

Matrices

AS

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What you need to know

By the end of this topic you should be able to:

1. **Add, subtract and multiply matrices**, including non-square ones (never bigger than 3 rows by 3 columns), and know the **zero matrix** $\mathbf{0}$ and the **identity matrix** $\mathbf{I}$.
2. **Know what "singular" and "non-singular" mean** for a square matrix, **work out determinants** of 2×2 and 3×3 matrices (written $\det \mathbf{M}$), and **find the inverse** of any non-singular 2×2 or 3×3 matrix.
3. **Use** $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, and its longer versions such as $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.
4. **Use 2×2 matrices for transformations of the x - y plane**: rotations, reflections, enlargements, stretches and shears. In particular:
 - $\mathbf{A}^{-1}$ is the transformation that undoes $\mathbf{A}$;
 - $\mathbf{AB}$ means "do $\mathbf{B}$ first, then $\mathbf{A}$ ";
 - areas are multiplied by $|\det \mathbf{M}|$;
 - find the matrix for a given transformation, or a sequence of them.
5. **Understand "invariant"** for points and lines, and solve problems about invariant points and invariant lines.

Other transformations can appear in a question, but you are only expected to know these five by name. Eigenvalues and eigenvectors are not in this topic: they are in Further Pure Mathematics 2.

Understand it

What a matrix is

A matrix is a rectangle of numbers in rows and columns. A matrix with 2 rows and 3 columns is a 2×3 matrix ("rows by columns"). Two matrices are equal only when they are the same size and every entry matches.

Adding and subtracting work entry by entry, and only for matrices of the same size. **Multiplying by a number** multiplies every entry: $3 \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ 0 & 12 \end{pmatrix}$.

Multiplying matrices

To find the entry in row i , column j of $\mathbf{AB}$, run along **row i of $\mathbf{A}$** and down **column j of $\mathbf{B}$** , multiply the pairs and add:

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + (-1) \cdot 0 & 2 \cdot 5 + (-1)(-2) \\ 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 5 + 4(-2) \end{pmatrix} = \begin{pmatrix} 2 & 12 \\ 3 & 7 \end{pmatrix}.$$

This only works when each row of $\mathbf{A}$ is as long as each column of $\mathbf{B}$: an $m \times n$ matrix times an $n \times p$ matrix gives an $m \times p$ matrix. The inner numbers must match; the outer numbers give the size of the answer. For example, a 2×3 times a 3×1 gives a 2×1 :

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 + 0 + 4 \\ -3 + 3 + 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}.$$

Order matters. Swapping the two matrices above gives $\begin{pmatrix} 1 & 5 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 17 & 19 \\ -6 & -8 \end{pmatrix}$, a different answer. In general $\mathbf{AB} \neq \mathbf{BA}$, and $\mathbf{BA}$ may not even exist. A product of three matrices can be done in either grouping, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$, but the order left to right must stay the same.

The **zero matrix $\mathbf{0}$** has every entry 0. The **identity matrix $\mathbf{I}$** has 1s down the leading diagonal (top left to bottom right) and 0s elsewhere; it does nothing when you multiply by it: $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$.

Determinants

The **determinant** of a 2×2 matrix is "leading diagonal minus the other diagonal":

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

For a 3×3 matrix, expand along the top row. Each top-row entry is multiplied by the 2×2 determinant left when you cover up its row and column (its **minor**), with the signs $+$ $-$ $+$:

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

You may expand along any row or column instead, using the sign pattern $\begin{smallmatrix} + & - & + \\ - & + & - \\ + & - & + \end{smallmatrix}$. A row or column with zeros in it saves work.

A square matrix is **singular** if its determinant is 0 and **non-singular** if its determinant is **not** 0. Only non-singular matrices have inverses.

Two facts are used again and again: $\det(\mathbf{AB}) = \det \mathbf{A} \times \det \mathbf{B}$, and so $\det(\mathbf{A}^{-1}) = \frac{1}{\det \mathbf{A}}$.

Inverses

The **inverse $\mathbf{A}^{-1}$** is the matrix with $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. For a 2×2 matrix: **swap** the leading diagonal, **change the sign** of the other two entries, and **divide by the determinant**:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For a 3×3 matrix there are four steps:

1. Find the determinant (if it is 0, stop: there is no inverse).
2. Replace every entry by its minor, then apply the sign pattern $\begin{smallmatrix} + & - & + \\ - & + & - \\ + & - & + \end{smallmatrix}$. This is the matrix of **cofactors**.
3. **Transpose** it: rows become columns (reflect it in the leading diagonal).
4. Divide by the determinant.

Example 2 below works one through. Always check one entry of $\mathbf{A}\mathbf{A}^{-1}$: it should come out 1 on the diagonal and 0 off it.

Solving matrix equations. Matrices don't divide. To get $\mathbf{N}$ out of $\mathbf{M}\mathbf{N} = \mathbf{P}$, multiply both sides **on the left** by $\mathbf{M}^{-1}$: $\mathbf{N} = \mathbf{M}^{-1}\mathbf{P}$. From $\mathbf{N}\mathbf{M} = \mathbf{P}$, multiply **on the right**: $\mathbf{N} = \mathbf{P}\mathbf{M}^{-1}$.

