

Momentum

A2

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Read first: [Motion of a projectile](#)

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What you need to know

By the end of this topic you should be able to:

1. **State Newton's experimental law** (the law of restitution) and what the **coefficient of restitution** e means: the speed at which two objects separate after an impact is e times the speed at which they approached, measured along the line in which they push on each other.
2. **Know that** $0 \leq e \leq 1$, that $e = 1$ is called **perfectly elastic** (no kinetic energy is lost) and that $e = 0$ is called **inelastic** (the objects do not separate: they move on together, or a ball does not bounce).
3. **Solve problems about direct and oblique impacts**, using conservation of linear momentum and/or Newton's experimental law:
 - two smooth spheres colliding **directly** (moving along the line joining their centres) or **obliquely** (at an angle to it);
 - a smooth sphere or particle hitting a **fixed surface**, such as a wall, a barrier, the ground or an inclined plane, head-on or at an angle.

Questions often add kinetic energy, angles of deflection, or a projectile that bounces. Impulse and collisions of more than two spheres at once are not in this topic.

Understand it

Momentum and why it is conserved

The **momentum** of an object is its mass times its velocity, mv . It is a vector, so direction matters: pick a positive direction and give anything moving the other way a negative velocity.

When two spheres collide, each pushes on the other with an equal and opposite force for the same short time. So whatever momentum one gains, the other loses, and the **total momentum is the same before and after**:

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B.$$

This is **conservation of linear momentum**. It holds whatever the spheres are made of, but it only tells you one equation. With two unknown velocities after the impact you need a second fact, and that is Newton's experimental law.

Newton's experimental law

Drop a tennis ball and a lump of clay: the ball bounces and the clay doesn't. How "bouncy" a collision is depends on the materials, and experiments show a simple rule:

$$\text{speed of separation} = e \times \text{speed of approach},$$

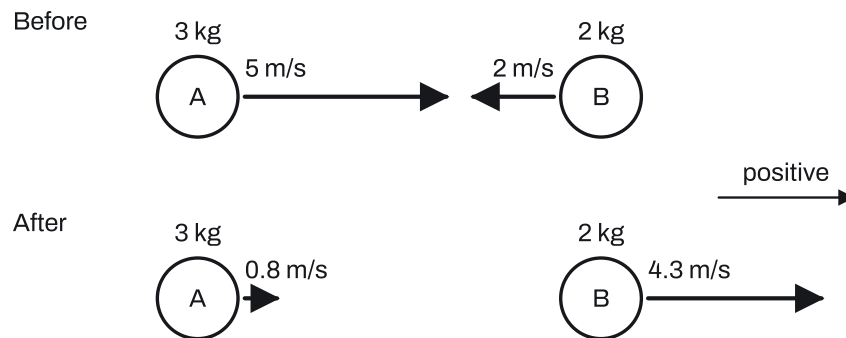
where e , the **coefficient of restitution**, is a constant for the two materials, with $0 \leq e \leq 1$.

With velocities measured in the same positive direction, **A behind B**:

$$v_B - v_A = e(u_A - u_B).$$

The left side is how fast B pulls away from A after the impact; the right side is how fast A was closing on B before it.

- $e = 1$: **perfectly elastic**. They separate as fast as they approached, and no kinetic energy is lost.
- $e = 0$: **inelastic**. They don't separate at all, so they move on with the same velocity (they coalesce or stay together). The most kinetic energy is lost.
- Anything in between: some kinetic energy turns into heat and sound. Momentum is still conserved.



Example. A of mass 3 kg moves at 5 m/s towards B of mass 2 kg moving at 2 m/s the other way; $e = 0.5$. Take right as positive, so B's velocity is -2 :

- momentum: $3(5) + 2(-2) = 3v_A + 2v_B$, so $3v_A + 2v_B = 11$;
- restitution: $v_B - v_A = 0.5(5 - (-2)) = 3.5$.

Substituting $v_B = v_A + 3.5$ gives $5v_A + 7 = 11$, so $v_A = 0.8$ m/s and $v_B = 4.3$ m/s, both to the right. The kinetic energy falls from 41.5 J to 19.45 J, a loss of 22.05 J. If the spheres had coalesced ($e = 0$) they would have moved on together at $\frac{11}{5} = 2.2$ m/s.

