

Matrices A2

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Read first: [Matrices](#), [Vectors](#)

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What you need to know

By the end of this topic you should be able to:

1. **Switch between three simultaneous equations and one matrix equation.** Three linear equations in x, y, z can be written as $\mathbf{Ax} = \mathbf{b}$, and a matrix equation like that can be written out as three equations.
2. **Know what can happen to three equations in three unknowns.** Link it to whether the matrix $\mathbf{A}$ is singular: a unique solution, infinitely many solutions (consistent) or no solution (inconsistent). Solve consistent systems, and describe each case with planes: meeting at one point, meeting in a common line, or having no point in common.
3. **Know the words "characteristic equation", "eigenvalue" and "eigenvector"** for a square matrix $\mathbf{A}$, and use the definition $\mathbf{Ae} = \lambda\mathbf{e}$ to prove simple facts, such as λ^n being an eigenvalue of $\mathbf{A}^n$.
4. **Find eigenvalues and eigenvectors of 2×2 and 3×3 matrices.** In this course the eigenvalues are always real and all different.
5. **Write a square matrix as $\mathbf{QDQ}^{-1}$,** where $\mathbf{D}$ is a diagonal matrix of eigenvalues and the columns of $\mathbf{Q}$ are matching eigenvectors, and use it, for example to find powers of the matrix.
6. **Use the fact that a square matrix satisfies its own characteristic equation** (the Cayley–Hamilton theorem) for 2×2 and 3×3 matrices, for example to find higher powers or the inverse.

Multiplying matrices, determinants and inverses are in Matrices (Further Pure 1); planes, normals and the vector product are in Vectors. This note uses them without teaching them again.

Understand it

Three equations as one matrix equation

The equations

$$\begin{array}{l} x + 2y - z = 3 \\ 2x + y + z = 1 \\ x - y + 2z = 0 \end{array} \quad \text{are the same as} \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

Row i of the matrix holds the coefficients of equation i ; a missing letter is a 0. Going the other way, multiply out $\mathbf{Ax}$ row by row to get the three equations.

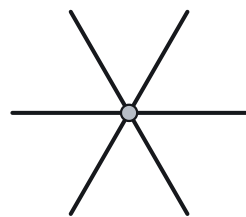
If $\mathbf{A}$ is **non-singular** ($\det \mathbf{A} \neq 0$), multiply both sides on the left by $\mathbf{A}^{-1}$: $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. There is exactly one solution, whatever $\mathbf{b}$ is.

If $\mathbf{A}$ is **singular** ($\det \mathbf{A} = 0$), there is never a unique solution. Either there are infinitely many solutions (the equations are **consistent**) or none at all (they are **inconsistent**). Which one depends on $\mathbf{b}$, so you have to look at the equations themselves.

The planes behind the equations

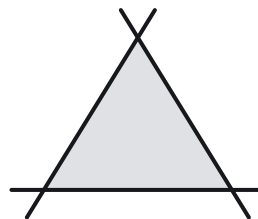
Each equation $ax + by + cz = d$ is a plane with normal $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, and a solution is a point on all three planes.

- $\det \mathbf{A} \neq 0$: the three planes meet at **one point**.
- $\det \mathbf{A} = 0$, **consistent**: usually the planes all contain one **common line**, like pages meeting at the spine of a book. This is called a **sheaf**. (If two or all three equations are multiples of each other, the planes are the same plane, and the solutions fill a line or a whole plane.)
- $\det \mathbf{A} = 0$, **inconsistent**: there is no point on all three. Look at the normals:
 - no two normals parallel: each pair of planes meets in a line, but the three lines are parallel and separate. The planes form a **triangular prism**;
 - two normals parallel: those two planes are **parallel and distinct**, and the third plane cuts both;
 - all three normals parallel: three parallel planes (at least two of them distinct).



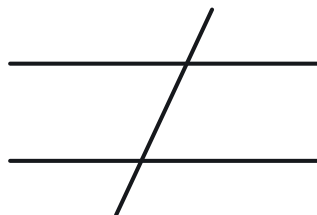
Sheaf

consistent: one common line



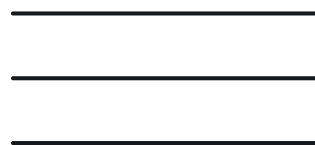
Triangular prism

inconsistent: no common point



Two parallel planes

inconsistent: the third cuts both



Three parallel planes

inconsistent: no two meet

How to tell consistent from inconsistent. When $\det \mathbf{A} = 0$, one row of $\mathbf{A}$ is a combination of the other two. Find that combination and apply it to the whole equations, right-hand sides included. For the system above (its matrix has $\det = 0$): equation 2 minus equation 1 gives $x - y + 2z = -2$. Equation 3 says $x - y + 2z = 0$. Both can't be true, so the system is inconsistent, and no two of the normals $(1, 2, -1)$, $(2, 1, 1)$, $(1, -1, 2)$ are parallel: a triangular prism. Had equation 3 been $x - y + 2z = -2$, it would add nothing new, and the planes would form a sheaf. Example 1 finishes that case.

Homogeneous systems (every right-hand side 0) are always consistent, because $x = y = z = 0$ works. "No unique solution" then means infinitely many.