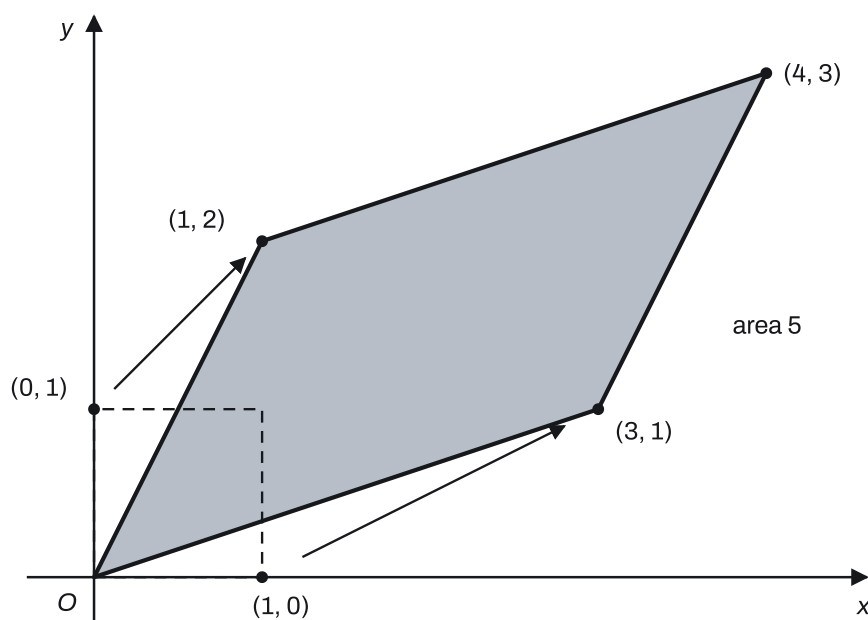
The inverse of a product. $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, with the order reversed. You can see why by checking: $(\mathbf{A}\mathbf{B})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{B}\mathbf{B}^{-1})\mathbf{A}^{-1} = \mathbf{A}\mathbf{I}\mathbf{A}^{-1} = \mathbf{I}$. Think of socks and shoes: you put socks on first, but take shoes off first. The same holds for more matrices: $(\mathbf{A}\mathbf{B}\mathbf{C})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

Matrices as transformations

A 2×2 matrix $\mathbf{M}$ moves every point (x, y) of the plane to a new point (X, Y) , its **image**:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix}.$$

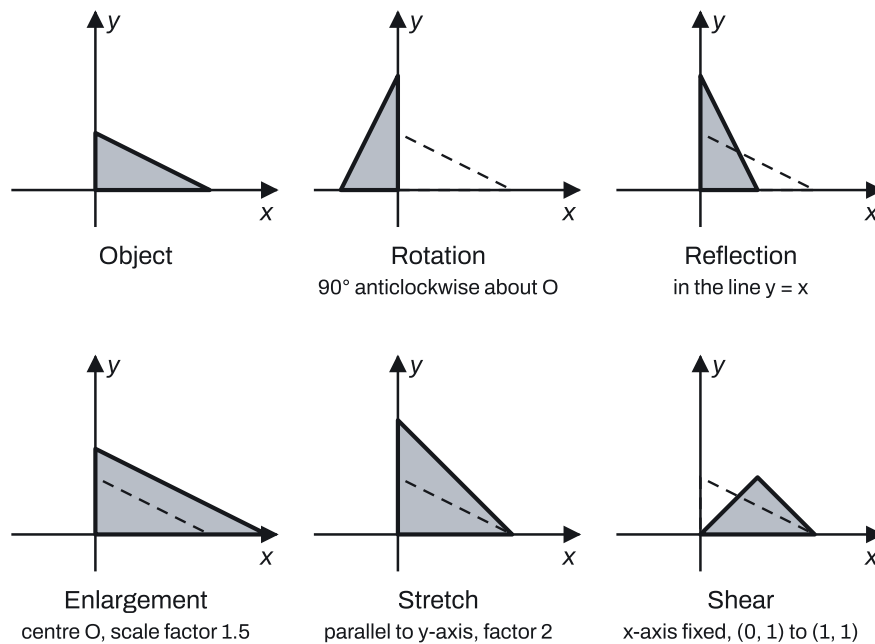
The origin never moves. The key idea: **the first column of $\mathbf{M}$ is the image of $(1, 0)$, and the second column is the image of $(0, 1)$** . So to find the matrix of a transformation, ask where $(1, 0)$ and $(0, 1)$ go, and write those as the columns. To recognise a matrix, read its columns and picture the unit square moving.



