

Cambridge AS & A Level · Further Mathematics 9231 · Notes

Integration A2

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Read first: [Hyperbolic functions](#), [Differentiation](#), [Summation of series](#), [Polar coordinates](#)

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What you need to know

By the end of this topic you should be able to:

1. **Integrate hyperbolic functions**, such as $\sinh x$, $\cosh x$, $\operatorname{sech}^2 x$ and $\tanh x$, and powers such as $\cosh^2 x$ with an identity.
2. **Recognise the three standard forms** $\frac{1}{\sqrt{a^2 - x^2}}$, $\frac{1}{\sqrt{x^2 + a^2}}$ and $\frac{1}{\sqrt{x^2 - a^2}}$, completing the square first when the quadratic under the root is not already in that form.
3. **Integrate related functions with a trigonometric or hyperbolic substitution**, such as $x = a \sin \theta$, $x = a \sinh u$ or $x = a \cosh u$.
4. **Derive and use reduction formulae** to find definite integrals I_n for particular values of n .
5. **Use rectangles to approximate the area under a curve**: find upper and lower bounds for an integral, and use rectangles to prove inequalities for sums or find the limit of a sum.
6. **Find arc lengths** of curves given in Cartesian form, in parametric form or in polar coordinates.
7. **Find the area of a surface of revolution** when a curve given in Cartesian or parametric form is rotated about the x -axis or the y -axis. (Polar curves are not asked for here.)

Harder integration from A Level Mathematics can appear inside any of these, for example integration by parts twice, partial fractions or the substitution $t = \tan \frac{1}{2}x$. The identities and logarithmic forms of the hyperbolic functions are in the Hyperbolic functions note, and their derivatives in the Differentiation note; this note uses them.

Understand it

Integrating hyperbolic functions

Each derivative in the Differentiation note read backwards gives an integral:

$$\int \sinh x \, dx = \cosh x + c, \quad \int \cosh x \, dx = \sinh x + c, \quad \int \operatorname{sech}^2 x \, dx = \tanh x + c.$$

With an inside function kx , divide by k : $\int \cosh 3x \, dx = \frac{1}{3} \sinh 3x + c$. Two more come from $\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)|$:

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx = \ln \cosh x + c, \quad \int \coth x \, dx = \ln |\sinh x| + c.$$

(No modulus is needed for $\ln \cosh x$, because $\cosh x \geq 1$.)

Squares need a double-angle identity, just as $\cos^2 x$ does in trigonometry. From $\cosh 2x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$:

$$\int \cosh^2 x \, dx = \int \frac{1}{2}(\cosh 2x + 1) \, dx = \frac{1}{4} \sinh 2x + \frac{1}{2}x + c, \quad \int \sinh^2 x \, dx = \frac{1}{4} \sinh 2x - \frac{1}{2}x + c.$$

For example, since $\sinh(2 \ln 3) = \sinh(\ln 9) = \frac{9 - \frac{1}{9}}{2} = \frac{40}{9}$,

$$\int_0^{\ln 3} \sinh^2 x \, dx = \frac{1}{4} \times \frac{40}{9} - \frac{1}{2} \ln 3 = \frac{10}{9} - \frac{1}{2} \ln 3.$$

Products such as $x \cosh x$ need integration by parts: $\int x \cosh x \, dx = x \sinh x - \int \sinh x \, dx = x \sinh x - \cosh x + c$.

The three standard forms

The derivatives of $\sin^{-1}$, $\sinh^{-1}$ and $\cosh^{-1}$ give three integrals that MF19 lists (with $a > 0$):

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}, \quad \int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a}, \quad \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} \quad (x > a).$$

You can see why with a substitution. For the middle one, put $x = a \sinh u$, so $dx = a \cosh u \, du$ and $\sqrt{x^2 + a^2} = \sqrt{a^2(\sinh^2 u + 1)} = a \cosh u$. The integrand becomes $\frac{a \cosh u}{a \cosh u} = 1$, so the integral is $u = \sinh^{-1} \frac{x}{a}$. The others work the same way with $x = a \sin \theta$ ($\sqrt{a^2 - a^2 \sin^2 \theta} = a \cos \theta$) and $x = a \cosh u$ ($\sqrt{a^2 \cosh^2 u - a^2} = a \sinh u$).

To give an exact answer, turn the inverse hyperbolic function into a logarithm with MF19, for example

$\sinh^{-1} \frac{3}{4} = \ln \left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1} \right) = \ln 2$. Don't mix these up with the two forms without a square root: $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$, and $\int \frac{dx}{x^2 - a^2}$ is a logarithm (both also in MF19).

Completing the square turns any quadratic under the root into one of the three shapes. For example, $6x - x^2 = 9 - (x - 3)^2$, so

$$\int \frac{dx}{\sqrt{6x - x^2}} = \int \frac{dx}{\sqrt{3^2 - (x - 3)^2}} = \sin^{-1} \frac{x - 3}{3} + c.$$

The substitution $u = x - 3$ has $du = dx$, so the standard form applies with $x - 3$ in place of x .