Eigenvalues and eigenvectors

Most vectors change direction when a matrix multiplies them. An **eigenvector** of a square matrix $\mathbf{A}$ is a **non-zero** vector $\mathbf{e}$ that keeps its direction: it is only stretched (or reversed) by a number λ , its **eigenvalue**:

$$\mathbf{A}\mathbf{e} = \lambda\mathbf{e}, \quad \mathbf{e} \neq \mathbf{0}.$$

For $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$: $\mathbf{M} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 5. Any non-zero multiple, such as $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$, is also an eigenvector for 5; you give the simplest one.

Finding the eigenvalues. $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$ is the same as $(\mathbf{A} - \lambda\mathbf{I})\mathbf{e} = \mathbf{0}$. A non-zero solution $\mathbf{e}$ exists only when $\mathbf{A} - \lambda\mathbf{I}$ is singular (otherwise its inverse would give $\mathbf{e} = \mathbf{0}$). So the eigenvalues are the roots of the **characteristic equation**

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

$\mathbf{A} - \lambda\mathbf{I}$ is $\mathbf{A}$ with λ taken off each diagonal entry. For $\mathbf{M}$ above:

$$\begin{vmatrix} 4 - \lambda & 1 \\ 2 & 3 - \lambda \end{vmatrix} = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5) = 0,$$

so the eigenvalues are 2 and 5. For a 2×2 matrix the characteristic equation is always $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$: the sum of the diagonal entries, then the determinant. For a 3×3 matrix it is a cubic; solve it by spotting a root with the factor theorem, then factorising.

Triangular matrices are quick. If every entry below the leading diagonal is 0 (or every entry above it), $\det(\mathbf{A} - \lambda\mathbf{I})$ is just the product of the diagonal entries, so **the eigenvalues are the diagonal entries**. State them; there is no need to expand a determinant.

Finding an eigenvector for each eigenvalue: solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{e} = \mathbf{0}$. For $\lambda = 2$: $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$ gives $2x + y = 0$ (both rows say the same, as they must), so $\mathbf{e} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

For a 3×3 matrix the fastest way is the **vector product**. Each row $\mathbf{r}$ of $\mathbf{A} - \lambda\mathbf{I}$ has $\mathbf{r} \cdot \mathbf{e} = 0$, so $\mathbf{e}$ is perpendicular to every row. Take two rows that are not parallel: their vector product is perpendicular to both, so it is an eigenvector. Divide out any common factor. If you get $\mathbf{0}$, the two rows were parallel: use a different pair.

An eigenvector is never 0: if every pair gives **0**, your eigenvalue is wrong. Check the answer by multiplying: **Ae** should be λe .

Proving facts from $\mathbf{A}e = \lambda e$

Start from $\mathbf{A}e = \lambda e$ and multiply on the **left** by a matrix. Numbers like λ can move to the front; matrices can't swap places.

- **Powers:** $\mathbf{A}^2 e = \mathbf{A}(\mathbf{A}e) = \mathbf{A}(\lambda e) = \lambda(\mathbf{A}e) = \lambda^2 e$. Repeating this, $\mathbf{A}^n e = \lambda^n e$: λ^n is an eigenvalue of $\mathbf{A}^n$, with the same eigenvector.
- **Adding $k\mathbf{I}$:** $(\mathbf{A} + k\mathbf{I})e = \lambda e + ke = (\lambda + k)e$.
- **The inverse** (for non-singular $\mathbf{A}$, so $\lambda \neq 0$): multiply $\mathbf{A}e = \lambda e$ on the left by $\mathbf{A}^{-1}$ to get $e = \lambda \mathbf{A}^{-1}e$, so $\mathbf{A}^{-1}e = \frac{1}{\lambda}e$.

So any polynomial in $\mathbf{A}$, such as $(\mathbf{A} - 3\mathbf{I})^2$, has the **same eigenvectors** as $\mathbf{A}$, and its eigenvalues are the same polynomial of λ : here $(\lambda - 3)^2$.

Diagonalising: $\mathbf{A} = \mathbf{QDQ}^{-1}$

Put the eigenvectors of $\mathbf{A}$ side by side as the columns of $\mathbf{Q}$, and the eigenvalues, **in the same order**, down the diagonal of $\mathbf{D}$. Then $\mathbf{AQ} = \mathbf{QD}$ (each column says $\mathbf{A}e = \lambda e$), and because the eigenvalues are distinct, $\mathbf{Q}$ is non-singular, so

$$\mathbf{A} = \mathbf{QDQ}^{-1}.$$

For **M**: $\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$. (Swapping the columns of $\mathbf{Q}$ is fine as long as you swap the eigenvalues in $\mathbf{D}$ too.)

Why this helps with powers. In $\mathbf{A}^2 = (\mathbf{QDQ}^{-1})(\mathbf{QDQ}^{-1})$ the middle $\mathbf{Q}^{-1}\mathbf{Q}$ is $\mathbf{I}$, so $\mathbf{A}^2 = \mathbf{QD}^2\mathbf{Q}^{-1}$, and in general

$$\mathbf{A}^n = \mathbf{QD}^n\mathbf{Q}^{-1}, \quad \mathbf{D}^n = \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix}.$$

Powers of a diagonal matrix are just powers of its diagonal entries. For **M**, with $\mathbf{Q}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$:

$$\mathbf{M}^n = \frac{1}{3} \begin{pmatrix} 2^n + 2 \cdot 5^n & 5^n - 2^n \\ 2 \cdot 5^n - 2^{n+1} & 2^{n+1} + 5^n \end{pmatrix}.$$

Putting $n = 1$ gives back **M**. The same idea works for any polynomial in $\mathbf{A}$: $(\mathbf{A} - 3\mathbf{I})^2 = \mathbf{Q}(\mathbf{D} - 3\mathbf{I})^2\mathbf{Q}^{-1}$, with the same $\mathbf{Q}$.