The five transformations you need, with where $(1, 0)$ and $(0, 1)$ go:

Transformation	Matrix	Full description needs
Rotation through θ anticlockwise about O	$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$	"rotation", centre the origin, angle and direction
Reflection in the x -axis / y -axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} / \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	"reflection" and the mirror line
Reflection in $y = x$ / $y = -x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} / \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	"reflection" and the mirror line
Reflection in the line through O at angle θ to the x -axis	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$	"reflection" and the line $y = x \tan \theta$
Enlargement, centre O , scale factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	"enlargement", centre the origin, scale factor
Stretch parallel to the x -axis, factor k	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$	"stretch", the direction, scale factor
Stretch parallel to the y -axis, factor k	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$	"stretch", the direction, scale factor
Shear, x -axis fixed, $(0, 1) \mapsto (k, 1)$	$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$	"shear", the fixed line, and where one point goes
Shear, y -axis fixed, $(1, 0) \mapsto (1, k)$	$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$	"shear", the fixed line, and where one point goes

A **stretch** moves points away from one axis in one direction only: $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ multiplies every x -coordinate by k and leaves y alone (it is also called a one-way stretch in the x -direction). A **shear** slides points parallel to a fixed line, by an amount that grows with the distance from that line. Watch where the k sits: on the diagonal for a stretch, off it for a shear.



Doing one transformation after another

$\mathbf{AB} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A} \left(\mathbf{B} \begin{pmatrix} x \\ y \end{pmatrix} \right)$: the point meets **B** first. So **AB** is "**B**, then **A**". Read a product from right to left.

"A rotation followed by a stretch" is **SR**, with the stretch on the **left**.

The inverse matrix gives the transformation that **undoes** the original, and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ says: to undo "**B** then **A**", undo **A** first, then undo **B**. Inverses of the basic transformations are easy to state: a rotation through θ is undone by a rotation through $-\theta$; a reflection undoes itself; an enlargement or stretch with factor k is undone by the same kind with factor $\frac{1}{k}$; a shear with $(0, 1) \mapsto (k, 1)$ is undone by the shear with $(0, 1) \mapsto (-k, 1)$.

Area and the determinant

In the diagram above, the unit square (area 1) becomes a parallelogram of area $3 \times 2 - 1 \times 1 = 5$. In general, **every area is multiplied by $|\det \mathbf{M}|$** :

$$\text{area of image} = |\det \mathbf{M}| \times \text{area of object}.$$

A negative determinant means the shape has also been flipped over (as a reflection does). Rotations, reflections and shears have $|\det| = 1$, so they keep areas the same. A determinant of 0 squashes the whole plane onto a line or a point, which is why a singular matrix has no inverse: you cannot undo it.

Invariant points

A point is **invariant** if the transformation leaves it where it is: $\mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Write out the two equations and solve them together.

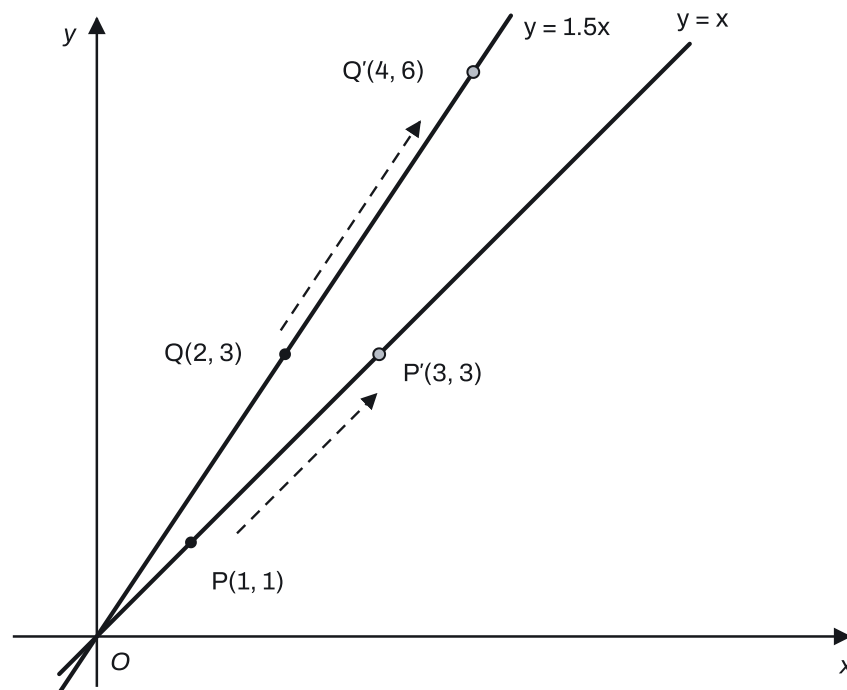
- Usually the only solution is $x = y = 0$: the origin is the only invariant point.

- Sometimes the two equations are the **same line**, and every point on that line is invariant: a **line of invariant points**. For $\begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix}$, the equations are $4x + 3y = x$ and $-2x - y = y$, which both simplify to $x + y = 0$. So every point of $y = -x$ stays put: $(1, -1) \mapsto (4 - 3, -2 + 1) = (1, -1)$.

A line of invariant points exists exactly when $\mathbf{M} - \mathbf{I}$ is singular, that is $\det(\mathbf{M} - \mathbf{I}) = 0$ (and $\mathbf{M} \neq \mathbf{I}$). The mirror line of a reflection and the fixed line of a shear are lines of invariant points.