Substitutions for related functions

The same substitutions work on other integrands containing these square roots, such as $\sqrt{a^2 - x^2}$ or

$$\frac{x^2}{\sqrt{x^2 + a^2}}.$$

Contains	Substitute	Because
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$a^2 - a^2 \sin^2 \theta = a^2 \cos^2 \theta$
$\sqrt{x^2 + a^2}$	$x = a \sinh u$	$a^2 \sinh^2 u + a^2 = a^2 \cosh^2 u$
$\sqrt{x^2 - a^2}$	$x = a \cosh u$	$a^2 \cosh^2 u - a^2 = a^2 \sinh^2 u$

Every substitution has the same four steps: replace dx (here $dx = 4 \cos \theta d\theta$), replace the rest of the integrand, change the limits, then integrate. For example, with $x = 4 \sin \theta$, the limits $x = 0$ and $x = 2$ become $\theta = 0$ and $\theta = \frac{\pi}{6}$:

$$\int_0^2 \sqrt{16 - x^2} dx = \int_0^{\pi/6} 4 \cos \theta \times 4 \cos \theta d\theta = 8 \int_0^{\pi/6} (1 + \cos 2\theta) d\theta = 8 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/6} = \frac{4\pi}{3} + 2\sqrt{3}.$$

After a hyperbolic substitution you often need $\cosh^2 u$ or $\sinh^2 u$ (use the double-angle forms above), and then $\sinh 2u = 2 \sinh u \cosh u$ to get back to values you know (Example 2).

Reduction formulae

A reduction formula links I_n , an integral that depends on a whole number n , to I_{n-1} or I_{n-2} . You find one simple integral such as I_0 or I_1 directly, then use the formula again and again to climb up to the one you want.

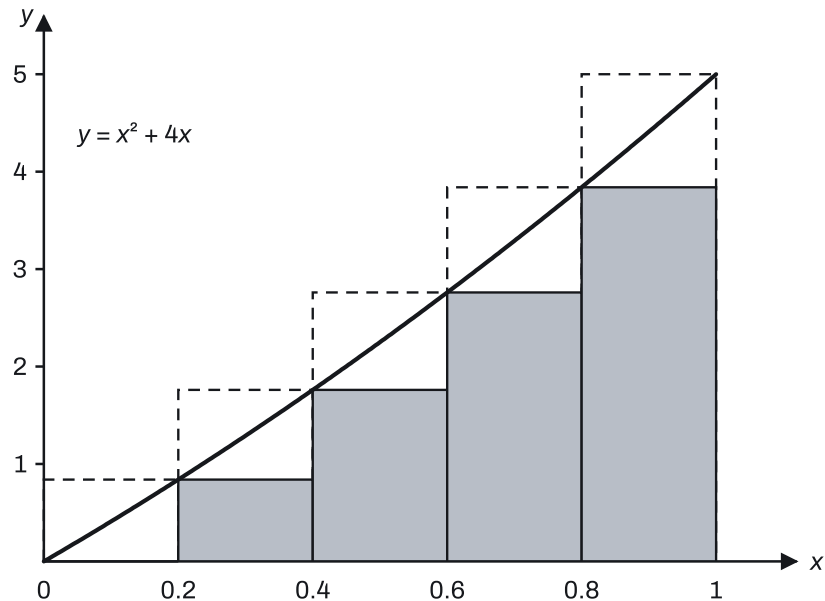
The usual tool is **integration by parts**, chosen so that differentiating lowers the power. For $I_n = \int_1^e (\ln x)^n dx$ (Example 3), differentiating $(\ln x)^n$ gives $n(\ln x)^{n-1} \cdot \frac{1}{x}$, and the $\frac{1}{x}$ cancels with the x from integrating 1. Other integrals need **an identity instead**. For $I_n = \int_0^{\pi/4} \tan^n x dx$, write $\tan^n x = \tan^{n-2} x (\sec^2 x - 1)$:

$$I_n = \int_0^{\pi/4} \tan^{n-2} x \sec^2 x dx - I_{n-2} = \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2}.$$

The first integral is of the form $\int f(x)^k f'(x) dx = \frac{f(x)^{k+1}}{k+1}$ with $f = \tan x$. Then $I_1 = [\ln \sec x]_0^{\pi/4} = \frac{1}{2} \ln 2$, $I_3 = \frac{1}{2} - I_1 = \frac{1}{2} - \frac{1}{2} \ln 2$, and $I_5 = \frac{1}{4} - I_3 = \frac{1}{2} \ln 2 - \frac{1}{4}$.

A question may give a hint such as "by considering $\frac{d}{dx}(x(1+x^3)^n)$ ". Differentiate that expression, rewrite the result in terms of the integrands of I_n and I_{n-1} , then integrate both sides between the limits.