Going backwards, if you are told the eigenvalues and eigenvectors, you can **build** the matrix: $\mathbf{A} = \mathbf{QDQ}^{-1}$.

A matrix satisfies its own characteristic equation

The **Cayley–Hamilton theorem** says: replace λ by $\mathbf{A}$ in the characteristic equation (and the number term c by $c\mathbf{I}$), and the result is true. For $\mathbf{M}$, $\lambda^2 - 7\lambda + 10 = 0$ becomes

$$\mathbf{M}^2 - 7\mathbf{M} + 10\mathbf{I} = \mathbf{0}.$$

Check: $\mathbf{M}^2 = \begin{pmatrix} 18 & 7 \\ 14 & 11 \end{pmatrix}$, and $\begin{pmatrix} 18 & 7 \\ 14 & 11 \end{pmatrix} - \begin{pmatrix} 28 & 7 \\ 14 & 21 \end{pmatrix} + \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \mathbf{0}.$

This gives two tools:

- **Higher powers.** $\mathbf{M}^2 = 7\mathbf{M} - 10\mathbf{I}$. Multiply by $\mathbf{M}$ to get $\mathbf{M}^3 = 7\mathbf{M}^2 - 10\mathbf{M}$, then replace $\mathbf{M}^2$ again: $\mathbf{M}^3 = 7(7\mathbf{M} - 10\mathbf{I}) - 10\mathbf{M} = 39\mathbf{M} - 70\mathbf{I}$. Every power comes down to the lower ones.
- **The inverse.** Multiply $\mathbf{M}^2 - 7\mathbf{M} + 10\mathbf{I} = \mathbf{0}$ by $\mathbf{M}^{-1}$: $\mathbf{M} - 7\mathbf{I} + 10\mathbf{M}^{-1} = \mathbf{0}$, so $\mathbf{M}^{-1} = \frac{1}{10}(7\mathbf{I} - \mathbf{M}) = \frac{1}{10} \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$. For a 3×3 matrix this gives $\mathbf{A}^{-1}$ from $\mathbf{A}^2$, $\mathbf{A}$ and $\mathbf{I}$, with no cofactors.

Key facts and formulas

The MF19 list of formulae has nothing on matrices, so everything here must be learned.

Formula	Status
Three equations $\Leftrightarrow \mathbf{Ax} = \mathbf{b}$; if $\det \mathbf{A} \neq 0$, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ (a unique point)	Learn it
$\det \mathbf{A} = 0$: no unique solution; consistent (sheaf, a common line) or inconsistent (triangular prism, or parallel planes)	Learn it
$\mathbf{Ae} = \lambda \mathbf{e}$, with $\mathbf{e} \neq \mathbf{0}$	Learn it
Characteristic equation: $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$	Learn it
2×2 : $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$	Learn it
Triangular matrix: eigenvalues are the diagonal entries	Learn it
Eigenvector of a 3×3 : vector product of two non-parallel rows of $\mathbf{A} - \lambda \mathbf{I}$	Learn it
$\mathbf{A}^n \mathbf{e} = \lambda^n \mathbf{e}$; $(\mathbf{A} + k\mathbf{I})\mathbf{e} = (\lambda + k)\mathbf{e}$; $\mathbf{A}^{-1}\mathbf{e} = \frac{1}{\lambda}\mathbf{e}$	Learn it
$\mathbf{A} = \mathbf{QDQ}^{-1}$: columns of $\mathbf{Q}$ are eigenvectors, $\mathbf{D}$ the matching eigenvalues	Learn it

Formula	Status
$\mathbf{A}^n = \mathbf{Q}\mathbf{D}^n\mathbf{Q}^{-1}$; $\mathbf{D}^n$ has diagonal $\lambda_1^n, \lambda_2^n, \lambda_3^n$	Learn it
Cayley–Hamilton: $\mathbf{A}$ satisfies its characteristic equation ($\lambda \rightarrow \mathbf{A}$, number $c \rightarrow c\mathbf{I}$)	Learn it
Vector product $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$	Learn it

How to do it

We have tagged 46 questions for this topic (2021–2025); 30 are different, because 16 appear on two papers. The table counts the 30 different ones. 21 of them mix types, so the counts add up to more than 30. One question also has vectors parts about planes; only its matrix parts are counted.

Question type	Questions
Using the characteristic equation (Cayley–Hamilton)	19
Three simultaneous equations: unique, consistent, inconsistent	15
Diagonalising: find $\mathbf{P}$ and $\mathbf{D}$	15
Finding or checking eigenvalues and eigenvectors	12
Finding a matrix from its eigenvalues and eigenvectors	3
Proving eigenvalue facts from $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$	2

Questions call the eigenvector matrix $\mathbf{P}$ (or $\mathbf{Q}$); it is the same thing.

