

Linear motion under a variable force

A2

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What you need to know

By the end of this topic you should be able to:

1. **Model straight-line motion under a force that changes**, where the force depends on the time t , the displacement x or the velocity v (or on two of these). Write Newton's second law as a **differential equation**, choosing the right form of the acceleration: $\frac{dv}{dt}$, or $v \frac{dv}{dx}$ when the question links v and x .
2. **Solve that differential equation by separating the variables**, use the starting conditions to find the constant, and answer the question: v in terms of t or x , x in terms of t , a time, a distance, a greatest height or a limiting speed.

Only equations whose variables separate are tested, and the integration needed is the integration from Pure Mathematics 3 (9709): powers, $\ln$, exponentials, partial fractions, substitution, integration by parts, $\tan^{-1}$ and simple trigonometry. The constant-acceleration (suvat) formulas **never** apply here, because the acceleration is not constant.

Understand it

Why suvat stops working

The suvat formulas come from assuming the acceleration is the same at every instant. Air or water resistance, a pull that depends on position, or a driving force that fades all change as the particle moves, so the acceleration changes too. Newton's second law is still true **at every instant**:

$$\text{resultant force} = m \times \text{acceleration},$$

but now the acceleration is a derivative, so the law becomes a differential equation. Solving it gives the motion.

Three ways to write the acceleration

Velocity is the rate of change of displacement, and acceleration is the rate of change of velocity:

$$v = \frac{dx}{dt}, \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

There is a third form. By the chain rule,

$$a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v \frac{dv}{dx}.$$

It is also $\frac{d}{dx} \left(\frac{1}{2} v^2 \right)$, since differentiating $\frac{1}{2} v^2$ with respect to x gives $v \frac{dv}{dx}$. For example, if $v = 4 \sin x$ then $v \frac{dv}{dx} = 4 \sin x \times 4 \cos x = 8 \sin 2x$.

Which form to use depends on the two variables the question links:

The question links

Write the acceleration as

v and t ("find v in terms of t ", "find the time when ...")

$$\frac{dv}{dt}$$

v and x ("find v in terms of x ", "find the distance when ...")

$$v \frac{dv}{dx}$$

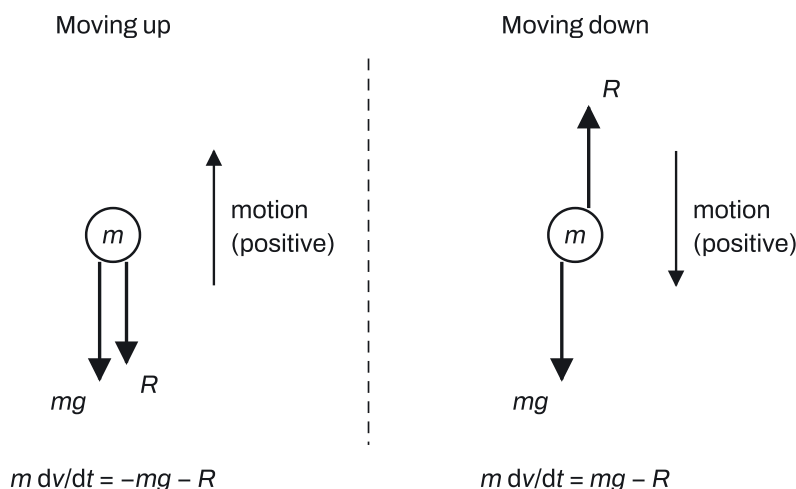
x and t

first find v , then use $v = \frac{dx}{dt}$

A force that depends on x **must** use $v \frac{dv}{dx}$, because $\frac{dv}{dt} = f(x)$ has three variables and will not separate.

Setting up the equation: signs matter

Choose a positive direction (normally the direction the particle is moving, or the direction of increasing x) and add up the forces with signs: a force acting that way is $+$, a force acting the other way is $-$. A **resistance** always acts against the motion.



On the way **up**, weight and resistance both slow the particle down, so both are negative. On the way **down**, the weight speeds it up and the resistance slows it down. That is why rising and falling need different equations, and why a question about a ball thrown upwards says "while it is moving upwards".

A force "towards O " when the particle is on the positive side of O is negative. Draw a small sketch with the direction of motion and every force before writing anything.

