{13}{5}\right) = \ln 5$.
5. $\tanh^2 A$ contains $\sin^2 A$, so its sign changes: $\tanh 2A \equiv \frac{2 \tanh A}{1 + \tanh^2 A}$. With $A = \ln 2$: $\tanh A = \frac{3}{5}$, so the right-hand side is $\frac{\frac{6}{5}}{1 + \frac{9}{25}} = \frac{\frac{6}{5}}{\frac{34}{25}} = \frac{15}{17}$, which matches $\tanh(2 \ln 2) = \tanh(\ln 4) = \frac{15}{17}$ from question 1.

Past questions to try

- **9231/22/O/N/21 Q8**: an identity with $\tanh$ and sech , then a reduction formula and exact values at a logarithm.
- **9231/22/O/N/23 Q7**: an identity for $\sinh^2 A$, then a surface area with a $\sinh$ substitution and an exact logarithm.
- **9231/23/M/J/25 Q2**: an identity with $\tanh$ and sech , then the length of a parametric curve.
- **9231/22/M/J/21 Q7**: an inverse sech function written as $\cosh^{-1}$, and its exact logarithmic value.