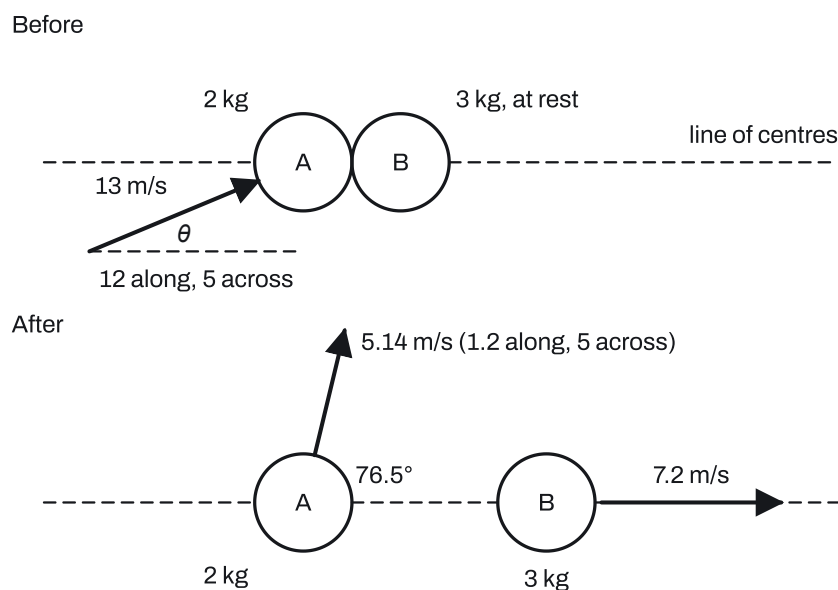
A positive answer means the sphere moves in your positive direction; a negative one means it moves the other way. You don't need to guess directions in advance, as long as **both equations use the same positive direction**.

Oblique impact of two smooth spheres

When the spheres meet at an angle, the push between them acts along the **line of centres**, the line through both centres at the moment of impact. Because the spheres are **smooth**, there is no sideways (frictional) force. That splits the problem in two:

- **Perpendicular to the line of centres:** no force acts, so **each sphere keeps its own velocity component unchanged**.
- **Along the line of centres:** use conservation of momentum and Newton's experimental law exactly as for a direct impact, with the components along that line.

So: resolve each velocity along and perpendicular to the line of centres, do a direct-impact calculation along the line, carry the perpendicular parts across unchanged, then recombine.



Example. A (2 kg) moves at 13 m/s at angle θ to the line of centres, where $\tan \theta = \frac{5}{12}$, and hits B (3 kg) at rest; $e = \frac{1}{2}$. A's components are $13 \cos \theta = 12$ along the line and $13 \sin \theta = 5$ across it.

- Along the line: $2(12) = 2v_A + 3v_B$ and $v_B - v_A = \frac{1}{2}(12 - 0) = 6$, so $5v_A + 18 = 24$, giving $v_A = 1.2$ and $v_B = 7.2$.
- Across the line: A keeps 5 m/s; B had 0 and still has 0.

After the impact A moves at $\sqrt{1.2^2 + 5^2} = \sqrt{26.44} \approx 5.14$ m/s at $\tan^{-1} \frac{5}{1.2} \approx 76.5^\circ$ to the line of centres, and B moves **along the line of centres** at 7.2 m/s. A sphere that was at rest always moves off along the line of centres, because that is the only direction it is pushed.

Kinetic energy

Kinetic energy is $\frac{1}{2}mv^2$, where v^2 is the **square of the full speed**, so for an oblique impact it includes both components: $\frac{1}{2}m(v_{\text{along}}^2 + v_{\text{across}}^2)$. Since the "across" parts don't change, the loss of kinetic energy comes only from the "along" parts. In the example the loss is

$$\frac{1}{2}(2)(12^2) - \left[\frac{1}{2}(2)(1.2^2) + \frac{1}{2}(3)(7.2^2) \right] = 144 - 79.2 = 64.8 \text{ J.}$$

Kinetic energy is **not** conserved unless $e = 1$. Never write a kinetic energy equation to replace the momentum equation.