Separating the variables

A boat of mass 400 kg is moving at 6 m s^{-1} when its engine is switched off. The water then gives a resistance of $80v$ newtons and there is no other horizontal force. Taking the direction of motion as positive,

$$400 \frac{dv}{dt} = -80v \quad \Rightarrow \quad \frac{dv}{dt} = -0.2v.$$

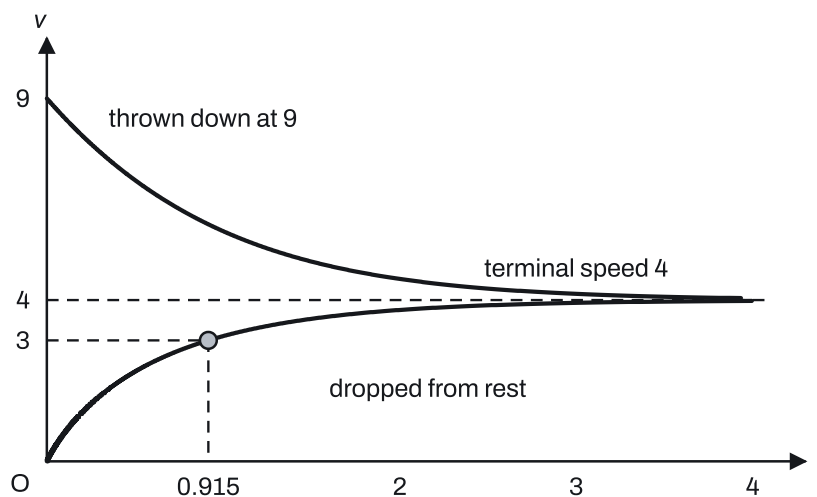
Write the mass first: Newton's second law has m in it. Now put everything with v on one side and everything with t on the other, then integrate:

$$\int \frac{1}{v} dv = \int -0.2 dt \quad \Rightarrow \quad \ln v = -0.2t + c.$$

When $t = 0$, $v = 6$, so $c = \ln 6$. Then $\ln \frac{v}{6} = -0.2t$, which gives $v = 6e^{-0.2t}$.

Limiting (terminal) speed

When a constant force (a weight, or an engine's driving force) acts against a resistance that grows with speed, the speed settles towards a value where the two balance. You can find it without solving anything: the speed stops changing when the acceleration is zero. For a seed falling with resistance $5m\sqrt{v}$ (Example 4 below), $m \frac{dv}{dt} = mg - 5m\sqrt{v}$, and $10 - 5\sqrt{v} = 0$ gives $v = 4$. This is the **terminal speed** (or limiting speed): the forces balance. A particle that starts slower than this speeds up towards it; one that starts faster slows down towards it. It never quite reaches it.



The diagram shows Example 4 below: both curves flatten towards the same dashed line, from opposite sides.

From velocity to displacement

Once you have v in terms of t , integrate again using $v = \frac{dx}{dt}$, with a second constant from the starting position. For the boat, measuring x from where the engine stopped, $x = \int 6e^{-0.2t} dt = -30e^{-0.2t} + c'$, and $x = 0$ at $t = 0$ gives $c' = 30$. So $x = 30(1 - e^{-0.2t})$: the boat never glides as far as 30 m.

If instead you have v in terms of x , write $\frac{dx}{dt} = v(x)$ and separate again: $\int \frac{1}{v(x)} dx = \int 1 dt$.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
Newton's second law: resultant force = ma	Learn it
$v = \frac{dx}{dt}, \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	Learn it
$a = v \frac{dv}{dx}$ (use when the question links v and x)	Learn it
Weight = mg , with $g = 10 \text{ m s}^{-2}$	Learn it
Terminal (limiting) speed: the value of v that makes the acceleration zero	Learn it
$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln ax+b + c$	Learn it
$\int x^n dx = \frac{x^{n+1}}{n+1} + c \ (n \neq -1), \quad \int e^{kx} dx = \frac{1}{k} e^{kx} + c$	Learn it
$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$	Given
$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln \left \frac{a+x}{a-x} \right $	Given
$\int \tan x dx = \ln \sec x $	Given
suvat formulas	Given, but not valid when the force varies

Useful rearrangements: $\frac{v}{v+b} = 1 - \frac{b}{v+b}$ (divide before integrating), $\frac{1}{v(v+b)} = \frac{1}{b} \left(\frac{1}{v} - \frac{1}{v+b} \right)$ (partial fractions), and $\int \frac{v}{a+bv^2} dv = \frac{1}{2b} \ln(a+bv^2)$ (the top is a multiple of the derivative of the bottom).