Rectangles and bounds

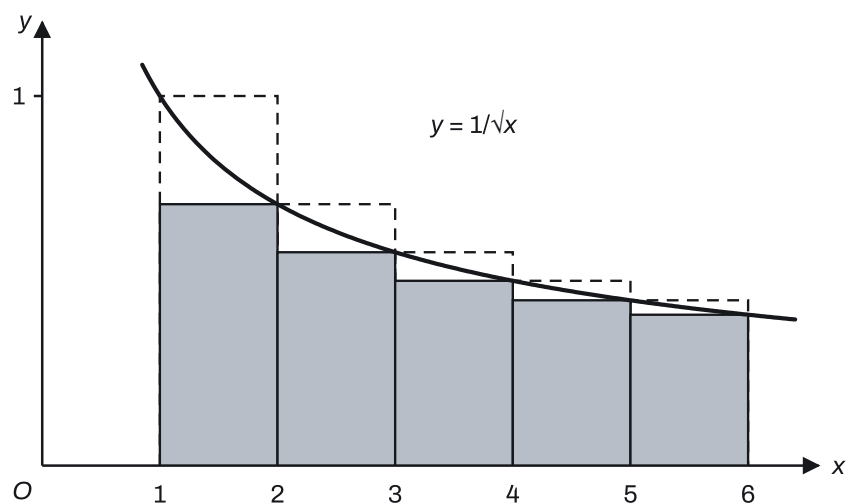


Split $[0, 1]$ into n strips of width $\frac{1}{n}$. On an **increasing** curve, a rectangle whose height is the curve's value at the **left** end of its strip lies under the curve, and one whose height is taken at the **right** end reaches above it. So

$$\underbrace{\frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)}_{\text{lower bound } L_n} < \int_0^1 f(x) dx < \underbrace{\frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right)}_{\text{upper bound } U_n}.$$

For a **decreasing** curve the two swap: right ends give the lower bound. The difference $U_n - L_n$ is just the first and last rectangles' difference, $\frac{1}{n}(f(1) - f(0))$, so it tends to 0 as $n \rightarrow \infty$. Both bounds then tend to the integral, which is how exam questions find the limit of a sum.

The sums are evaluated with MF19 results: $\sum r$, $\sum r^2$, $\sum r^3$ for polynomials, and the sum of a geometric progression for exponentials such as a^x , where the heights are $a^{1/n}$, $a^{2/n}$, $\dots$ with common ratio $a^{1/n}$.



Unit-width rectangles turn a sum into an integral. A rectangle of width 1 and height $f(r)$ has area $f(r)$, so a sum $\sum f(r)$ is the area of a row of rectangles. For the decreasing curve $y = \frac{1}{\sqrt{x}}$:

- the dashed rectangles on $[r, r+1]$, for $r = 1$ to n , lie above the curve, so $\sum_{r=1}^n \frac{1}{\sqrt{r}} > \int_1^{n+1} \frac{dx}{\sqrt{x}} = 2\sqrt{n+1} - 2$;
- the shaded rectangles on $[r-1, r]$, for $r = 2$ to n , lie under it, so $\sum_{r=2}^n \frac{1}{\sqrt{r}} < \int_1^n \frac{dx}{\sqrt{x}} = 2\sqrt{n} - 2$. Adding the $r = 1$ term, 1, to both sides gives $\sum_{r=1}^n \frac{1}{\sqrt{r}} < 2\sqrt{n} - 1$.

For $n = 5$ the sum is about 3.232, between $2\sqrt{6} - 2 \approx 2.899$ and $2\sqrt{5} - 1 \approx 3.472$. The limits of the integral must match exactly the strips the rectangles cover.

Arc length

Zoom in on a curve until a tiny piece looks straight. Its length δs is the hypotenuse of a right-angled triangle with sides δx and δy , so $(\delta s)^2 \approx (\delta x)^2 + (\delta y)^2$. Dividing by $(\delta x)^2$, $(\delta t)^2$ or $(\delta \theta)^2$ and adding up the pieces gives the three formulas:

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \quad s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt, \quad s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

In exam questions the expression under the root is designed to be a perfect square, so **simplify it before integrating**. For the cardioid $r = 1 + \cos \theta$:

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 1 + 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 + 2\cos\theta = 4\cos^2\frac{\theta}{2},$$

so the whole curve, $-\pi \leq \theta \leq \pi$ (where $\cos \frac{\theta}{2} \geq 0$), has length $\int_{-\pi}^{\pi} 2\cos\frac{\theta}{2} d\theta = \left[4\sin\frac{\theta}{2}\right]_{-\pi}^{\pi} = 8$.

Surface area of revolution

When a small piece of curve of length δs at height y is rotated about the x -axis, it sweeps out a thin band of radius y , whose area is about $2\pi y \delta s$. Adding the bands:

$$S = 2\pi \int y ds = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{or} \quad 2\pi \int_{t_1}^{t_2} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

About the y -axis the radius is x instead, so y becomes x in front of the root. The ds part is exactly the arc length's integrand. **Check:** the semicircle $y = \sqrt{a^2 - x^2}$ has $1 + \left(\frac{dy}{dx}\right)^2 = \frac{a^2}{a^2 - x^2}$, so $y\sqrt{\dots} = a$, and $S = 2\pi \int_{-a}^a a dx = 4\pi a^2$, the surface area of a sphere.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\int \sinh x \, dx = \cosh x, \int \cosh x \, dx = \sinh x, \int \operatorname{sech}^2 x \, dx = \tanh x$	Given
$\int \tanh x \, dx = \ln \cosh x, \int \coth x \, dx = \ln \sinh x $	Learn it
$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x, \int \operatorname{cosech}^2 x \, dx = -\coth x$	Learn it
$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1), \sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$	Learn it
$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \, (x < a), \int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a}, \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} \, (x > a)$	Given
$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}, \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left \frac{x-a}{x+a} \right , \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left \frac{a+x}{a-x} \right $	Given
$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$	Given
Integration by parts $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx; \int \frac{f'(x)}{f(x)} dx = \ln f(x) $	Given
$\int \sec x \, dx = \ln \sec x + \tan x , \int \operatorname{cosec} x \, dx = -\ln \operatorname{cosec} x + \cot x $	Given
$t = \tan \frac{1}{2}x: \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$	Given
Substitutions $x = a \sin \theta, x = a \sinh u, x = a \cosh u$ for $\sqrt{a^2 - x^2}, \sqrt{x^2 + a^2}, \sqrt{x^2 - a^2}$	Learn it
$\sum r = \frac{1}{2}n(n+1), \sum r^2 = \frac{1}{6}n(n+1)(2n+1), \sum r^3 = \frac{1}{4}n^2(n+1)^2; S_n = \frac{a(1-r^n)}{1-r}$	Given
Increasing f : $\frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) < \int_0^1 f(x) \, dx < \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right)$ (reversed if decreasing)	Learn it
Arc length $s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \int \sqrt{\dot{x}^2 + \dot{y}^2} dt, \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$	Learn it
Surface area about the x -axis $S = 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ or $2\pi \int y \sqrt{\dot{x}^2 + \dot{y}^2} dt$ (about the y -axis, x in place of y)	Learn it