1. Using the characteristic equation (Cayley–Hamilton) (19 questions)

- Get the characteristic equation**, multiplied out: $\lambda^3 + a\lambda^2 + b\lambda + c = 0$. For a triangular matrix, multiply out $(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$ from the diagonal entries. If it is given ("show that"), you still need the determinant working.
- Write it with the matrix**: " $\mathbf{A}$ satisfies its characteristic equation, so $\mathbf{A}^3 + a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{0}$." Don't forget the $\mathbf{I}$.
- For a higher power**: make the top power the subject, $\mathbf{A}^3 = -a\mathbf{A}^2 - b\mathbf{A} - c\mathbf{I}$. Multiply by $\mathbf{A}$ to get $\mathbf{A}^4$, then replace the $\mathbf{A}^3$ that appears. Repeat until only the powers asked for are left, and collect terms.
- For the inverse**: multiply the equation by $\mathbf{A}^{-1}$, which lowers every power by one and turns $c\mathbf{I}$ into $c\mathbf{A}^{-1}$. Make $\mathbf{A}^{-1}$ the subject, then work out $\mathbf{A}^2$ and substitute to get the actual matrix.

Variations you will meet:

- "Show that $\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{A} + r\mathbf{I}$ "**: only the numbers p, q, r are wanted; don't work out any matrices. Example 2 does this.

- **Missing terms:** if the characteristic equation has no λ term or no number term, the answer has fewer terms too (for example, a zero eigenvalue gives no $\mathbf{I}$ term, as in $\mathbf{A}^3 = 4\mathbf{A}^2 - 3\mathbf{A}$; then $\mathbf{A}$ is singular and has no inverse).
- **A polynomial in $\mathbf{A}$,** such as $(\mathbf{A} - 2\mathbf{I})^3 = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I}$: expand the bracket like algebra (matrices $\mathbf{A}$ and $\mathbf{I}$ can be multiplied in any order), then replace $\mathbf{A}^3$ using Cayley–Hamilton. Or find the characteristic equation of $\mathbf{A} - 2\mathbf{I}$ (its eigenvalues are $\lambda - 2$) and use that.
- $(k\mathbf{A} + m\mathbf{I})^2 = \mathbf{A}^n(\dots)$ or $\mathbf{A}^{-2}$: rearrange the equation so the expression you want is the subject **before** squaring or inverting.
- **Letters in the matrix**, such as eigenvalues $a, 2a, -3a$: multiply out $(\lambda - a)(\lambda - 2a)(\lambda + 3a)$ carefully, and the inverse is in terms of a .
- **A 2×2 matrix:** the characteristic equation is $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$, and the same steps work.

2. Three simultaneous equations: unique, consistent, inconsistent (15 questions)

1. **Write the matrix of coefficients** and find its determinant, expanding along the row with most zeros. If there is an unknown k , you get an expression in k .
2. **Unique solution** $\Leftrightarrow \det \neq 0$. "Find the values of k for which there is no unique solution" means solve $\det = 0$; "the set of values for a unique solution" means $k \neq$ those values.
3. **Interpret:** a unique solution means "the three planes meet at a single point".
4. **When $\det = 0$,** put the value of k in and find a combination of two equations whose left-hand side equals the third (or eliminate one unknown from two different pairs). Compare the right-hand sides:
 - they agree: **consistent**, a **sheaf** of planes through a common line. Show it by reducing all three equations to two, then write the line with a parameter if asked;
 - they disagree: **inconsistent**, a clear contradiction such as $0 = 4$. Then check the normals: a **triangular prism** if no two are parallel; **two parallel planes** (not the same plane) and a third that is not parallel to them, if two are parallel.

Variations you will meet:

- **Two special values of k** from a quadratic determinant: treat each separately. They often give different pictures (one a prism, the other two parallel planes).
- **Parallel planes are easy to spot:** two equations whose left-hand sides are multiples, such as $2x - y + 3z = 1$ and $4x - 2y + 6z = 5$. Double the first: same left side $4x - 2y + 6z$, different right sides (2 and 5), so the planes are parallel and distinct.
- **"Show there is no unique solution" with no k to find:** show the determinant is 0 for every value of the letter in it.
- **"Show the system is consistent for all values of a ":** when $\det \neq 0$ it is consistent automatically; for the value that makes $\det = 0$, show consistency by elimination (or by finding one solution).
- **"Show there is a unique solution" with conditions** (such as $a \neq 0$, $a \neq 1$, or a an integer): factorise the determinant and show the conditions stop it being 0.
- **Homogeneous equations** (all right-hand sides 0): always consistent; $\det = 0$ means infinitely many solutions.

- **Given a solution, find a constant:** substitute x, y, z into the equation containing it.

3. Diagonalising: find $\mathbf{P}$ and $\mathbf{D}$ (15 questions)

1. **Eigenvalues:** read them off a triangular matrix, or solve $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
2. **Eigenvectors:** for each λ , the vector product of two non-parallel rows of $\mathbf{A} - \lambda\mathbf{I}$. Simplify by dividing out common factors.
3. **Write $\mathbf{P}$** with the eigenvectors as columns and **$\mathbf{D}$** with the matching values in the same order.