Impact with a fixed surface

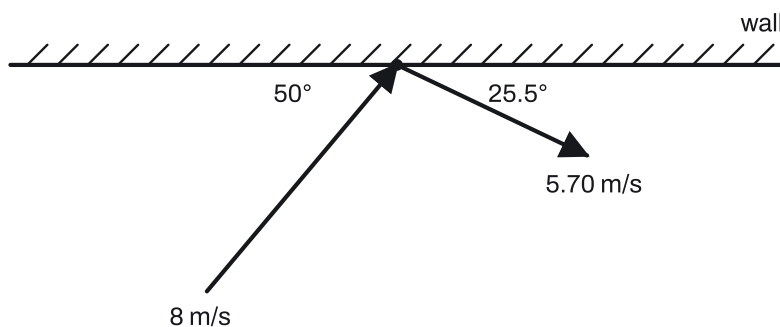
A wall, a barrier or the ground doesn't move, so it has no momentum equation of its own. For a smooth surface:

- **Parallel to the surface:** no force, so the component is unchanged.
- **Perpendicular to the surface:** the component reverses and is multiplied by e .

If a particle hits at speed u at angle α **to the wall** and leaves at speed v at angle β to the wall:

$$v \cos \beta = u \cos \alpha, \quad v \sin \beta = e u \sin \alpha, \quad \text{so } \tan \beta = e \tan \alpha.$$

Since $e \leq 1$, $\beta \leq \alpha$: the particle leaves closer to the wall than it arrived.



Along the wall: $8 \cos 50^\circ \approx 5.14 \rightarrow 5.14$ (unchanged)

Into the wall: $8 \sin 50^\circ \approx 6.13 \rightarrow 0.4 \times 6.13 \approx 2.45$ (reversed)

Example. A particle hits a wall at 8 m/s at 50° to it; $e = 0.4$. Parallel: $8 \cos 50^\circ \approx 5.142$. Perpendicular: $0.4 \times 8 \sin 50^\circ \approx 2.451$. It leaves at $\sqrt{5.142^2 + 2.451^2} \approx 5.70$ m/s at $\tan^{-1}(0.4 \tan 50^\circ) \approx 25.5^\circ$ to the wall. The fraction of kinetic energy it keeps is $\cos^2 50^\circ + 0.4^2 \sin^2 50^\circ \approx 0.507$.

A head-on impact ($\alpha = 90^\circ$) is the simplest case: a ball that hits the floor at speed w bounces up at ew .

Key facts and formulas

None of these are in the MF19 list of formulae, so you must learn them all. (The constant-acceleration formulas you may need for a projectile that bounces are given.)

Formula	Status
Momentum = mv (a vector: signs show direction)	Learn it
Conservation of momentum: $m_A u_A + m_B u_B = m_A v_A + m_B v_B$	Learn it

Formula	Status
Newton's experimental law: $v_B - v_A = e(u_A - u_B)$ (separation = $e \times$ approach)	Learn it
$0 \leq e \leq 1$; $e = 1$ perfectly elastic, $e = 0$ inelastic	Learn it
Smooth spheres: velocity components perpendicular to the line of centres are unchanged	Learn it
Smooth fixed surface: parallel component unchanged, perpendicular component $\times e$ and reversed	Learn it
Wall, angles measured from the wall: $\tan \beta = e \tan \alpha$	Learn it
Kinetic energy = $\frac{1}{2}mv^2$, with v^2 = sum of the squares of the components	Learn it
$v^2 = u^2 + 2as$, $s = ut + \frac{1}{2}at^2$, $v = u + at$	Given

How to do it

In the Paper 3 questions we have tagged for this topic (36 questions, 2021–2025; 25 different questions, as some appear on two papers), the types came up this often. 2 of the 36 questions mix two types, so the counts add up to 38.