How to do it

In the Paper 2 questions we have tagged for this topic (78 questions, 2021–2025; 24 of them appear in two papers of the same session, so there are 54 different questions), the types came up this often. 10 of the 78 questions mix two or three types, so the counts add up to more than 78. Some also have parts on hyperbolic

identities, differentiation, complex numbers or differential equations; those parts are taught in their own topics, and only the integration is counted here.

Question type	Questions
Rectangles: bounds for integrals and sums, and limits	32
Standard integrals and substitutions	19
Reduction formulae	15
Arc length	14
Surface area of revolution	9

1. Rectangles: bounds for integrals and sums, and limits (32 questions)

- Decide whether the curve is increasing or decreasing** on the interval (from the diagram, or from the sign of $f'(x)$). That tells you which ends give rectangles above the curve.
- Write the sum of the areas**, each term as width $\times$ height, showing at least the first two terms and the last one, for example $\frac{1}{n}f\left(\frac{1}{n}\right) + \frac{1}{n}f\left(\frac{2}{n}\right) + \cdots + \frac{1}{n}f\left(\frac{n}{n}\right)$.
- State the comparison with the integral**, with the inequality the right way round and the correct limits, and say why (the rectangles lie above or below the curve).
- Evaluate the sum** with the MF19 results for $\sum r$, $\sum r^2$, $\sum r^3$ or the geometric series, and the integral if it is needed, then simplify to the printed form.
- For the other bound, **repeat with the other ends** (step 2 with $r = 0$ to $n - 1$ instead of $r = 1$ to n). Its algebra is almost the same, so adapt your working.

Variations you will meet:

- $\lim(U_n - L_n) = 0$: simplify $U_n - L_n$ to a single term such as $\frac{c}{n}$ (Example 4), then say it tends to 0 as $n \rightarrow \infty$.
- The least n with $U_n - L_n < 10^{-k}$** : solve the inequality for n and give the next whole number.
- Bounds for a sum from an integral**, where the sum has n inside it, such as $\sum_{r=1}^n \frac{r}{n^2+r^2}$. Write $\frac{1}{n}f\left(\frac{r}{n}\right)$ for the curve given and simplify until it matches the sum; for this one $f(x) = \frac{x}{1+x^2}$, because $\frac{1}{n} \cdot \frac{r/n}{1+r^2/n^2} = \frac{r}{n^2+r^2}$, and $\int_0^1 f(x) dx = \frac{1}{2} \ln 2$. The integral is often a standard form (type 2). For the other bound, the sum over $r = 0$ to $n - 1$ is the sum you want plus the $r = 0$ term minus the $r = n$ term, so add and subtract those.
- "Deduce the exact value of $\lim_{n \rightarrow \infty}$ of the sum"**: both bounds tend to the same integral, so the limit is the integral's exact value.
- Unit-width rectangles** for $\sum_{r=1}^N f(r)$: draw or read which strips the rectangles cover, integrate over exactly those strips, and add any terms left out (as with $\frac{1}{\sqrt{x}}$ in "Understand it"). The integral often needs a technique from type 2, integration by parts, or a hyperbolic integral such as $\int \tanh x dx = \ln \cosh x$.
- Estimates of a constant**: if the exact integral contains a constant such as $\ln 2$ or π , rearrange $L_n < \int < U_n$ to get bounds on that constant (reversing the inequalities if you take reciprocals).

- **A series in the bound:** a limit involving $e^{\pm 1/n}$ is found by putting $x = \pm \frac{1}{n}$ into the MF19 series for e^x and keeping the first few terms.

2. Standard integrals and substitutions (19 questions)

1. **Match the integrand to a form:** a hyperbolic function (or its square), one of the three square-root forms, or a root that needs a substitution. For a quadratic under a root, **complete the square** first.
2. **With a substitution** (given, or from the table in "Understand it"): replace dx using $\frac{dx}{du}$, rewrite everything else in terms of u with an identity, and **change the limits**.
3. **Integrate** and substitute the limits.
4. **Give the answer in the form asked for:** an inverse hyperbolic function becomes a logarithm with MF19, and a $\sinh 2u$ becomes $2 \sinh u \cosh u$ so that known values can be used.