How to do it

In the Paper 3 questions we have tagged for this topic (32 questions, 2021–2025), the types came up this often. 29 of the 32 questions mix more than one type, so the counts add up to more than 32.

Question type	Questions
Velocity in terms of time	22
Displacement in terms of time	18
Velocity in terms of displacement	12
Time, distance or height at a key moment	7

Question type	Questions
Limiting (terminal) speed	5
Force from a given motion	4

Every type starts the same way:

1. **Sketch** the particle with its direction of motion and every force.
2. **Write Newton's second law with m visible**, for example $m \frac{dv}{dt} = -mg - R$ for a particle moving up against a resistance R . Resistances are negative in the direction of motion.
3. **Choose the form of a** from what the question links ($\frac{dv}{dt}$ or $v \frac{dv}{dx}$).

1. Velocity in terms of time (22 questions)

1. Write $m \frac{dv}{dt} = F$ and simplify.
2. **Separate:** $\int \frac{1}{f(v)} dv = \int 1 dt$ (or $\int g(t) dt$).
3. **Integrate**, including $+c$.
4. **Find c** from the starting values (usually v at $t = 0$). Using limits on a definite integral is just as good.
5. **Make v the subject:** exponentiate a log, take tan of an inverse tangent.

Variations you will meet:

- **A resistance proportional to v** , with or without a constant force, gives a log: $\frac{dv}{dt} = -(p + qv)$ separates to $\int \frac{1}{p+qv} dv = -\int 1 dt$, so $\frac{1}{q} \ln(p + qv) = -t + c$.
- **A resistance that is a power of v with no other force:** $\frac{dv}{dt} = -kv^n$ gives $\int v^{-n} dv$, a power of v and no logs at all.
- **A resistance with two terms**, such as $v^2 + bv$: factorise and use partial fractions, $\frac{1}{v(v+b)} = \frac{1}{b} \left(\frac{1}{v} - \frac{1}{v+b} \right)$ (Example 3).
- **Resistance kv^2 plus weight or a driving force.** Take out the constant so the integrand has the MF19 shape:
 - speeds **increasing** towards a limit give $a^2 - v^2$: for example $\frac{1}{36-v^2}$, which is on MF19 as $\frac{1}{12} \ln \frac{6+v}{6-v}$ (partial fractions give the same thing);
 - moving **up** against weight and kv^2 gives $a^2 + v^2$, so $\tan^{-1}$: $\int \frac{1}{9+v^2} dv = \frac{1}{3} \tan^{-1} \frac{v}{3}$. Find the constant in radians, then take tan of both sides.
- **$\sqrt{v}$ in the force:** substitute $u = \sqrt{v}$, so $dv = 2u du$, and the integrand becomes a fraction in u (Example 4).
- **The force depends only on t :** just integrate. For example $a = \frac{6}{2t+3}$ gives $v = 3 \ln(2t+3) + c$; a product such as te^{-t} needs integration by parts (Example 1). If the force acts **towards** O while the particle moves away, put a minus sign in.
- **The force depends on v and t together:** separate into a v part and a t part. For example $\frac{dv}{dt} = -t(v+2)$ gives $\int \frac{1}{v+2} dv = -\int t dt$, so $\ln(v+2) = -\frac{1}{2}t^2 + c$. With $v = 6$ at $t = 0$ this gives $v = 8e^{-t^2/2} - 2$. Use the laws of logs to combine terms before exponentiating.

- **"In terms of t and an arbitrary constant"**: stop before using any starting values, and write the answer as $v = A \times \dots$.
- **Negative velocity**. $\ln v$ only exists for $v > 0$. If the particle moves in the negative direction, integrate to $\ln|v|$ and use $\ln(-v)$: for $\frac{dv}{dt} = -v$ with $v = -2$ at $t = 0$, the answer is $v = -2e^{-t}$.
- **"Show that $t = \dots$ "**: the answer is given as t in terms of v , so you may leave the log in place, but show the integration, the constant and at least one line of rearranging.