Invariant lines

A line is **invariant** if every point on it is mapped to a point **on the same line**. The points may move along the line; only the line as a whole must stay put. So a line of invariant points is always an invariant line, but an invariant line is usually **not** made of invariant points.



To find the invariant lines through the origin, take the line $y = mx$. A point on it is (x, mx) . Find its image (X, Y) and demand that the image is on the same line: $Y = mX$. The x cancels and you get a **quadratic in m** :

- two real roots: two invariant lines through O ;
- one repeated root: exactly one;
- no real roots: none of the form $y = mx$.

The line $x = 0$ has no gradient, so $y = mx$ misses it. It is invariant only when the top-right entry of $\mathbf{M}$ is 0; check it separately in that case.

Key facts and formulas

The MF19 list of formulae has nothing on matrices, so everything here must be learned.

Formula	Status
AB : row of A times column of B ; $(m \times n)(n \times p) = m \times p$	Learn it
$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$	Learn it
3×3 determinant: expand along a row with signs + − +	Learn it
Singular $\Leftrightarrow \det \mathbf{M} = 0$; non-singular $\Leftrightarrow \det \mathbf{M} \neq 0$	Learn it
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$	Learn it
3×3 inverse = $\frac{1}{\det \mathbf{A}} \times$ (transpose of the matrix of cofactors)	Learn it
$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$; $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$	Learn it
$\det(\mathbf{AB}) = \det \mathbf{A} \det \mathbf{B}$; $\det(\mathbf{A}^{-1}) = \frac{1}{\det \mathbf{A}}$	Learn it
Columns of M = images of (1, 0) and (0, 1)	Learn it
Rotation θ anticlockwise about O : $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$	Learn it
Stretch, enlargement, shear and reflection matrices (table above)	Learn it
AB = "B first, then A"	Learn it
Area of image = $ \det \mathbf{M} \times$ area of object	Learn it
Invariant point: $\mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$	Learn it
Invariant line $y = mx$: image (X, Y) of (x, mx) satisfies $Y = mX$	Learn it
$\cos(A \pm B)$, $\sin(A \pm B)$ (for exact values such as $\cos \frac{\pi}{12}$)	Given
Exact values worth knowing: $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\sin \frac{\pi}{6} = \frac{1}{2}$, $\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos \frac{\pi}{3} = \frac{1}{2}$, $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.	

How to do it

In the Paper 1 questions we have tagged for this topic (37 questions, 2021–2025), 26 are different: the other 11 appear on two papers. The table counts the 26 different ones. Nearly every question is in several parts, and 24 of the 26 mix types, so the counts add up to more than 26.

Question type	Questions
Invariant lines through the origin	16
Describing the transformations a matrix represents	16
Inverses, $(\mathbf{AB})^{-1}$ and matrix equations	15
Area scale factor	9
Determinants and singular matrices	8

Question type	Questions
Matrix products, including non-square matrices	7
Finding the matrix of a transformation or sequence	6
Invariant points and lines of invariant points	5

1. Invariant lines through the origin (16 questions)

1. **Multiply out** the image of a general point: $\mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$, writing X and Y in terms of x and y .
2. **Put** $y = mx$ into both, and use $Y = mX$.
3. **Cancel** x (it isn't 0 for a general point on the line) to get a quadratic in m .
4. **Solve** it and write the lines as equations, $y = \dots x$, not just values of m .

For $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ this is done in Quick check 4; for $\begin{pmatrix} 5 & -2 \\ 3 & 0 \end{pmatrix}$ in Example 3.

Variations you will meet:

- **The matrix is a product** (such as $\mathbf{CAB}$ or $\mathbf{MN}$, often from an earlier part). Multiply it out first; use a given answer if there is one.
- **"Show that the line with gradient g is invariant"**: take the point (x, gx) , find its image, and show that $Y \div X = g$.
- **Unknown constant**, such as "find the set of values of k for which there are no invariant lines through the origin". Form the quadratic in m with k in it, then use the discriminant $b^2 - 4ac$: < 0 for none, $= 0$ for exactly one, > 0 for two distinct lines. Example 4 does this.
 - If the m^2 coefficient contains k , the value that makes it 0 needs separate thought (the quadratic becomes linear, and $x = 0$ may become invariant).
 - If $\mathbf{M}$ is singular for some k , one of the lines is squashed to the origin rather than mapped onto itself, so leave that value of k out when two invariant lines are needed.
- **Trigonometric entries** (a rotation through θ in the product): the quadratic has $\sin \theta$ and $\cos \theta$ in it. "Exactly one invariant line" means the discriminant is 0, which gives a trigonometric equation to solve in the stated interval. Double angle formulae and $\cos^2 \theta + \sin^2 \theta = 1$ tidy it up.
- **A common factor**. If both sides share a factor such as $(1 + m)$, don't cancel it: take it out, and it gives one of the answers ($m = -1$).