Question type	Questions
Oblique impact of two smooth spheres	20
Particle or sphere hitting a fixed wall or barrier	8
Direct impact of two spheres	5
A projectile that bounces off a surface	5

1. Oblique impact of two smooth spheres (20 questions)

- Draw the line of centres** and mark each sphere's velocity before, split into a component **along** it and one **across** it (usually $u \cos \theta$ and $u \sin \theta$). Choose a positive direction along the line (say from A towards B) and give a component the other way a minus sign.
- Label the unknowns after:** v_A , v_B along the line. The across components stay as they were.
- Momentum along the line of centres**, with the masses: $m_A(\dots) + m_B(\dots) = m_A v_A + m_B v_B$.
- Newton's law along the line:** $v_B - v_A = e \times$ (approach speed). The approach speed is the difference of the "along" components before, so if they move towards each other you **add** their sizes.
- Solve** the two equations for v_A and v_B , then **use the extra condition** the question gives (below).

If a result is given ("show that the speed of B is ..."), solve for it by adding or subtracting the equations and show every line.

Variations you will meet:

- "After the collision A moves perpendicular to the line of centres."** Its along-the-line component is 0: put $v_A = 0$. (It then moves with its unchanged across component only.) Say this in words when you use it.

- **"B moves perpendicular to its original direction", or "A is deflected through a right angle":** the velocity after is at 90° to the velocity before. Write both as (along, across) pairs; perpendicular means (along before)(along after) + (across before)(across after) = 0. Since the across component p is unchanged, this is $av + p^2 = 0$, where a and v are the along components before and after, so $v = -\frac{p^2}{a}$: the along component must reverse. A sketch of the two velocity triangles gives the same result.
- **"A and B move in the same direction (or parallel directions) after."** Their velocity vectors are parallel, so the ratio across : along is the same for both: $\frac{p_A}{v_A} = \frac{p_B}{v_B}$, where p is the across component.
- **"A moves at angle ϕ to the line of centres after."** Then $\tan \phi = \frac{\text{across}}{v_A}$. If you know A's speed after instead, use the across component: $v \sin \phi = u \sin \theta$.
- **Kinetic energy conditions** ("the kinetic energies after are equal", "a fraction is lost"): write each kinetic energy with **both** components for a sphere that has an across component, and the correct mass for each sphere. With $\tan \theta$ given, replace $\cos^2 \theta$ and $\sin^2 \theta$ by numbers from a right-angled triangle, then solve (often a quadratic, where you reject the root that makes e , k or a speed impossible).
- **Loss of kinetic energy.** Total before minus total after. Because the across components don't change, you may use only the along components, but **say so**.
- **Angle of deflection** ("the angle through which A's direction is turned"). Find the angle the velocity makes with the line of centres before and after, then work out the angle between the two directions on a sketch; it is not always a simple difference. In the example in "Understand it", A moves at 22.6° to the line before and 76.5° after, on the same side, so it is deflected through 53.9° .
- **Unknown mass km or unknown e , and "the range of values of k ".** Find e in terms of k , then use $0 \leq e \leq 1$. For example $e = \frac{4-k}{2k}$ gives $0 \leq 4 - k$ and $4 - k \leq 2k$, so $\frac{4}{3} \leq k \leq 4$.
- **Both spheres moving.** Resolve both velocities; the one moving towards the other along the line gets the minus sign in the momentum equation and adds to the approach speed. If the two angles are linked (for example $\alpha + \beta = 90^\circ$), replace $\cos \beta$ with $\sin \alpha$ early.
- **Walls after the collision** ("each sphere then takes the same time to reach a wall"): time = distance to the wall $\div$ the velocity component **towards that wall**.

2. Particle or sphere hitting a fixed wall or barrier (8 questions)

1. Draw the wall, the path in and the path out, and **mark the angles to the wall** (check whether the question measures them from the wall or from the normal).
2. **Parallel to the wall:** $v \cos \beta = u \cos \alpha$.
3. **Perpendicular:** $v \sin \beta = e u \sin \alpha$.
4. **Divide** to get $\tan \beta = e \tan \alpha$, or **square and add** to get the speed: $v^2 = u^2(\cos^2 \alpha + e^2 \sin^2 \alpha)$.