2. Displacement in terms of time (18 questions)

When you already have v in terms of t :

1. Write $\frac{dx}{dt} = v(t)$ and integrate, with a new constant.
2. Use the starting position (usually $x = 0$ at $t = 0$) to find the constant.

When you have v in terms of x :

1. Write $\frac{dx}{dt} = v(x)$ and separate: $\int \frac{1}{v(x)} dx = \int 1 dt$.
2. Integrate (often a power, a log, $\tan^{-1}$, or partial fractions), use the starting values, and make x the subject.

Variations you will meet:

- **$\tan$ in v** integrates to a log of a cosine: $\int \tan(\alpha - t) dt = \ln|\cos(\alpha - t)|$ (MF19 gives $\int \tan x dx = \ln|\sec x|$; watch the sign from the $-t$).
- **Top = a multiple of the derivative of the bottom**: $\int \frac{6t}{t^2+4} dt = 3 \ln(t^2 + 4)$.
- **$\frac{dx}{dt}$ given as x times a function of t** , such as $\frac{v}{x} = \frac{3}{2t+1}$: separate as $\int \frac{1}{x} dx = \int \frac{3}{2t+1} dt$, so $\ln x = \frac{3}{2} \ln(2t + 1) + c$ and $x = A(2t + 1)^{3/2}$. If the top and bottom of the t fraction have the same degree, divide first: $\frac{2t+5}{t+1} = 2 + \frac{3}{t+1}$.
- **Displacement at one time only**: a definite integral is quickest. For $v = \sqrt{2t + 1}$, the displacement in the first 4 seconds is $\int_0^4 v dt = \left[\frac{1}{3}(2t + 1)^{3/2} \right]_0^4 = \frac{26}{3} \text{ m}$.
- **Two log terms**: combine them into one log before exponentiating, for example $x = 2 \ln(t + 2) - \ln 4$ becomes $x = \ln \frac{(t+2)^2}{4}$.

3. Velocity in terms of displacement (12 questions)

1. Write $mv \frac{dv}{dx} = F$.
2. **Separate**: if F depends on x , $\int v dv = \int \frac{F(x)}{m} dx$; if F depends on v , $\int \frac{v}{f(v)} dv = \int 1 dx$; if F is a v part times an x part, split it the same way (Quick check 3).
3. Integrate, find the constant from the starting values, make v (or v^2) the subject.
4. **Take the square root with care** (see below).

Variations you will meet:

- **Force depends on x :** $\int v \, dv = \frac{1}{2}v^2$, so you get v^2 first (Example 2). Look for a perfect square: $v^2 = 1 - \sin 2x$ is $(\cos x - \sin x)^2$, so $v = \cos x - \sin x$ while that is positive. The simpler v is then used in type 2.
- **Choosing the sign of the square root.** If $v^2 = 9(2 + \sin x)^2$ and $v = -6$ when $x = 0$, then $v = -3(2 + \sin x)$: $2 + \sin x$ is never zero, so v never changes sign and stays negative. Give the reason in a few words: " v is negative initially and is never zero".
- **Force depends on v :** if a factor v cancels, do it first; otherwise the v on top often needs dividing: $\frac{v^2}{v+1} = v - 1 + \frac{1}{v+1}$, so $\int \frac{v^2}{v+1} \, dv = \frac{1}{2}v^2 - v + \ln(v+1)$. When the top is a multiple of the derivative of the bottom it is one log: $\int \frac{v}{a+bv^2} \, dv = \frac{1}{2b} \ln(a+bv^2)$.
- **A higher power:** if the equation gives v^3 (or v^2) after exponentiating, finish by taking the cube root (or square root). Leaving $v^3 = \dots$ loses the last mark.
- **"Show that $x = \dots$ ":** the constant must come from the starting values, and the last line must match the printed answer exactly.