2. Describing the transformations a matrix represents (16 questions)

1. **Recognise each matrix** from the table in "Understand it": read the columns as the images of $(1, 0)$ and $(0, 1)$.
2. **Give full details** when asked: the type, and for a rotation the centre (the origin), angle and direction; for a reflection the mirror line; for an enlargement the centre and scale factor; for a stretch the direction and scale factor; for a shear the fixed axis and where one point goes.

3. **For a product, state the order:** the right-hand matrix happens first. " $\mathbf{M} = \mathbf{PQ}$ is $\mathbf{Q}$ followed by $\mathbf{P}$."

For example, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ is a shear with the x -axis fixed and $(0, 1) \mapsto (3, 1)$, followed by a reflection in the x -axis.

Variations you will meet:

- **One diagonal matrix** $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, perhaps with letters and a condition: if $p = q$ it is an enlargement, centre the origin, scale factor p ; if one of p, q is 1, it is a stretch in the direction of the other one.
- **Describe $\mathbf{M}^{-1}$ when $\mathbf{M}$ is a sequence:** undo each step, in the reverse order. If $\mathbf{M}$ is "stretch then reflect", $\mathbf{M}^{-1}$ is "reflect, then the stretch with factor $\frac{1}{k}$ ".
- **"Show that $\mathbf{M}$ does not represent a rotation":** compare with $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. A rotation has equal diagonal entries and off-diagonal entries that are **negatives** of each other; show $\mathbf{M}$'s entries can't satisfy both (for example, one would have to be a negative square).
- **Recognise a rotation from exact values**, such as $\frac{1}{2} \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$, which is a rotation of $\frac{\pi}{6}$ anticlockwise. Using the compound angle formulae first may be part of the question.
- **"Write $\mathbf{M}$ (or $\mathbf{M}^{-1}$) as the product of two matrices, neither of which is $\mathbf{I}$ ":** if $\mathbf{M} = \mathbf{PQ}$ is given, then $\mathbf{M}^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$. Invert each simple matrix separately; don't multiply out and invert the product.

3. Inverses, $(\mathbf{AB})^{-1}$ and matrix equations (15 questions)

1. **2×2 inverse:** swap, change signs, divide by the determinant.
2. **3×3 inverse:** determinant, cofactors, transpose, divide (Example 2).
3. **Matrix equations:** $\mathbf{MN} = \mathbf{P} \Rightarrow \mathbf{N} = \mathbf{M}^{-1}\mathbf{P}$; $\mathbf{NM} = \mathbf{P} \Rightarrow \mathbf{N} = \mathbf{PM}^{-1}$. Keep the inverse on the same side as $\mathbf{M}$ was.

Variations you will meet:

- **"The matrix that transforms the image back onto the object"** is $\mathbf{M}^{-1}$. With a letter k in $\mathbf{M}$, the answer is in terms of k .
- **Only part of a 3×3 inverse**, such as its top row. Find the determinant and just the cofactors you need: the top row of $\mathbf{A}^{-1}$ comes from the cofactors of the **first column** of $\mathbf{A}$ (because of the transpose).
- **An unknown in $\mathbf{A}$ and $\mathbf{A}^{-1}$ given.** Use $\det(\mathbf{A}^{-1}) = \frac{1}{\det \mathbf{A}}$, or multiply one row of $\mathbf{A}$ by one column of $\mathbf{A}^{-1}$ and set the result equal to the matching entry of $\mathbf{I}$.
- **Powers:** if $\mathbf{A}^n$ is known (for example from induction), invert it like any other matrix; its determinant is $(\det \mathbf{A})^n$.
- **Non-square matrices have no inverse.** To find an unknown matrix $\mathbf{C}$ in, say, $\mathbf{BAC} = \mathbf{P}$ where $\mathbf{BA}$ is not square, work out the size $\mathbf{C}$ must be, write it with unknown entries, and choose values that work. "Give an example" means any one correct matrix will do.

4. Area scale factor (9 questions)

1. Find $\det \mathbf{M}$ (for a product, multiply the determinants).
2. **Image area** = $|\det \mathbf{M}| \times \text{object area}$. Check which triangle is the object: to go from image back to object, divide.

Variations you will meet:

- **"The unit square is mapped to a parallelogram of area ..."**: the unit square has area 1, so the parallelogram's area is $|\det \mathbf{M}|$. Set it equal to the given value and solve, keeping the modulus: $|\det \mathbf{M}| = 3$ means $\det \mathbf{M} = 3$ or -3 .
- **"Show the two triangles have the same area"**: show $|\det| = 1$ (a product of shears, for example).
- **"Find k so the areas are equal"**: $|\det \mathbf{M}| = 1$, so check both $+1$ and -1 against any conditions on k given in the question.

5. Determinants and singular matrices (8 questions)

1. **Expand** the determinant along a row or column (the one with most zeros).
2. **Singular**: set it equal to 0 and solve. **Non-singular**: show it can never be 0 (or find the values of k that make it 0 and exclude them).