Variations you will meet:

- **Kinetic energy lost or kept.** The fraction kept is $\frac{v^2}{u^2} = \cos^2 \alpha + e^2 \sin^2 \alpha$. "Loses 15%" means it **keeps** 85%. For example, at 45° with $e = \frac{1}{2}$ it keeps $\frac{1}{2} + \frac{1}{8} = \frac{5}{8}$ of its kinetic energy.
- **Angle of deflection.** The particle turns through $180^\circ - \alpha - \beta$ (sketch it to check).

- **Two walls at right angles.** The first impact scales one component by e , the second scales the other, so the particle ends up moving **parallel to its original path, in the opposite direction**, at speed eu , keeping e^2 of its kinetic energy (with $e = 0.8$, that is 64%). Show this with the components; don't just state it.
- **Two walls at another angle.** After the first impact, use the triangle formed by the two walls and the path: the angles in the triangle add to 180° , so the angle at which it meets the second wall is $180^\circ - (\text{angle between the walls}) - \beta$. For walls at 100° with $\beta = 35^\circ$, it meets the second wall at 45° . Then apply the same two equations again with the new speed.
- **Two parallel walls a distance d apart.** Only the component across the gap carries it from one wall to the other: time to cross $= \frac{d}{\text{perpendicular component}}$, and the distance travelled along the path $= \text{speed} \times \text{time}$. The component parallel to the walls never changes.
- **Speed after known, angles unknown.** Write $\cos \beta$ and $\sin \beta$ from the two equations, square them and use $\sin^2 \beta + \cos^2 \beta = 1$ to get an equation in α alone; then find β and any combination the question asks for with the compound-angle formulas.
- **A sphere from an earlier collision** goes on to hit a barrier: use its velocity after the first collision as u here.

3. Direct impact of two spheres (5 questions)

1. Draw both spheres before and after with arrows, all pointing in the **positive** direction for the unknowns.
2. **Momentum:** $m_A u_A + m_B u_B = m_A v_A + m_B v_B$, with a minus sign on any velocity the other way.
3. **Restitution:** $v_B - v_A = e(u_A - u_B)$, where A is the one behind.
4. Solve, and interpret signs: negative means that sphere moves the other way.

Variations you will meet:

- **A condition after the impact** (a ratio of momenta, of speeds or of kinetic energies): write it in terms of v_A and v_B and solve with the momentum equation, then find e from the restitution equation. Or solve in terms of e first and substitute into the condition. Take square roots carefully in a kinetic energy ratio.
- **Coalescing or "stays together":** $e = 0$, one common velocity V from momentum alone: $m_A u_A + m_B u_B = (m_A + m_B)V$. The combined object then carries on, for example compressing a spring (use energy with the **total** mass).
- **A later impact.** A sphere goes on to hit a wall (it rebounds at e' times its speed) or the other sphere again. **There is a further collision** if, after an impact, the sphere behind is moving faster in the same direction as the one in front, or they are moving towards each other. Show the velocities and say which case applies.
- **A range of e** for something to happen, such as "A's direction is reversed": write the condition as an inequality in v_A (for example $v_A < 0$), then combine with $0 \leq e \leq 1$.

4. A projectile that bounces off a surface (5 questions)

The impact is instant, so handle the flight before and after with projectile methods (see Motion of a projectile) and apply the surface rule at the moment of impact.

1. Find the velocity components just before impact from the projectile equations ($v_y = u \sin \theta - gt$, $v_x = u \cos \theta$).

2. **Horizontal ground:** the horizontal component is unchanged; the vertical component reverses and is multiplied by e . For a ball dropped from 3.2 m ($g = 10$): it lands at $\sqrt{2 \times 10 \times 3.2} = 8$ m/s; with $e = 0.75$ it rebounds at 6 m/s and rises $\frac{6^2}{2 \times 10} = 1.8$ m.
3. **Vertical barrier:** the vertical component is unchanged, so the **time of flight is unchanged**; the horizontal component reverses and is multiplied by e . The time to the barrier is $\frac{\text{distance}}{u \cos \theta}$ and the time back is $\frac{\text{distance}}{e u \cos \theta}$; their sum is the whole time of flight.
4. **Inclined plane:** resolve the velocity **along and perpendicular to the plane** (not horizontally and vertically). Along the plane it is unchanged; perpendicular to it, it reverses and is multiplied by e . For example, a particle falling vertically at 5 m/s onto a plane inclined at 30° has $5 \sin 30^\circ = 2.5$ m/s along the plane and $5 \cos 30^\circ \approx 4.33$ m/s into it; with $e = 0.5$ it leaves with 2.5 along and about 2.17 away from the plane. Then convert back to horizontal and vertical if you need heights or distances.
5. The speed after impact comes from both components, $\sqrt{v_x^2 + v_y^2}$.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
-