4. Time, distance or height at a key moment (7 questions)

These are the last part of a longer question. Translate the moment into a value, then substitute into the right expression:

The question says	Use
comes to rest / reaches its greatest height	$v = 0$
returns to O / passes through O again	$x = 0$ (and $t > 0$)
when its speed is . . .	$v = \dots$
average speed while moving	total distance $\div$ total time

1. **For a time**, use the v - t (or x - t) relation. For a greatest height, solve $v(t) = 0$.
2. **For a distance at a given speed**, either use the v - x relation directly (from $v \frac{dv}{dx}$), or find the time from $v(t)$ and substitute it into $x(t)$. Both routes score; the first is often shorter.
3. Give a decimal answer to 3 significant figures unless the question asks for something else; keep exact forms such as $\frac{1}{2} \ln \frac{3}{2}$ until the end.

For a distance or time **before coming to rest**, you are allowed to use the formula from an earlier "show that", even if you could not prove it.

5. Limiting (terminal) speed (5 questions)

1. **Quickest:** the limiting speed is where the acceleration is zero. For a car of mass 800 kg with driving force 1200 N and resistance $(10v + v^2)$ N, $1200 - 10v - v^2 = 0$ gives $(v + 40)(v - 30) = 0$, so $v = 30 \text{ m s}^{-1}$.
2. **From your answer:** let $t \rightarrow \infty$ (or $x \rightarrow \infty$) in your expression, using $e^{-kt} \rightarrow 0$.

Use the first method when you can: it does not depend on the differential equation being solved correctly.

"Deduce" or "state" parts are usually worth 1 mark and need only the value.

6. Force from a given motion (4 questions)

Here you work backwards: the motion is given (or found), and you need the force.

1. Find v (differentiate x , or solve the given equation first).
2. **Differentiate again** for the acceleration: $a = \frac{dv}{dt}$. Use the product or quotient rule when v is a product or a quotient.
3. **Force** = ma at the time asked. Give the **magnitude** (a positive number) if the question says "magnitude".

Variations you will meet:

- v contains an arbitrary constant A and you are told the value of a at some time: substitute to find A first, then write $F = ma$ in terms of m and t .
- **Given $\frac{v}{x}$ as a function of t** : solve for x as in type 2, differentiate twice, then multiply by the mass.
- **Given v as a function of x** : the force is $mv \frac{dv}{dx}$, with no need to find t .

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
-

Example 1

A particle P of mass 2 kg is at rest at a point O on a smooth horizontal line when $t = 0$. It then moves along the line under a single horizontal force of magnitude $4te^{-t}$ N, acting in the direction of motion, where t s is the time. The displacement of P from O is x m and its velocity is v m s⁻¹.

- (a) Find an expression for v in terms of t . **[4]**
- (b) State the limiting speed of P . **[1]**
- (c) Find the displacement of P from O when $t = 2$. **[4]**

(a) Newton's second law: $2 \frac{dv}{dt} = 4te^{-t}$, so $\frac{dv}{dt} = 2te^{-t}$. **B1** (the mass must be seen)

By parts, $\int 2te^{-t} dt = -2te^{-t} + \int 2e^{-t} dt = -2te^{-t} - 2e^{-t}$, so $v = -2(t+1)e^{-t} + c$. **M1 A1**

At $t = 0$, $v = 0$: $c = 2$. So

$$v = 2 - 2(t + 1)e^{-t}.$$

A1

(b) As $t \rightarrow \infty$, $(t + 1)e^{-t} \rightarrow 0$, so $v \rightarrow 2 \text{ m s}^{-1}$. **B1**

(c) $\frac{dx}{dt} = 2 - 2(t + 1)e^{-t}$. By parts again, $\int 2(t + 1)e^{-t} dt = -2(t + 2)e^{-t}$, so **M1**

$$x = 2t + 2(t + 2)e^{-t} + c'.$$

A1

At $t = 0$, $x = 0$: $0 = 4 + c'$, so $c' = -4$. **M1** At $t = 2$: $x = 4 + 8e^{-2} - 4 = 8e^{-2} = 1.08 \text{ m}$ (3 s.f.). **A1**

Example 2

A particle P of mass 3 kg moves along a horizontal straight line. At time t s its displacement from a fixed point O on the line is x m and its velocity is $v \text{ m s}^{-1}$. The only horizontal force on P acts in the direction OP and has magnitude $12(x + 2)$ N. When $t = 0$, $x = 0$ and $v = 4$.