Variations you will meet:

- **An indefinite integral with a given substitution:** give the answer in terms of x , not u , and include $+c$. For example, $x = a \sinh u$ gives $u = \sinh^{-1} \frac{x}{a}$.
- **A coefficient on x^2 :** take it out or choose the substitution to cancel it. $\int \frac{dx}{\sqrt{9x^2+4}} = \frac{1}{3} \int \frac{dx}{\sqrt{x^2+\frac{4}{9}}} = \frac{1}{3} \sinh^{-1} \frac{3x}{2}$, or use $x = \frac{2}{3} \sinh u$ directly.
- **An integral in part (a) that a later part needs**, such as for the integrating factor or the right-hand side of a differential equation: find it here, with the constant, and use it there.
- **An identity from an earlier part:** if a power of $\cos \theta$ has been written as a sum of multiple angles and a constant, integrate term by term with $\int \cos k\theta d\theta = \frac{1}{k} \sin k\theta$. The same applies to a hyperbolic identity from part (a), such as a double-angle formula.
- **A rational function after substituting:** divide first if the top has the same degree as the bottom, for example $\frac{u^2}{u^2+4} = 1 - \frac{4}{u^2+4}$, then use the $\tan^{-1}$ form. With $u^2 - a^2$ on the bottom, use partial fractions or the MF19 form for $\frac{1}{x^2-a^2}$.
- **Recognition:** $\int f(x)^k f'(x) dx = \frac{f(x)^{k+1}}{k+1}$ when the rest of the integrand is the derivative of the function being raised to a power, for example $\int \cos \theta \sin^4 \theta d\theta = \frac{1}{5} \sin^5 \theta + c$.

3. Reduction formulae (15 questions)

1. **Choose the parts** so that the power goes down when you differentiate: $u = x^n$ with $\frac{dv}{dx} = \sinh x$ or e^x , $u = (\ln x)^n$ with $\frac{dv}{dx} = 1$, $u = \cos^{n-1} \theta$ with $\frac{dv}{dx} = \cos \theta$.
2. **Integrate by parts**, writing the $[uv]$ term with its limits.
3. **If the new integral is not yet I_{n-1} or I_{n-2}** , either integrate by parts again (for x^n times $\sinh x$, $\cosh x$, $\sin x$ or $\cos x$, two rounds reach I_{n-2}) or use an identity, such as $\sin^2 \theta = 1 - \cos^2 \theta$ or $\tanh^2 x = 1 - \operatorname{sech}^2 x$, to split it into I_n and I_{n-2} .
4. **Evaluate the square-bracket terms** at the limits (terms with x^n at $x = 0$ vanish for $n \geq 1$) and rearrange to the printed formula. If I_n appears on both sides, collect it.
5. **To find a particular I_n :** work out the simplest one (I_0 or I_1) directly, then apply the formula step by step.

Variations you will meet:

- **"By considering $\frac{d}{dx}(x(1+x^k)^n)$ ":** differentiate with the product rule, write $x^k = (1+x^k) - 1$ so that the result is a combination of $(1+x^k)^n$ and $(1+x^k)^{n-1}$, then integrate both sides between the limits. Reports say this was more successful than integrating by parts.
- **Negative or zero n :** a formula may be used with $n = 0$ or a negative n , starting from an I_n that is a standard integral (for example a $\tan^{-1}$ or $\sinh^{-1}$ form).
- **A hint for a derivative**, such as the derivative of a power of a hyperbolic function: use it for the $\frac{du}{dx}$ in the integration by parts.
- **Linked to another part:** a reduction formula combined with a power of $\cos \theta$ from de Moivre's theorem, or an arc length s that turns out to be one of the I_n after a substitution.

4. Arc length (14 questions)

1. **Pick the formula** for the form of the curve: Cartesian, parametric or polar.
2. **Differentiate**, then **simplify the expression under the root before integrating**. Expand, collect, and look for a perfect square, such as $(1+t^2)^2$ or $4\cos^2 \frac{\theta}{2}$, or use an identity ($1 + \sinh^2 x = \cosh^2 x$, $\tanh^2 t + \operatorname{sech}^2 t = 1$, $1 + \tan^2 x = \sec^2 x$).
3. **Take the square root** (check it is positive over the interval) and integrate between the limits.
4. **Give an exact answer** in the form asked for.

Variations you will meet:

- **Polar curves such as $r = e^{k\theta}$:** $r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1+k^2)e^{2k\theta}$, so the root is $\sqrt{1+k^2}e^{k\theta}$.
- **Find a limit from the length**, such as "given the length is s , find a in terms of s ": integrate with the unknown limit, set equal to s and rearrange, often with logarithms.
- **"Show that $s = \int \dots$ "** in a new variable: do the simplification, then the substitution with its limits, and reach the printed integral exactly.
- **With a series:** if the integral can't be done exactly, a Maclaurin series of the integrand, integrated term by term, gives an approximation.
- **Trigonometric curves:** if the root becomes $\sqrt{1+\tan^2 x} = \sec x$, the integral $\int \sec x \, dx$ is in MF19.

5. Surface area of revolution (9 questions)

1. **Write the formula with the right radius:** y for rotation about the x -axis, x for the y -axis, in front of the arc length root. Include 2π and the limits.
2. **Simplify the root** as for arc length.
3. **Multiply out and integrate**, often needing integration by parts or a double-angle identity.
4. **Give the answer** in the form asked for (in terms of π and e , or π and logarithms).