Variations you will meet:

- **A power or polynomial:** "such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$ " or " $(\mathbf{A} + 3\mathbf{I})^2 = \mathbf{PDP}^{-1}$ ". $\mathbf{P}$ is the same as for $\mathbf{A}$; put the power or polynomial of each eigenvalue into $\mathbf{D}$: λ^5 or $(\lambda + 3)^2$. Leaving plain eigenvalues in $\mathbf{D}$ loses the last mark. Example 3 does this.
- **$\mathbf{A}^{-1}$ or $\mathbf{A}^{-2}$:** $\mathbf{D}$ holds $\frac{1}{\lambda}$ or $\frac{1}{\lambda^2}$.
- **$\mathbf{A}^n$ for a general n :** $\mathbf{D}$ has $\lambda_1^n, \lambda_2^n, \lambda_3^n$; a zero eigenvalue stays 0.
- **Letters in the matrix:** eigenvectors may contain the letter; keep it, and simplify each vector.
- **Eigenvalues or eigenvectors given in an earlier part:** use them; you only need to find the missing ones.

4. Finding or checking eigenvalues and eigenvectors (12 questions)

1. **"Show that $\mathbf{v}$ is an eigenvector and state its eigenvalue":** multiply $\mathbf{A}\mathbf{v}$ and show the result is a multiple of $\mathbf{v}$. The multiple is the eigenvalue.
2. **"Show that the characteristic equation is ...":** write $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ with the λ s on the diagonal, expand showing the 2×2 determinants (a row or column with zeros helps), and multiply out to the given cubic, " $= 0$ " included.
3. **"Hence find the eigenvalues":** solve the cubic. If one eigenvalue is known, divide it out as a factor; otherwise try small integers.
4. **"State the eigenvalues"** of a triangular matrix: the diagonal entries, no working needed.

If one eigenvector is given and others must be found, you still use the rows of $\mathbf{A} - \lambda\mathbf{I}$. If eigenvectors are given and you need the eigenvalues, multiply each by $\mathbf{A}$.

5. Finding a matrix from its eigenvalues and eigenvectors (3 questions)

1. **Write $\mathbf{P}$** (the given eigenvectors as columns, in the given order) and **$\mathbf{D}$** (the eigenvalues in that order).
2. **Find $\mathbf{P}^{-1}$:** often an earlier part hands it to you (for example, $\mathbf{P}$ was the matrix whose inverse you found with Cayley–Hamilton).
3. **Multiply $\mathbf{PDP}^{-1}$,** two matrices at a time, showing $\mathbf{PD}$ (each column of $\mathbf{P}$ times its eigenvalue) first.

"Find $\mathbf{A}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ " is the same question: $\mathbf{A} = \mathbf{PDP}^{-1}$. Then the eigenvalues of $\mathbf{A}$ are the diagonal of $\mathbf{D}$, and its eigenvectors are the columns of $\mathbf{P}$. Example 4 builds a matrix this way.

6. Proving eigenvalue facts from $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$ (2 questions)

1. **Start** from $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$.
2. **Multiply on the left** by $\mathbf{A}$ (for powers) or by $\mathbf{A}^{-1}$ (for the inverse). Move numbers to the front and use $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$ again.
3. **Finish** with a line of the form (matrix) $\mathbf{e} =$ (number) $\mathbf{e}$ and a sentence saying $\mathbf{e}$ is an eigenvector with that eigenvalue.

Never divide by a vector or "invert" $\mathbf{e}$. For the inverse, say why $\lambda \neq 0$ (the matrix is non-singular).

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
-

Example 1

The system of equations

$$\begin{aligned}x + 2y - z &= 3 \\2x + ky + z &= 1 \\x - y + 2z &= c\end{aligned}$$

has constants k and c .

- (a) Find the value of k for which the system does not have a unique solution. **[2]**
- (b) For this value of k , find the value of c for which the system is consistent. Find the general solution in this case and interpret it geometrically. **[4]**
- (c) For this value of k and any other value of c , show that the system is inconsistent and interpret this geometrically. **[2]**

(a) Expand along the top row: **M1**

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & k & 1 \\ 1 & -1 & 2 \end{vmatrix} = 1(2k + 1) - 2(4 - 1) - 1(-2 - k) = 3k - 3.$$

No unique solution when $3k - 3 = 0$, so $k = 1$. **A1**

(b) With $k = 1$, equation 2 minus equation 1 gives $x - y + 2z = 1 - 3 = -2$. **M1**

This matches the left-hand side of equation 3, so the system is consistent only when $c = -2$. **A1**

Then equation 3 adds nothing, and equations 1 and 3 remain. Subtract them: $3y - 3z = 5$, so $y = \frac{5}{3} + z$.

From equation 1, $x = 3 - 2y + z = -\frac{1}{3} - z$. With $z = t$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{5}{3} \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

A1. The three planes meet in this common line: they form a **sheaf**. **B1**

(Check: the direction is the vector product of the normals $(1, 2, -1) \times (1, -1, 2) = (3, -3, -3)$, a multiple of $(-1, 1, 1)$.)

(c) If $c \neq -2$, equations 1 and 2 force $x - y + 2z = -2$ while equation 3 says $x - y + 2z = c$: a contradiction, so there is no solution. **B1**

No two of the normals $(1, 2, -1)$, $(2, 1, 1)$, $(1, -1, 2)$ are parallel, so each pair of planes meets in a line but nothing lies on all three: a **triangular prism**. **B1**

Example 2

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{pmatrix}$.