(a) Show that $v = 2(x + 2)$. **[4]**

(b) Find an expression for x in terms of t . **[3]**

(c) Find the time at which the force on P has magnitude 60 N. **[2]**

(a) The force depends on x and the answer links v and x , so use $v \frac{dv}{dx}$:

$$3v \frac{dv}{dx} = 12(x + 2) \Rightarrow \int v dv = \int 4(x + 2) dx.$$

B1 M1

$\frac{1}{2}v^2 = 2(x + 2)^2 + c$. **A1** At $x = 0$, $v = 4$: $8 = 8 + c$, so $c = 0$ and $v^2 = 4(x + 2)^2$. The force acts in the direction OP , so P keeps speeding up away from O and v stays positive:

$$v = 2(x + 2).$$

A1 (given answer, with the constant and the choice of sign shown)

(b) $\frac{dx}{dt} = 2(x + 2)$, so $\int \frac{1}{x + 2} dx = \int 2 dt$ and $\ln(x + 2) = 2t + c'$. **M1 A1**

At $t = 0$, $x = 0$: $c' = \ln 2$. So $x + 2 = 2e^{2t}$, that is $x = 2e^{2t} - 2$. **A1**

(c) $12(x + 2) = 60$ gives $x = 3$. Then $2e^{2t} = 5$, so $t = \frac{1}{2} \ln \frac{5}{2} = 0.458 \text{ s}$ (3 s.f.). **M1 A1**

Example 3

A particle P of mass 2 kg moves along a horizontal straight line. The only horizontal force on P is a resistance of magnitude $(v^2 + 4v)$ N, where $v \text{ m s}^{-1}$ is the velocity of P at time t s. When $t = 0$, P is at the point O and $v = 4$. The displacement of P from O is x m.

(a) Find an expression for v in terms of t . [6]

(b) Find the time taken for the speed of P to halve. [2]

(c) Find v in terms of x , and hence show that P never travels more than $2 \ln 2$ m from O . [5]

(a) Newton's second law: $2 \frac{dv}{dt} = -(v^2 + 4v)$. **B1**

Separate: $\int \frac{1}{v(v+4)} dv = - \int \frac{1}{2} dt$. Partial fractions: $\frac{1}{v(v+4)} = \frac{1}{4} \left(\frac{1}{v} - \frac{1}{v+4} \right)$. **M1**

$$\frac{1}{4} \ln \frac{v}{v+4} = -\frac{1}{2}t + c.$$

A1

At $t = 0$, $v = 4$: $c = \frac{1}{4} \ln \frac{1}{2}$. **M1** So $\ln \frac{v}{v+4} = -2t + \ln \frac{1}{2}$, and $\frac{v}{v+4} = \frac{1}{2}e^{-2t}$. **M1** Rearranging, $2v = (v+4)e^{-2t}$.

$$v = \frac{4e^{-2t}}{2 - e^{-2t}}.$$

A1

(b) $v = 2$ when $\frac{2}{6} = \frac{1}{2}e^{-2t}$, so $e^{-2t} = \frac{2}{3}$ and $t = \frac{1}{2} \ln \frac{3}{2} = 0.203$ s (3 s.f.). **M1 A1**

(c) This links v and x , so use $2v \frac{dv}{dx} = -v(v+4)$. Since $v > 0$, cancel v : $\frac{dv}{dx} = -\frac{1}{2}(v+4)$. **M1**

$$\int \frac{1}{v+4} dv = - \int \frac{1}{2} dx \Rightarrow \ln(v+4) = -\frac{1}{2}x + c.$$

A1

At $x = 0$, $v = 4$: $c = \ln 8$. **M1** So $v+4 = 8e^{-x/2}$, that is $v = 8e^{-x/2} - 4$. **A1**

From part (a), $v > 0$ for every t , so $8e^{-x/2} > 4$, which gives $e^{-x/2} > \frac{1}{2}$ and $x < 2 \ln 2$. As $t \rightarrow \infty$, $v \rightarrow 0$ and $x \rightarrow 2 \ln 2 = 1.39$ m, but P never gets there. **A1**

Example 4

A seed of mass m kg is released from rest and falls vertically. The forces on it are its weight and a resistance of magnitude $5m\sqrt{v}$ N, where v m s⁻¹ is its speed at time t s. Take $g = 10$.