Variations you will meet:

- **A given substitution to reach a printed integral:** find $\frac{du}{dx}$, replace every x part, including the radius, and **show the working for the new limits** (for example, evaluate $\operatorname{sech} x$ at the top limit). The result is often a rational function in u (type 2), or needs a second, hyperbolic substitution.
- **Parametric curves:** multiply out $y(\dots)$ before integrating. A product with t may need parts, and an odd function over a symmetrical interval integrates to 0.
- **"In terms of $\sinh 2$ and $\cosh 2$ " or " π and e ":** convert a $\cosh 4$ or $\sinh 4$ with a double-angle formula, or write each hyperbolic value in exponentials.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
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Example 1

The curve C has equation $y = \cosh x$ for $0 \leq x \leq \ln 2$.

(a) Find the exact length of C . **[3]**

(b) Find the exact area of the surface generated when C is rotated through 2π radians about the x -axis. **[4]**

(a) $\frac{dy}{dx} = \sinh x$, so $\sqrt{1 + \sinh^2 x} = \cosh x$. **B1**

$$s = \int_0^{\ln 2} \cosh x \, dx = \left[\sinh x \right]_0^{\ln 2}.$$

M1 for the arc length formula with the root simplified.

$$\sinh(\ln 2) = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}, \text{ so } s = \frac{3}{4}. \text{ **A1**}$$

(b) The radius is $y = \cosh x$:

$$S = 2\pi \int_0^{\ln 2} \cosh x \times \cosh x \, dx = \pi \int_0^{\ln 2} (1 + \cosh 2x) \, dx.$$

M1 for the surface area formula, **M1** for the double-angle identity.

$$= \pi \left[x + \frac{1}{2} \sinh 2x \right]_0^{\ln 2} = \pi \left(\ln 2 + \frac{1}{2} \sinh(\ln 4) \right) = \pi \left(\ln 2 + \frac{1}{2} \times \frac{15}{8} \right) = \pi \left(\ln 2 + \frac{15}{16} \right).$$

A1 for the integral, **A1** for the exact answer.

Example 2

(a) Find $\int_{-2}^2 \frac{1}{\sqrt{x^2 + 4x + 13}} dx$, giving your answer in the form $\ln a$. **[4]**

(b) Use the substitution $x = 3 \sinh u$ to show that $\int_0^4 \frac{x^2}{\sqrt{x^2 + 9}} dx = 10 - \frac{9}{2} \ln 3$. **[6]**

(a) $x^2 + 4x + 13 = (x + 2)^2 + 9$. **B1**

$$\int_{-2}^2 \frac{dx}{\sqrt{(x+2)^2 + 3^2}} = \left[\sinh^{-1} \frac{x+2}{3} \right]_{-2}^2 = \sinh^{-1} \frac{4}{3} - \sinh^{-1} 0.$$

M1 A1 for the standard form with $a = 3$ and $x + 2$.

$$\sinh^{-1} \frac{4}{3} = \ln \left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1} \right) = \ln \left(\frac{4}{3} + \frac{5}{3} \right) = \ln 3.$$

A1

(b) $dx = 3 \cosh u \, du$ and $\sqrt{9 \sinh^2 u + 9} = 3 \cosh u$. **B1**

$$\int \frac{9 \sinh^2 u}{3 \cosh u} \times 3 \cosh u \, du = 9 \int \sinh^2 u \, du = \frac{9}{2} \int (\cosh 2u - 1) \, du.$$

M1 for substituting fully, **M1** for the double-angle identity.

$$= \frac{9}{2} \left[\frac{1}{2} \sinh 2u - u \right] = \frac{9}{2} \left[\sinh u \cosh u - u \right].$$

A1

Limits: $x = 0$ gives $u = 0$. $x = 4$ gives $\sinh u = \frac{4}{3}$, so $\cosh u = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$ and $u = \sinh^{-1} \frac{4}{3} = \ln 3$ (from part (a)). **M1** for the new limits.

$$\frac{9}{2} \left(\frac{4}{3} \times \frac{5}{3} - \ln 3 \right) - 0 = \frac{9}{2} \times \frac{20}{9} - \frac{9}{2} \ln 3 = 10 - \frac{9}{2} \ln 3.$$

A1, the given answer.

Example 3

The integral I_n is defined by $I_n = \int_1^e (\ln x)^n dx$, where n is a non-negative integer.