(a) State the eigenvalues of **A**. **[1]**

(b) Use the characteristic equation of **A** to find $\mathbf{A}^{-1}$. **[4]**

(c) Use the characteristic equation of **A** to show that $\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{A} + r\mathbf{I}$, where p , q and r are integers to be found. **[4]**

(a) **A** is upper triangular, so its eigenvalues are its diagonal entries: 3, -1 and 4. **B1**

(b) The characteristic equation is **B1**

$$(\lambda - 3)(\lambda + 1)(\lambda - 4) = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 5\lambda + 12 = 0.$$

A satisfies it: $\mathbf{A}^3 - 6\mathbf{A}^2 + 5\mathbf{A} + 12\mathbf{I} = \mathbf{0}$. Multiply by $\mathbf{A}^{-1}$: **M1**

$$\mathbf{A}^2 - 6\mathbf{A} + 5\mathbf{I} + 12\mathbf{A}^{-1} = \mathbf{0} \Rightarrow \mathbf{A}^{-1} = \frac{1}{12} (-\mathbf{A}^2 + 6\mathbf{A} - 5\mathbf{I}).$$

$$\mathbf{A}^2 = \begin{pmatrix} 9 & 4 & 17 \\ 0 & 1 & 15 \\ 0 & 0 & 16 \end{pmatrix}, \text{ M1 so}$$

$$\mathbf{A}^{-1} = \frac{1}{12} \begin{pmatrix} 4 & 8 & -11 \\ 0 & -12 & 15 \\ 0 & 0 & 3 \end{pmatrix}.$$

A1. Check: row 1 of **A** times column 3 is $\frac{1}{12}(3(-11) + 2(15) + 1(3)) = 0$, as it should be.

(c) $\mathbf{A}^3 = 6\mathbf{A}^2 - 5\mathbf{A} - 12\mathbf{I}$. B1

Multiply by **A**: $\mathbf{A}^4 = 6\mathbf{A}^3 - 5\mathbf{A}^2 - 12\mathbf{A}$. **M1**

Replace $\mathbf{A}^3$: **M1**

$$\mathbf{A}^4 = 6(6\mathbf{A}^2 - 5\mathbf{A} - 12\mathbf{I}) - 5\mathbf{A}^2 - 12\mathbf{A} = 31\mathbf{A}^2 - 42\mathbf{A} - 72\mathbf{I}.$$

So $p = 31, q = -42, r = -72$. **A1**

Example 3

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 4 & 1 \\ -3 & 3 & 2 \end{pmatrix}$.

(a) Show that the characteristic equation of **A** is $\lambda^3 - 7\lambda^2 + 7\lambda + 15 = 0$, and find the eigenvalues of **A**. **[4]**

(b) Find a matrix **P** and a diagonal matrix **D** such that $(\mathbf{A} - \mathbf{I})^3 = \mathbf{PDP}^{-1}$. **[7]**

(a) Expand $\det(\mathbf{A} - \lambda\mathbf{I})$ along the top row: **M1**

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 4-\lambda & 1 \\ -3 & 3 & 2-\lambda \end{vmatrix} = (1-\lambda)[(4-\lambda)(2-\lambda) - 3] - 2[-(2-\lambda) + 3] - 2[-3 + 3(4-\lambda)] \\ = (1-\lambda)(\lambda^2 - 6\lambda + 5) - 2(\lambda + 1) - 2(9 - 3\lambda) = -\lambda^3 + 7\lambda^2 - 7\lambda - 15.$$

Setting this to 0 and multiplying by -1 : $\lambda^3 - 7\lambda^2 + 7\lambda + 15 = 0$. **A1** (the given answer, with the working shown)

$\lambda = -1$ gives $-1 - 7 - 7 + 15 = 0$, so $(\lambda + 1)$ is a factor: **M1**

$$(\lambda + 1)(\lambda^2 - 8\lambda + 15) = (\lambda + 1)(\lambda - 3)(\lambda - 5) = 0.$$

The eigenvalues are $-1, 3$ and 5 . **A1**

(b) For each eigenvalue, take the vector product of the first two rows of $\mathbf{A} - \lambda\mathbf{I}$: **M1**

- $\lambda = -1$: rows $(2, 2, -2)$ and $(-1, 5, 1)$ give $(12, 0, 12)$, so $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. **A1**
- $\lambda = 3$: rows $(-2, 2, -2)$ and $(-1, 1, 1)$ give $(4, 4, 0)$, so $\mathbf{e}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. **A1**
- $\lambda = 5$: rows $(-4, 2, -2)$ and $(-1, -1, 1)$ give $(0, 6, 6)$, so $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. **A1**

Check one: $\mathbf{A}\mathbf{e}_3 = (2 - 2, 4 + 1, 3 + 2) = (0, 5, 5) = 5\mathbf{e}_3$.

$\mathbf{A} - \mathbf{I}$ has the same eigenvectors, with eigenvalues $\lambda - 1$: that is $-2, 2, 4$. So $(\mathbf{A} - \mathbf{I})^3$ has eigenvalues $(-2)^3 = -8, 2^3 = 8$ and $4^3 = 64$. **M1**

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 64 \end{pmatrix}.$$

M1 A1 for the columns of $\mathbf{P}$ matched to the entries of $\mathbf{D}$, with the powers taken.