(a) Show that $t = \frac{4}{5} \ln \left(\frac{2}{2 - \sqrt{v}} \right) - \frac{2}{5} \sqrt{v}$. [6]

(b) State the limiting speed of the seed. [1]

(c) Find the time taken for the seed to reach a speed of 3 m s⁻¹. [2]

(a) Downwards is positive: $m \frac{dv}{dt} = mg - 5m\sqrt{v}$, so $\frac{dv}{dt} = 10 - 5\sqrt{v}$. **B1**

Separate: $t = \int \frac{1}{10 - 5\sqrt{v}} dv$. **M1** Substitute $u = \sqrt{v}$, so $v = u^2$ and $dv = 2u du$:

$$t = \int \frac{2u}{10 - 5u} du = \frac{2}{5} \int \frac{u}{2 - u} du = \frac{2}{5} \int \left(-1 + \frac{2}{2 - u} \right) du = \frac{2}{5} (-u - 2 \ln(2 - u)) + c.$$

M1 for the substitution, **A1** for the integral.

At $t = 0$, $v = 0$ so $u = 0$: $0 = -\frac{4}{5} \ln 2 + c$, so $c = \frac{4}{5} \ln 2$. **M1** So

$$t = \frac{4}{5} \ln 2 - \frac{4}{5} \ln(2 - \sqrt{v}) - \frac{2}{5} \sqrt{v} = \frac{4}{5} \ln \left(\frac{2}{2 - \sqrt{v}} \right) - \frac{2}{5} \sqrt{v}.$$

A1 (given answer, with the constant and the combining of the logs shown)

(b) The acceleration is zero when $10 - 5\sqrt{v} = 0$, so the limiting speed is 4 m s^{-1} . (Or: as $\sqrt{v} \rightarrow 2$, $t \rightarrow \infty$.) **B1**

(c) At $v = 3$: $\frac{2}{2 - \sqrt{3}} = 2(2 + \sqrt{3}) = 4 + 2\sqrt{3}$, so $t = \frac{4}{5} \ln(4 + 2\sqrt{3}) - \frac{2}{5} \sqrt{3} = 0.915 \text{ s (3 s.f.)}$. **M1 A1**

The second diagram in "Understand it" shows this motion: the marked point is where $v = 3$.

Common mistakes

- **Using suvat.** The acceleration changes, so the constant-acceleration formulas are wrong here. Mark schemes give **no marks** to a part answered with suvat.
- **Getting the sign of a force wrong.** A resistance, or a force "towards O " or "in the direction PO ", acts against the motion and must be negative. Examiners mention this in several years' reports, and it leads to a wrong constant and a wrong answer even when the integration is good. Draw the sketch first.
- **Leaving out the mass.** Writing $\frac{dv}{dt} = \text{force}$ (without m) loses the first mark, and is wrong whenever $m \neq 1$. Start with $m \frac{dv}{dt} = \dots$ and cancel afterwards.
- **Leaving out the weight** in vertical motion, or giving it the wrong sign on the way up.
- **Using $\frac{dv}{dt}$ when the question links v and x** (or $v \frac{dv}{dx}$ for v and t). Check which two variables the answer must contain before you write the acceleration.
- **Separating wrongly**, for example writing $\int (p - qv) dv$ instead of $\int \frac{1}{p - qv} dv$, or differentiating instead of integrating.
- **Integration slips**, such as $\int \frac{1}{(t+1)^2} dt = \ln(t+1)$. It is $-\frac{1}{t+1}$: only a power of -1 gives a log.
- **Forgetting the constant**, or finding it from the wrong starting values. Every integration needs its own constant.
- **Mishandling logs** when making v the subject: $\ln(a) + \ln(b) = \ln(ab)$, not $\ln(a+b)$. Combine into one log before exponentiating.
- **Taking a square root without thinking about the sign**, or stopping at v^2 or v^3 when the question asks for v .
- **Restarting from scratch** to find x when you already have $v(t)$. Integrating $v(t)$ is usually quicker and safer than a new $v \frac{dv}{dx}$ equation.