(a) Show that, for $n \geq 1$, $I_n = e - nI_{n-1}$. **[3]**

(b) Find the exact value of I_4 . **[3]**

(a) Integrate by parts with $u = (\ln x)^n$, $\frac{dv}{dx} = 1$, so $\frac{du}{dx} = \frac{n(\ln x)^{n-1}}{x}$ and $v = x$: **M1**

$$I_n = \left[x(\ln x)^n \right]_1^e - \int_1^e x \times \frac{n(\ln x)^{n-1}}{x} dx.$$

A1

At $x = e$, $e(\ln e)^n = e$; at $x = 1$, $1 \times 0^n = 0$ (for $n \geq 1$). So $I_n = e - nI_{n-1}$. **A1**, the given answer with the limits shown.

$$(b) I_0 = \int_1^e 1 \, dx = e - 1. \text{ **B1**}$$

$I_1 = e - (e - 1) = 1$, $I_2 = e - 2 \times 1 = e - 2$, $I_3 = e - 3(e - 2) = 6 - 2e$. **M1** for using the formula repeatedly.

$$I_4 = e - 4(6 - 2e) = 9e - 24. \text{ **A1**}$$

Example 4

The curve $y = x^2 + 4x$, for $0 \leq x \leq 1$, is drawn with a set of n rectangles of width $\frac{1}{n}$ whose heights are the values of y at the right-hand end of each strip.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 (x^2 + 4x) \, dx < U_n$, where $U_n = \frac{(n+1)(14n+1)}{6n^2}$. **[4]**

(b) Use a similar method to find, in terms of n , a lower bound L_n for $\int_0^1 (x^2 + 4x) \, dx$. **[3]**

(c) Find the least value of n for which $U_n - L_n < 10^{-2}$. **[2]**

(a) The curve is increasing on $[0, 1]$ ($\frac{dy}{dx} = 2x + 4 > 0$), so rectangles with right-end heights lie above it:

$$\int_0^1 (x^2 + 4x) \, dx < \frac{1}{n} \left[\left(\frac{1}{n^2} + \frac{4}{n} \right) + \left(\frac{4}{n^2} + \frac{8}{n} \right) + \cdots + \left(\frac{n^2}{n^2} + \frac{4n}{n} \right) \right] = \frac{1}{n^3} \sum_{r=1}^n r^2 + \frac{4}{n^2} \sum_{r=1}^n r.$$

M1 A1 for the sum of the areas and the comparison.

$$= \frac{(n+1)(2n+1)}{6n^2} + \frac{2(n+1)}{n} = \frac{(n+1)[(2n+1) + 12n]}{6n^2} = \frac{(n+1)(14n+1)}{6n^2}.$$

M1 for the MF19 sums, **A1** for the given answer.

(b) Left-end heights ($r = 0$ to $n - 1$) give rectangles under the curve. **M1**

$$L_n = \frac{1}{n^3} \sum_{r=1}^{n-1} r^2 + \frac{4}{n^2} \sum_{r=1}^{n-1} r = \frac{(n-1)(2n-1)}{6n^2} + \frac{2(n-1)}{n} = \frac{(n-1)(14n-1)}{6n^2}.$$

M1 for the sums with $n - 1$, **A1**.

$$(c) U_n - L_n = \frac{(14n^2 + 15n + 1) - (14n^2 - 15n + 1)}{6n^2} = \frac{30n}{6n^2} = \frac{5}{n}. \text{ **M1**}$$

$\frac{5}{n} < 0.01 \Rightarrow n > 500$, so the least value is $n = 501$. **A1**

(Check: as $n \rightarrow \infty$ both bounds tend to $\frac{14}{6} = \frac{7}{3}$, and $\int_0^1 (x^2 + 4x) \, dx = \frac{1}{3} + 2 = \frac{7}{3}$.)

Common mistakes

- **Not justifying the comparison with the integral.** Reports often say conclusions were reached without justification, especially with rectangles. State which rectangles lie above or below the curve and why, and write the inequality with its limits.
- **The wrong number of rectangles, or limits that don't match.** Count the terms of your sum, and make the integral cover exactly the strips your rectangles cover. For unit-width rectangles, add back any term left out.
- **Not simplifying before integrating.** Reports praise candidates who simplified $\dot{x}^2 + \dot{y}^2$ (or $r^2 + r'^2$) to a perfect square before putting it into the formula; those who didn't made more errors.
- **Forgetting the radius or the 2π .** A surface area needs $2\pi y$ (or $2\pi x$) inside the integral, with the ds root. Taking y outside the integral earns no method mark.
- **Forgetting to change the limits,** or leaving the answer in terms of u . When you substitute, change the limits or go back to x ; an indefinite integral must be in x with $+c$ (reports note the constant often goes missing).
- **Using $\ln$ where there is no derivative on top.** $\int \frac{dx}{x^2+4x+13}$ is not $\ln(x^2 + 4x + 13)$: complete the square and use a standard form.
- **Losing signs in integration by parts.** In a reduction formula a second round of parts has a minus sign in front of a bracket; keep the brackets and the limits in every line, since a given answer loses its final mark if they are missing.
- **Starting the reduction from the wrong place.** Find the simplest integral (I_0 or I_1) correctly first; a wrong I_0 spoils every later value.
- **Answers in the wrong form.** If the question asks for π and e , or for $\sinh 2$ and $\cosh 2$, convert $\cosh 4$ or $\sinh(\ln 3)$ values before stopping.

How to get the marks

- **"Show that" and given answers:** the marks are for the working. For a reduction formula, write the $[uv]$ term and the new integral, then the limits substituted, then the rearrangement. For a given substitution, show $\frac{dx}{du}$ (or $\frac{du}{dx}$), the integrand in u , and how each new limit was found.
- **Rectangles:** the usual pattern is **M1 A1** for the sum of areas (at least the first and last terms, with the right width and number of rectangles), **M1 A1** for using the MF19 sum formula or the geometric series, and a final **A1** for a clear comparison with the integral.
- **"Hence"** means use the previous result, such as the substitution's integral inside a bound, or the identity from part (a).
- **"Exact"** means fractions, surds, π , e and logarithms, with inverse hyperbolic values turned into logarithms when a form such as $\ln a$ is asked for. Decimals score nothing.
- **"Least value of n ":** show the simplified $U_n - L_n$, solve the inequality, and state a whole number.
- **Arc length and surface area:** the method marks are for a correct formula with correct limits, for simplifying the root and for integrating; the answer mark needs the exact form.