Variations you will meet:

- **"Show that $\mathbf{A}$ is non-singular for all k "**: the determinant is often a quadratic in k . Show it has no real roots (discriminant < 0) or that it is always negative or always positive, for example $3k^2 + 8 > 0$ for every k .
- **"Find the set of values of k for which $\mathbf{A}$ is non-singular"**: $\det \mathbf{A} \neq 0$, so the answer is " $k \neq \dots$ ", not an inequality like $k > \dots$.
- **A product such as $\mathbf{ABC}$ is singular**: multiply out (in either grouping) and find the 3×3 determinant, simplifying carefully until it is 0.
- **Roots of a polynomial in the matrix**: the determinant becomes an expression in α, β, γ ; use the symmetric functions from the cubic (see Roots of polynomial equations) to show it is 0 or to get a new fact.
- **$\mathbf{M} - \mathbf{I}$ singular**: subtract 1 from each diagonal entry, then take the determinant.

6. Matrix products, including non-square matrices (7 questions)

1. **Check the sizes** first: $(3 \times 2)(2 \times 2)(2 \times 3)$ is allowed and gives 3×3 .
2. **Multiply two at a time**, in the order written, showing each intermediate product. Either grouping is fine.
3. **"Show that $\mathbf{CAB} = \dots$ "**: write every intermediate matrix; the given answer must come from visible working.

If $\mathbf{A}$ contains an unknown k that is found from a condition (such as $\mathbf{A}$ singular), find k first. Powers of a matrix, $\mathbf{A}^{k+1} = \mathbf{A}^k \mathbf{A}$, come up in induction proofs.

7. Finding the matrix of a transformation or sequence (6 questions)

1. **Write the matrix for each step** from the table, or by finding the images of $(1, 0)$ and $(0, 1)$.
2. **Multiply in reverse order**: "**P** followed by **Q**" is **QP**.
3. Multiply out and simplify, using exact values for the trigonometry.

Variations you will meet:

- "**Show that $2\mathbf{M} = \dots$** ": find **M** with fractions such as $\frac{\sqrt{3}}{2}$, then double every entry.
- **Unknown factor**, such as a stretch with factor d : find **M** in terms of d , then use another fact (an area, a given matrix) to find d .
- **Split a given matrix** into named transformations, such as an enlargement times a stretch times a reflection: write each with an unknown factor, multiply, and compare entries.

8. Invariant points and lines of invariant points (5 questions)

1. Write $\mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ as **two equations**.
2. Rearrange both to the form $px + qy = 0$.
3. **Only the origin**: show the two equations are different lines through O , so they meet only at $(0, 0)$.
4. **A line of invariant points**: show both equations give the **same** line, and state it.

Variations you will meet:

- "**Find the value of θ (or k) for which there is a line of invariant points**": the two equations must be the same line, so $\det(\mathbf{M} - \mathbf{I}) = 0$. For a matrix with $\det \mathbf{M} = 1$ this is the same as "the diagonal entries add up to 2". Solve the resulting equation in the given interval, and check your answer doesn't make $\mathbf{M} = \mathbf{I}$.
- "**Show the invariant points lie on the line ...**": use **both** equations, and show each one gives that line. Don't find invariant lines instead; that is a different question.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
-

Example 1

The matrix $\mathbf{M}$ represents a rotation through 90° anticlockwise about the origin, followed by a stretch parallel to the y -axis with scale factor 4.

(a) Find $\mathbf{M}$. [3]

(b) The triangle T has area 6 units². Find the area of its image under $\mathbf{M}$. [2]

(c) Write $\mathbf{M}^{-1}$ as a product of two matrices, neither of which is $\mathbf{I}$, and describe the transformation it represents. [3]

(d) Find the matrix $\mathbf{N}$ such that $\mathbf{MN} = \begin{pmatrix} 4 & -2 \\ 8 & 12 \end{pmatrix}$. [2]

(a) Rotation: $\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **B1**. Stretch: $\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ **B1**.

The rotation happens first, so it goes on the right:

$$\mathbf{M} = \mathbf{SR} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}.$$

B1 for the correct order and product.

(b) $\det \mathbf{M} = 0 \times 0 - (-1)(4) = 4$. **M1** for using the determinant as the area factor.

Area of image = $4 \times 6 = 24$ units². **A1**

(c) $\mathbf{M}^{-1} = (\mathbf{SR})^{-1} = \mathbf{R}^{-1}\mathbf{S}^{-1}$:

$$\mathbf{M}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

B1 for both inverses, **B1** for the order.

It is a stretch parallel to the y -axis with scale factor $\frac{1}{4}$, followed by a rotation through 90° **clockwise** about the origin. **B1**

(d) Multiply on the left by $\mathbf{M}^{-1} = \begin{pmatrix} 0 & \frac{1}{4} \\ -1 & 0 \end{pmatrix}$: **M1**

$$\mathbf{N} = \mathbf{M}^{-1} \begin{pmatrix} 4 & -2 \\ 8 & 12 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 8 & 12 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -4 & 2 \end{pmatrix}.$$

A1. Check: $\mathbf{MN} = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 8 & 12 \end{pmatrix}$.

Example 2

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 2 & 1 & k \\ 0 & 3 & 1 \\ 1 & -1 & 2 \end{pmatrix}$, where k is a constant.