Example 1

Two uniform smooth spheres A and B of equal radii have masses $3m$ and $2m$ and move in the same straight line on a smooth horizontal surface, in the same direction. A has speed $4u$ and B, in front of it, has speed u . A catches up with B and the coefficient of restitution between them is $\frac{2}{3}$.

(a) Find the speeds of A and B after the collision and the loss of kinetic energy. **[5]**

B then hits a fixed smooth wall perpendicular to its path. The coefficient of restitution between B and the wall is $\frac{1}{4}$.

(b) Show that A and B collide again, and find their velocities after this second collision. **[5]**

(a) Take the direction of motion as positive.

Momentum: $3m(4u) + 2m(u) = 3mv_A + 2mv_B$, so $3v_A + 2v_B = 14u$. **M1**

Restitution: $v_B - v_A = \frac{2}{3}(4u - u) = 2u$. **M1**

Substituting $v_B = v_A + 2u$: $5v_A + 4u = 14u$, so $v_A = 2u$ and $v_B = 4u$. **A1**

$$\text{KE before} = \frac{1}{2}(3m)(16u^2) + \frac{1}{2}(2m)(u^2) = 25mu^2. \text{ KE after} = \frac{1}{2}(3m)(4u^2) + \frac{1}{2}(2m)(16u^2) = 22mu^2. \text{ M1}$$

$$\text{Loss} = 3mu^2. \text{ A1}$$

(b) B rebounds from the wall with speed $\frac{1}{4} \times 4u = u$, now moving back towards A. **B1**

A is still moving towards the wall at $2u$, so the spheres are moving towards each other and must collide again. **B1**

B1

Take the direction towards the wall as positive: A has velocity $2u$, B has $-u$.

$$\text{Momentum: } 3m(2u) + 2m(-u) = 3mv'_A + 2mv'_B, \text{ so } 3v'_A + 2v'_B = 4u. \text{ M1}$$

$$\text{Restitution (A is behind B): } v'_B - v'_A = \frac{2}{3}(2u - (-u)) = 2u. \text{ M1}$$

$$5v'_A + 4u = 4u, \text{ so } v'_A = 0: \text{ A comes to rest, and B moves towards the wall at } 2u. \text{ A1}$$

Example 2

A particle of mass 0.2 kg moves on a smooth horizontal floor with speed 5 m/s and strikes a fixed smooth vertical wall. Just before the impact its path makes an angle α with the wall, where $\tan \alpha = 4$. The coefficient of restitution is $\frac{3}{4}$.

(a) Find the speed of the particle after the impact and the angle its path then makes with the wall. **[4]**

(b) Find the angle through which the particle's direction of motion is turned, and the kinetic energy lost. **[3]**

$$\text{(a) From } \tan \alpha = 4: \sin \alpha = \frac{4}{\sqrt{17}} \text{ and } \cos \alpha = \frac{1}{\sqrt{17}}.$$

$$\text{Parallel to the wall (unchanged): } 5 \cos \alpha = \frac{5}{\sqrt{17}}. \text{ B1}$$

$$\text{Perpendicular (reversed, } \times e): \frac{3}{4} \times 5 \sin \alpha = \frac{3}{4} \times \frac{20}{\sqrt{17}} = \frac{15}{\sqrt{17}}. \text{ B1}$$

$$\text{Speed} = \sqrt{\frac{25}{17} + \frac{225}{17}} = \sqrt{\frac{250}{17}} \approx 3.83 \text{ m/s. M1}$$

$$\tan \beta = \frac{15/\sqrt{17}}{5/\sqrt{17}} = 3, \text{ so } \beta = 71.6^\circ \text{ to the wall. A1 (Or directly: } \tan \beta = e \tan \alpha = \frac{3}{4} \times 4 = 3.)$$