- **Method marks follow through.** A slip in the sum or in I_0 still leaves the later method marks available, so keep going.

Quick check

1. Find the exact values of (a) $\int_0^{\ln 3} \tanh x \, dx$, (b) $\int_3^5 \frac{dx}{\sqrt{x^2 - 9}}$, giving each answer as a logarithm.
2. Find the exact value of $\int_{5/2}^{11/2} \frac{dx}{\sqrt{8x - x^2 - 7}}$.
3. Let $I_n = \int_0^{\ln 2} \tanh^n x \, dx$. Show that, for $n \geq 2$, $I_n = I_{n-2} - \frac{1}{n-1} \left(\frac{3}{5}\right)^{n-1}$, and hence find I_4 .
4. A curve has parametric equations $x = t^2$, $y = t - \frac{1}{3}t^3$, for $0 \leq t \leq 1$. Find its length, and the area of the surface generated when it is rotated through 2π radians about the x -axis.
5. By considering suitable rectangles of unit width under and over the curve $y = \frac{x}{x^2 + 1}$ (which is decreasing for $x \geq 1$), show that $\frac{1}{2} \ln \frac{(n+1)^2 + 1}{2} < \sum_{r=1}^n \frac{r}{r^2 + 1} < \frac{1}{2} + \frac{1}{2} \ln \frac{n^2 + 1}{2}$.

Answers

1. (a) $\left[\ln \cosh x \right]_0^{\ln 3} = \ln \cosh(\ln 3) = \ln \frac{3 + \frac{1}{3}}{2} = \ln \frac{5}{3}$. (b) $\left[\cosh^{-1} \frac{x}{3} \right]_3^5 = \cosh^{-1} \frac{5}{3} - \cosh^{-1} 1 = \ln \left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1} \right) - 0 = \ln \left(\frac{5}{3} + \frac{4}{3} \right) = \ln 3$.

2. $8x - x^2 - 7 = 9 - (x - 4)^2$, so the integral is $\left[\sin^{-1} \frac{x-4}{3} \right]_{5/2}^{11/2} = \sin^{-1} \frac{1}{2} - \sin^{-1} \left(-\frac{1}{2} \right) = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$.

3. $\tanh^n x = \tanh^{n-2} x (1 - \operatorname{sech}^2 x)$, so $I_n = I_{n-2} - \int_0^{\ln 2} \tanh^{n-2} x \operatorname{sech}^2 x \, dx = I_{n-2} - \left[\frac{\tanh^{n-1} x}{n-1} \right]_0^{\ln 2}$, and $\tanh(\ln 2) = \frac{4-1}{4+1} = \frac{3}{5}$. Then $I_0 = \ln 2$, $I_2 = \ln 2 - \frac{3}{5}$, $I_4 = I_2 - \frac{1}{3} \times \frac{27}{125} = \ln 2 - \frac{3}{5} - \frac{9}{125} = \ln 2 - \frac{84}{125}$.

4. $\dot{x} = 2t$, $\dot{y} = 1 - t^2$, so $\dot{x}^2 + \dot{y}^2 = 4t^2 + 1 - 2t^2 + t^4 = (1 + t^2)^2$. Length $= \int_0^1 (1 + t^2) \, dt = \frac{4}{3}$. Surface area $= 2\pi \int_0^1 \left(t - \frac{1}{3}t^3 \right) (1 + t^2) \, dt = 2\pi \int_0^1 \left(t + \frac{2}{3}t^3 - \frac{1}{3}t^5 \right) \, dt = 2\pi \left(\frac{1}{2} + \frac{1}{6} - \frac{1}{18} \right) = \frac{11\pi}{9}$.

5. Let $f(x) = \frac{x}{x^2 + 1}$, with $\int f(x) \, dx = \frac{1}{2} \ln(x^2 + 1)$. Rectangles on $[r, r+1]$ with height $f(r)$, for $r = 1$ to n , lie above the curve, so $\sum_{r=1}^n f(r) > \int_1^{n+1} f(x) \, dx = \frac{1}{2} \ln \frac{(n+1)^2 + 1}{2}$. Rectangles on $[r-1, r]$ with height $f(r)$, for $r = 2$ to n , lie under it, so $\sum_{r=2}^n f(r) < \int_1^n f(x) \, dx = \frac{1}{2} \ln \frac{n^2 + 1}{2}$; adding $f(1) = \frac{1}{2}$ gives the upper bound.

Past questions to try

- **9231/23/O/N/24 Q3**: a surface of revolution with two substitutions, the second one hyperbolic.
- **9231/22/O/N/24 Q3**: the exact length of a parametric curve, simplifying to a perfect square.
- **9231/22/M/J/25 Q2**: a reduction formula with two rounds of integration by parts and a hyperbolic function.
- **9231/23/O/N/24 Q6**: upper and lower bounds from rectangles, the least n , and bounds for $\ln 2$.