How to get the marks

- **The first mark is for Newton's second law** with the mass shown and every force present with the right sign. Write it on its own line before simplifying.
- **Method marks for separating and integrating** need an integral of the right form (a log, $\tan^{-1}$, a power). If the form is right but a coefficient is wrong, you still earn the method mark.
- **The constant must be found from the starting values** (or by using limits). Later marks depend on this, so write the substitution clearly: "at $x = 0$, $v = 4$, so $c = \ln 8$ ".
- **Rearrange fully.** "Find v in terms of t " means $v = \dots$ with nothing else on the left. Any correct equivalent form is accepted.
- **"Show that"** means the answer is given: show the separation, the integration, the constant and a line of working before the final line, which must match the printed result. Justify the choice of sign when you take a square root.
- **"Deduce", "State":** one mark, just the value (and a brief reason such as " $e^{-t} \rightarrow 0$ " or "the acceleration is zero").
- **Keep m and g as letters** if the question uses letters; put $g = 10$ only when numbers are given.
- **Radians:** $\tan^{-1}$ constants are angles in radians ($\tan^{-1} 1 = \frac{\pi}{4}$).
- **Accuracy:** give non-exact answers to 3 significant figures unless told otherwise, and keep exact values (such as $\frac{1}{2} \ln \frac{3}{2}$) until the last step.
- **Follow-through:** a later part that uses your earlier (wrong) expression correctly can still earn method marks, so always carry on.

Quick check

1. A particle starts at O with velocity 1 m s^{-1} and moves in a straight line with acceleration $4 \cos 2t \text{ m s}^{-2}$ at time $t \text{ s}$. Find v and x in terms of t , and state the greatest speed of the particle.
2. A particle of mass 0.5 kg moves along a straight line with displacement $x = 2 \ln(1 + t^2) \text{ m}$ from O at time $t \text{ s}$. Find the magnitude and direction of the force acting on it when $t = 2$.
3. A particle of mass 2 kg moves along a horizontal line away from O . The only horizontal force on it acts towards O and has magnitude $2xv \text{ N}$, where $x \text{ m}$ is its displacement from O and $v \text{ m s}^{-1}$ its velocity. It leaves O with speed 8 m s^{-1} . Find v in terms of x , and the distance from O at which v would be zero.
4. A cyclist and bicycle of total mass 80 kg move along a horizontal road under a driving force of 320 N and a resistance of $5v^{3/2} \text{ N}$. Find the limiting speed.

Answers

1. $v = 1 + \int_0^t 4 \cos 2t \, dt = 1 + 2 \sin 2t$. Then $x = \int_0^t v \, dt = t + 1 - \cos 2t$. The greatest value of $\sin 2t$ is 1, so the greatest speed is 3 m s^{-1} .
2. $v = \frac{dx}{dt} = \frac{4t}{1+t^2}$ and, by the quotient rule, $a = \frac{4(1+t^2) - 4t \times 2t}{(1+t^2)^2} = \frac{4(1-t^2)}{(1+t^2)^2}$. At $t = 2$, $a = -\frac{12}{25} = -0.48$, so $F = 0.5 \times (-0.48) = -0.24$: a force of magnitude 0.24 N towards O (in the direction of decreasing x).
3. $2v \frac{dv}{dx} = -2xv$, so $\frac{dv}{dx} = -x$ and $v = -\frac{1}{2}x^2 + c$. At $x = 0$, $v = 8$: $c = 8$, so $v = 8 - \frac{1}{2}x^2$. $v = 0$ when $x^2 = 16$, so at 4 m from O . (There the force is zero too, so the particle creeps towards this point and never passes it.)
4. The speed stops changing when the acceleration is zero: $320 - 5v^{3/2} = 0$, so $v^{3/2} = 64$ and the limiting speed is $64^{2/3} = 16 \text{ m s}^{-1}$.

Past questions to try

- [9231/34/M/J/25 Q2](#): resistance kv^3 , then x at a given time.
- [9231/33/M/J/24 Q7](#): a parachutist slowing to a terminal speed, with v , x and a distance.
- [9231/32/O/N/22 Q4](#): v in terms of x with a log, then the limiting speed.
- [9231/33/M/J/22 Q5](#): a "show that" with a perfect square, then x in terms of t .