(a) Find the value of k for which $\mathbf{A}$ is singular. **[3]**

(b) Given that $k = 2$, find $\mathbf{A}^{-1}$. **[4]**

(a) Expand along the top row: **M1**

$$\det \mathbf{A} = 2 \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + k \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} = 2(7) - 1(-1) + k(-3) = 15 - 3k.$$

A1

Singular means $\det \mathbf{A} = 0$: $15 - 3k = 0$, so $k = 5$. **A1**

(b) With $k = 2$: $\det \mathbf{A} = 15 - 6 = 9$. **B1**

Minors with the signs $\begin{smallmatrix} + & - & + \\ - & + & - \\ + & - & + \end{smallmatrix}$ give the matrix of cofactors: **M1**

$$\begin{pmatrix} 7 & 1 & -3 \\ -4 & 2 & 3 \\ -5 & -2 & 6 \end{pmatrix}.$$

For example, the entry in row 2, column 1 covers row 2 and column 1 of $\mathbf{A}$, leaving $\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 4$, and the sign there is $-$, giving -4 . **A1** for all nine.

Transpose and divide by 9:

$$\mathbf{A}^{-1} = \frac{1}{9} \begin{pmatrix} 7 & -4 & -5 \\ 1 & 2 & -2 \\ -3 & 3 & 6 \end{pmatrix}.$$

A1. Check one entry of $\mathbf{A}\mathbf{A}^{-1}$: row 1 of $\mathbf{A}$ times column 1 is $\frac{1}{9}(14 + 1 - 6) = 1$.

Example 3

The matrix $\mathbf{M}$ is $\begin{pmatrix} 5 & -2 \\ 3 & 0 \end{pmatrix}$.

(a) Find the equations of the invariant lines, through the origin, of the transformation represented by $\mathbf{M}$. **[5]**

(b) Show that the origin is the only invariant point of this transformation. **[3]**

(a)

$$\begin{pmatrix} 5 & -2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x - 2y \\ 3x \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}.$$

B1

On $y = mx$: $X = 5x - 2mx$ and $Y = 3x$. Using $Y = mX$: **M1**

$$3x = m(5x - 2mx) \Rightarrow 3 = 5m - 2m^2 \Rightarrow 2m^2 - 5m + 3 = 0.$$

A1

$$(2m - 3)(m - 1) = 0, \text{ so } m = \frac{3}{2} \text{ or } m = 1. \text{ A1}$$

The invariant lines are $y = x$ and $y = \frac{3}{2}x$. **A1**

The top-right entry is $-2 \neq 0$, so $x = 0$ is not invariant: $(0, 1) \mapsto (-2, 0)$.

(b) For an invariant point, $5x - 2y = x$ and $3x = y$. **M1**

The first gives $y = 2x$; the second gives $y = 3x$. **A1**

These are different lines through the origin, so they meet only at $(0, 0)$: the origin is the only invariant point. **A1**

The diagram of invariant lines in "Understand it" shows this matrix: $(1, 1)$ moves to $(3, 3)$, still on $y = x$, but it has moved, so it is not an invariant point.

Example 4

The matrix $\mathbf{M}$ is $\begin{pmatrix} 3 & k \\ 1 & 1 \end{pmatrix}$, where k is a non-zero constant.

(a) Find the set of values of k for which the transformation represented by $\mathbf{M}$ has no invariant lines through the origin. **[5]**

(b) Given that $k = 8$, find the equations of the invariant lines through the origin. **[2]**

(a) The image of (x, mx) is $\begin{pmatrix} 3x + kmx \\ x + mx \end{pmatrix}$. **B1**

$Y = mX: x + mx = m(3x + kmx)$, so $1 + m = 3m + km^2$, that is **M1**

$$km^2 + 2m - 1 = 0.$$

A1

No real roots when the discriminant is negative: **M1**

$$2^2 - 4(k)(-1) < 0 \Rightarrow 4 + 4k < 0 \Rightarrow k < -1.$$

Since $k \neq 0$, the line $x = 0$ is never invariant ($(0, 1) \mapsto (k, 1)$), so the answer is $k < -1$. **A1**

(b) $8m^2 + 2m - 1 = 0 \Rightarrow (4m - 1)(2m + 1) = 0$, so $m = \frac{1}{4}$ or $m = -\frac{1}{2}$. **M1**

The invariant lines are $y = \frac{1}{4}x$ and $y = -\frac{1}{2}x$. **A1**

Here $\det \mathbf{M} = 3 - 8 = -5 \neq 0$, so neither line is squashed to the origin. (With $k = 3$, $\mathbf{M}$ would be singular: the root $m = -1$ would give the line $y = -x$, which is mapped to the origin, not onto itself, so it would not count.)

(When $k = -1$ the discriminant is 0 and there is exactly one invariant line, $y = x$.)