Example 4

(a) The square matrix $\mathbf{A}$ has an eigenvalue λ with corresponding eigenvector $\mathbf{e}$. Show that $\mathbf{e}$ is an eigenvector of $\mathbf{A}^2 + 2\mathbf{A}$, and state the corresponding eigenvalue. **[2]**

The 3×3 matrix $\mathbf{B}$ has eigenvalues $2, -1$ and 4 , with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

(b) Find $\mathbf{B}$. **[5]**

(c) State the eigenvalues of $\mathbf{B}^2 + 2\mathbf{B}$. **[1]**

(a) Multiply $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$ on the left by $\mathbf{A}$: $\mathbf{A}^2\mathbf{e} = \mathbf{A}(\lambda\mathbf{e}) = \lambda\mathbf{A}\mathbf{e} = \lambda^2\mathbf{e}$. **M1**

So $(\mathbf{A}^2 + 2\mathbf{A})\mathbf{e} = \lambda^2\mathbf{e} + 2\lambda\mathbf{e} = (\lambda^2 + 2\lambda)\mathbf{e}$: $\mathbf{e}$ is an eigenvector of $\mathbf{A}^2 + 2\mathbf{A}$ with eigenvalue $\lambda^2 + 2\lambda$. **A1**

(b) $\mathbf{B} = \mathbf{PDP}^{-1}$ with **B1**

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

$\det \mathbf{P} = 1(0 + 1) - 1(1 - 0) + 1(1 - 0) = 1$, and the cofactors, transposed, give **M1**

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix}.$$

A1. Multiply two at a time: $\mathbf{PD} = \begin{pmatrix} 2 & -1 & 4 \\ 2 & 0 & -4 \\ 0 & -1 & 4 \end{pmatrix}$ (each column of $\mathbf{P}$ times its eigenvalue), **M1** then

$$\mathbf{B} = \mathbf{PDP}^{-1} = \begin{pmatrix} 7 & -5 & -8 \\ -2 & 4 & 2 \\ 5 & -5 & -6 \end{pmatrix}.$$

A1. Check: $\mathbf{B} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$, twice the first eigenvector.

(c) By part (a), the eigenvalues are $\lambda^2 + 2\lambda$: $4 + 4 = 8, 1 - 2 = -1$ and $16 + 8 = 24$. **B1**

Common mistakes

- **Accepting a zero eigenvector.** $\mathbf{0}$ is never an eigenvector. If the vector product gives $\mathbf{0}$, try another pair of rows; if every pair does, go back and check the eigenvalue. Examiners report this every year.
- **Expanding a determinant for a triangular matrix.** The eigenvalues are on the diagonal; state them and save the time.
- **Not checking eigenvectors.** A sign slip in a vector product is common. Multiply $\mathbf{A}\mathbf{e}$ and see that you get $\lambda\mathbf{e}$.
- **Mismatching $\mathbf{P}$ and $\mathbf{D}$.** The eigenvalue in column i of $\mathbf{D}$ must belong to the eigenvector in column i of $\mathbf{P}$. A repeated column, or a column of zeros, loses the marks.
- **Plain eigenvalues in $\mathbf{D}$ for a power.** For $\mathbf{A}^5$ or $(\mathbf{A} - 3\mathbf{I})^2$, $\mathbf{D}$ must hold λ^5 or $(\lambda - 3)^2$.
- **Forgetting $\mathbf{I}$ in Cayley–Hamilton.** $\mathbf{A}^2 - 7\mathbf{A} + 10 = \mathbf{0}$ is wrong: a matrix can't be added to a number. Write $10\mathbf{I}$.
- **Doing unnecessary work.** "Find p, q and r " in $\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{A} + r\mathbf{I}$ needs no matrix arithmetic. Substituting the actual matrices wastes time and invites slips.
- **Sign errors when replacing powers.** When substituting for $\mathbf{A}^3$, put the whole expression in brackets before multiplying out.
- **Wrong "proof" of eigenvalue facts.** Multiply on the left by $\mathbf{A}$ or $\mathbf{A}^{-1}$. "Dividing by $\mathbf{e}$ " or dropping $\mathbf{e}$ from the working gets nothing.
- **Using only two equations for consistency.** Showing that two equations fit together says nothing about the third. Use all three, for example by eliminating one unknown from two different pairs.
- **Vague geometry.** "The planes meet at a point" is not enough for a unique solution: say they meet at a **single** point. For two parallel planes, say they are **not identical**, and say that the third plane is not parallel to them.
- **Mixing up sheaf and prism.** Sheaf: consistent, a common line. Prism: inconsistent, no common point. Decide consistency first.
- **Writing " $k > \dots$ " for a unique solution.** It is $\det \neq 0$, so the answer is " $k \neq \dots$ ".