$$\text{(b) } \alpha = \tan^{-1} 4 = 75.96^\circ \text{ and } \beta = 71.57^\circ. \text{ The particle turns through } 180^\circ - 75.96^\circ - 71.57^\circ = 32.5^\circ. \text{ B1}$$

$$\text{KE lost} = \frac{1}{2}(0.2)(5^2) - \frac{1}{2}(0.2)\left(\frac{250}{17}\right) = 2.5 - \frac{25}{17} = \frac{35}{34} \approx 1.03 \text{ J. M1 A1 (That is } \frac{7}{17} \text{ of its kinetic energy.)}$$

Example 3

Two uniform smooth spheres A and B of equal radii have masses $2m$ and $3m$. B is at rest on a smooth horizontal surface. A moves with speed u and hits B; just before the impact A's direction makes an angle θ with the line of centres, where $\cos \theta = \frac{2}{3}$. The coefficient of restitution is e .

$$\text{(a) Show that the speed of B after the collision is } \frac{4(1+e)u}{15}. \text{ [3]}$$

(b) Given that $e = \frac{1}{5}$, find the speed of A after the collision and the angle through which its direction is deflected. **[4]**

(c) Find, for $e = \frac{1}{5}$, the loss of kinetic energy. **[2]**

(a) Along the line of centres, A's component is $u \cos \theta = \frac{2}{3}u$. Let v_A, v_B be the components along the line after, from A towards B.

$$\text{Momentum: } 2m \left(\frac{2}{3}u \right) = 2mv_A + 3mv_B. \text{ M1}$$

$$\text{Restitution: } v_B - v_A = e \times \frac{2}{3}u. \text{ M1}$$

Multiply the second by 2 and add: $5v_B = \frac{4}{3}u + \frac{4}{3}eu$, so $v_B = \frac{4(1+e)u}{15}$. **A1** (B moves along the line of centres, so this is its speed.)

$$\text{(b) With } e = \frac{1}{5}: v_B = \frac{8u}{25} \text{ and } v_A = v_B - \frac{2}{15}u = \frac{8u}{25} - \frac{2u}{15} = \frac{14u}{75}. \text{ B1}$$

$$\text{A's across component is unchanged: } u \sin \theta = \frac{\sqrt{5}}{3}u. \text{ B1}$$

$$\text{Speed of A} = u \sqrt{\left(\frac{14}{75} \right)^2 + \frac{5}{9}} = u \sqrt{\frac{3321}{5625}} \approx 0.768u. \text{ M1}$$

Before, A moved at $\cos^{-1} \frac{2}{3} = 48.19^\circ$ to the line; after, at $\tan^{-1} \left(\frac{\sqrt{5}/3}{14/75} \right) = 75.94^\circ$, on the same side. It is deflected through $75.94^\circ - 48.19^\circ \approx 27.8^\circ$. **A1**

$$\text{(c) Only the components along the line change. KE before (along)} = \frac{1}{2}(2m) \left(\frac{2}{3}u \right)^2 = \frac{4}{9}mu^2. \text{ M1}$$

$$\text{KE after (along)} = \frac{1}{2}(2m) \left(\frac{14u}{75} \right)^2 + \frac{1}{2}(3m) \left(\frac{8u}{25} \right)^2 = \frac{196}{5625}mu^2 + \frac{864}{5625}mu^2 = \frac{1060}{5625}mu^2.$$

$$\text{Loss} = \frac{2500-1060}{5625}mu^2 = \frac{32}{125}mu^2. \text{ A1}$$

Example 4

Two uniform smooth spheres A and B of equal radii have masses $3m$ and $2m$ and move on a smooth horizontal surface. Just before they collide, A has speed $3u$ in a direction making an angle α with the line of centres, where $\cos \alpha = \frac{1}{3}$, and B moves along the line of centres towards A with speed $\frac{1}{3}u$. The coefficient of restitution is e .