Common mistakes

- **Multiplying in the wrong order.** "**P** followed by **Q**" is **QP**, not **PQ**. Examiners report this every year. Write the steps as they happen, then put them into the product from right to left.
- **Confusing a stretch with a shear.** A stretch has its factor on the leading diagonal, $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$; a shear has it off the diagonal, $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. Check with $(1, 0)$ and $(0, 1)$.
- **Incomplete descriptions.** "Rotation through 90° " is not enough: say "anticlockwise about the origin". A shear needs its fixed axis and where a point goes; a stretch needs its direction and scale factor.
- **Forgetting to divide by the determinant** in an inverse, or dividing by the wrong number. Check by multiplying back.
- **Inverting a product the long way, or in the wrong order.** For "write $\mathbf{M}^{-1}$ as a product of two matrices", invert each factor and reverse the order, $\mathbf{Q}^{-1}\mathbf{P}^{-1}$. Multiplying first and inverting the result doesn't answer the question.
- **Multiplying on the wrong side.** From $\mathbf{MN} = \mathbf{P}$, the answer is $\mathbf{M}^{-1}\mathbf{P}$, not $\mathbf{PM}^{-1}$.
- **"Non-singular" read as $\det > 0$.** It means $\det \neq 0$. The answer to "for which k is **A** non-singular?" is usually " $k \neq \dots$ ".
- **Finding the determinant of the wrong matrix**, such as $\det(\mathbf{CAB})$ when the question asks about **A**. Read which matrix is singular.
- **Mixing up invariant points and invariant lines.** Points: $\mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Lines: $Y = mX$. The points on an invariant line usually move.
- **Using only one equation for invariant points.** Both equations must hold; show what each one gives.
- **Cancelling a common factor** such as $(1 + m)$ and losing a solution, and so an invariant line.
- **Using $y = mx + c$** for invariant lines through the origin. Through the origin means $c = 0$.
- **Dropping the modulus with areas.** $|\det \mathbf{M}| = 1$ allows $\det \mathbf{M} = -1$ too; decide using the conditions in the question.
- **Multiplying when you should divide** with areas: check which triangle is the image.

How to get the marks

- **"Give full details"** means every property: type, centre, angle and direction, mirror line, scale factor, fixed line. **"State the type"** needs only the names, but still in the right order.
- **Order is marked separately.** In a sequence question one mark is usually for the order alone, so say "...", followed by ..." clearly.
- **"Show that"** (for example $\mathbf{CAB} = \dots$ or $2\mathbf{M} = \dots$) needs every intermediate matrix written out. The answer is printed, so the marks are for the working.
- **Invariant lines:** write the image $\begin{pmatrix} X \\ Y \end{pmatrix}$, write " $y = mx$, $Y = mX$ ", form the quadratic, solve it and **give the lines as equations**. Values of m alone lose the last mark.

- **Determinants:** show the expansion with the 2×2 determinants, not just the final number. For "show it is 0" or "show it is never 0", end with a clear sentence.
- **Inverses:** a correct inverse used in the right place earns the method mark even after an arithmetic slip, so write $\mathbf{N} = \mathbf{M}^{-1}\mathbf{P}$ before multiplying.
- **"Hence"** means use the previous part: the $\mathbf{M}$ you just found, or the product you just showed.
- **Exact answers:** keep surds and fractions in matrices. Angles in radians when the question is in radians, such as $\theta = \frac{\pi}{2}$.
- **Areas:** write "area of image = $|\det \mathbf{M}| \times$ area of object" with the numbers in, and include units if the question has them.
- **Follow-through:** later parts are often marked on your own earlier matrix, so keep going even if you are unsure of it.

Quick check

1. Describe fully the transformation represented by $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, and find its determinant.
2. Find the inverse of $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.
3. A triangle of area 7 cm^2 is transformed by $\begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}$. Find the area of its image.
4. Find the equations of the invariant lines through the origin of the transformation represented by $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$.
5. The matrices $\mathbf{P}$ and $\mathbf{Q}$ are non-singular, with $\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ and $\mathbf{Q}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Find $(\mathbf{PQ})^{-1}$.

Answers

- $(1, 0) \mapsto (0, -1)$ and $(0, 1) \mapsto (-1, 0)$: a reflection in the line $y = -x$. Its determinant is $0 \times 0 - (-1)(-1) = -1$ (areas unchanged, shape flipped).
- $\det = 8 - 3 = 5$, so the inverse is $\frac{1}{5} \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$.
- $\det = 15 - (-2) = 17$, so the image has area $17 \times 7 = 119 \text{ cm}^2$.
- The image of (x, mx) is $(2x + 2mx, x + 3mx)$. $Y = mX$: $1 + 3m = m(2 + 2m)$, so $2m^2 - m - 1 = 0$, $(2m + 1)(m - 1) = 0$ and $m = 1$ or $-\frac{1}{2}$. The lines are $y = x$ and $y = -\frac{1}{2}x$.
- $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$.

Past questions to try

- [9231/13/M/J/25 Q1](#): a stretch then a rotation, invariant lines, and the matrix that maps back.
- [9231/12/O/N/22 Q3](#): a shear and a stretch, the inverse as a product, invariant points and area.
- [9231/12/M/J/22 Q7](#): a 3×3 determinant and inverse, a non-square product, and invariant lines with a discriminant.
- [9231/13/O/N/21 Q4](#): a rotation and a stretch with exactly one invariant line.