How to get the marks

- **"Show that the characteristic equation is ...":** the equation is given, so the marks are for $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ written out and expanded, with the 2×2 determinants visible and the " $= 0$ " present. Then **find the roots** if asked: stopping at the equation loses the eigenvalue mark.
- **"State"** (the eigenvalues of a triangular matrix, or of $\mathbf{A}^2 + 2\mathbf{A}$) needs no working.
- **Eigenvectors:** one method mark for the method (vector product or equations), then an accuracy mark per eigenvector. Show the two rows you crossed.
- **$\mathbf{P}$ and $\mathbf{D}$:** write both matrices clearly at the end. Any order of columns earns full marks if $\mathbf{P}$ and $\mathbf{D}$ match.
- **Cayley–Hamilton:** say " $\mathbf{A}$ satisfies its characteristic equation" and write the matrix equation; that line earns a mark. Then show the multiplication by $\mathbf{A}$ (or $\mathbf{A}^{-1}$) and the substitution as separate steps.

- **"Use the characteristic equation"** means Cayley–Hamilton. Finding the inverse by cofactors instead earns nothing, even if it is right.
- **Systems:** show the determinant working, then the values. For consistency, show the elimination using **all three** equations and end with the contradiction (such as $0 = 4$) or with two equations left. Then give the geometry in words: "three planes meet at a single point", "the planes form a sheaf (a common line)", "a triangular prism", or "two parallel, non-identical planes; the third is not parallel to them". A clear sketch is also accepted.
- **"Hence"** means use the earlier part: the eigenvalues you found, or the inverse you just worked out.
- **Exact values:** keep fractions in matrices, such as $\frac{1}{12}$ in front.

Quick check

1. Find the eigenvalues of $\begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix}$ and an eigenvector for each.
2. Find the value of k for which the system $x + y + z = 1$, $2x - y + kz = 0$, $x + 4y + 3z = 5$ does not have a unique solution. For this k , show that the system is inconsistent and interpret it geometrically.
3. The 3×3 matrix $\mathbf{M}$ has characteristic equation $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$. Express $\mathbf{M}^{-1}$ in terms of $\mathbf{M}^2$, $\mathbf{M}$ and $\mathbf{I}$.
4. For $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 0 & -1 \end{pmatrix}$, find $\mathbf{P}$ and a diagonal $\mathbf{D}$ with $\mathbf{A}^5 = \mathbf{PDP}^{-1}$.
5. The vector $\mathbf{e}$ is an eigenvector of $\mathbf{A}$ with eigenvalue 3. Show that $\mathbf{e}$ is an eigenvector of $(\mathbf{A} - \mathbf{I})^2$ and find the eigenvalue.

Answers

1. $\lambda^2 - 7\lambda + (10 + 2) = \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4) = 0$, so $\lambda = 3$ or 4 . For 3 : $2x - 2y = 0$, eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For 4 : $x - 2y = 0$, eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
2. The determinant is $1(-3 - 4k) - 1(6 - k) + 1(8 + 1) = -3k$, so $k = 0$. Then $3 \times$ (equation 1) – (equation 2) gives $x + 4y + 3z = 3$, but equation 3 says $x + 4y + 3z = 5$: a contradiction, so the system is inconsistent. No two normals $(1, 1, 1)$, $(2, -1, 0)$, $(1, 4, 3)$ are parallel, so the planes form a triangular prism.
3. $\mathbf{M}^3 - 2\mathbf{M}^2 - 5\mathbf{M} + 6\mathbf{I} = \mathbf{0}$. Multiply by $\mathbf{M}^{-1}$: $\mathbf{M}^2 - 2\mathbf{M} - 5\mathbf{I} + 6\mathbf{M}^{-1} = \mathbf{0}$, so $\mathbf{M}^{-1} = \frac{1}{6}(-\mathbf{M}^2 + 2\mathbf{M} + 5\mathbf{I})$.
4. Triangular, so the eigenvalues are 2 and -1 . For 2 : $\begin{pmatrix} 0 & 4 \\ 0 & -3 \end{pmatrix}$ gives $y = 0$, eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. For -1 : $3x + 4y = 0$, eigenvector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$. So $\mathbf{P} = \begin{pmatrix} 1 & 4 \\ 0 & -3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 32 & 0 \\ 0 & -1 \end{pmatrix}$, since $2^5 = 32$ and $(-1)^5 = -1$.
5. $(\mathbf{A} - \mathbf{I})\mathbf{e} = \mathbf{A}\mathbf{e} - \mathbf{e} = 3\mathbf{e} - \mathbf{e} = 2\mathbf{e}$. Then $(\mathbf{A} - \mathbf{I})^2\mathbf{e} = (\mathbf{A} - \mathbf{I})(2\mathbf{e}) = 2(\mathbf{A} - \mathbf{I})\mathbf{e} = 4\mathbf{e}$, so $\mathbf{e}$ is an eigenvector with eigenvalue 4 .

Past questions to try

- **9231/24/M/J/25 Q7**: diagonalising a triangular matrix for a power, then Cayley–Hamilton for a higher power.
- **9231/22/O/N/24 Q8**: showing a characteristic equation, a power by Cayley–Hamilton, and $\mathbf{P}$ and $\mathbf{D}$ for a polynomial in $\mathbf{A}$.
- **9231/22/O/N/22 Q7**: the eigenvalues of $\mathbf{A}^{-1}$ by proof, eigenvectors from given information, and Cayley–Hamilton.
- **9231/23/O/N/22 Q2**: a system with no unique solution that is a sheaf for one value and a prism for others.