(a) Find, in terms of e and u , the components along the line of centres of the velocities of A and B after the collision. **[4]**

(b) After the collision, A moves perpendicular to the line of centres. Find e , and the loss of kinetic energy. **[4]**

(c) Find the range of values of e for which A's component along the line of centres is reversed by the collision. **[2]**

(a) A's components before: along $3u \times \frac{1}{3} = u$, across $3u \times \frac{2\sqrt{2}}{3} = 2\sqrt{2}u$. Take the direction from A to B as positive; B's velocity is $-\frac{1}{3}u$.

$$\text{Momentum: } 3m(u) + 2m \left(-\frac{1}{3}u \right) = 3mv_A + 2mv_B, \text{ so } 3v_A + 2v_B = \frac{7}{3}u. \text{ M1 A1}$$

Restitution: $v_B - v_A = e \left(u + \frac{1}{3}u \right) = \frac{4}{3}eu$. **M1**

Substituting: $5v_A + \frac{8}{3}eu = \frac{7}{3}u$, so

$$v_A = \frac{(7 - 8e)u}{15}, \quad v_B = \frac{(7 + 12e)u}{15}.$$

A1

(b) Perpendicular to the line of centres means $v_A = 0$, so $7 - 8e = 0$ and $e = \frac{7}{8}$. **M1 A1**

Then $v_B = \frac{(7+10.5)u}{15} = \frac{7}{6}u$, away from A. A loses its along component u and keeps $2\sqrt{2}u$ across. Only the along components change:

$$\text{Loss} = \left[\frac{1}{2}(3m)u^2 + \frac{1}{2}(2m)\left(\frac{u}{3}\right)^2 \right] - \frac{1}{2}(2m)\left(\frac{7u}{6}\right)^2 = \frac{29}{18}mu^2 - \frac{49}{36}mu^2 = \frac{1}{4}mu^2. \text{ **M1 A1**}$$

(c) Reversed means $v_A < 0$: $7 - 8e < 0$, so $e > \frac{7}{8}$. With $e \leq 1$: $\frac{7}{8} < e \leq 1$. **M1 A1**

Common mistakes

- **Leaving out the masses** in the momentum equation (writing $v_A + kv_B = u \cos \theta$ for spheres of masses m and km). Examiners say this every year: write every m , and use km , not m , for the second sphere. In a "show that" the missing masses cost the marks even when the answer comes out.
- **Inconsistent signs**. The momentum equation and the restitution equation must use the same positive direction. The most common error on both-spheres-moving questions is a sign that disagrees between the two. Draw arrows for the unknowns, all the same way.
- **Getting the approach speed wrong**. If the spheres move towards each other, the approach speed is the **sum** of the sizes of their along components, not the difference.
- **Forgetting the across component in kinetic energy**. After an oblique impact, a sphere's speed squared is $(\text{along})^2 + (\text{across})^2$. Omitting $u \sin \theta$ is the error examiners mention most often in kinetic energy parts.
- **Using cos where sin belongs**. The component along the line of centres is $u \cos \theta$ only when θ is measured from the line of centres. For walls, check whether the angle is to the wall or to the normal.
- **Misreading percentages**. "Loses 35% of its kinetic energy" means it keeps 65%. Write the equation as "KE after = $0.65 \times$ KE before".
- **Stating a deflection angle without showing where it comes from**. If you find the angle with the line of centres, say how it gives the angle turned through (for example " $180^\circ - \dots$ " from your sketch), even when the numbers happen to agree.
- **Claiming a direction result without evidence**, for example "the final path is parallel to the first" after two walls. Show the components that prove it.
- **Using kinetic energy instead of momentum** to find velocities after an impact. Energy is lost in every impact unless $e = 1$.
- **Keeping a root that can't be**. A quadratic in e or k often has a negative root, or an e above 1: reject it and say why.

How to get the marks

- **Momentum equation (M1)** needs every mass and the right components. **Restitution equation (M1)** needs the e on the approach side and signs that match your momentum equation. Write both in full before solving.
- **Say what you are resolving.** Headings such as "along the line of centres" and "perpendicular to the line of centres (unchanged)" make your method clear and earn the marks for the unchanged components, which are often a separate B1.
- **"Show that"** answers are given, so every algebraic step must be visible: the two equations, how you combined them and the final line exactly as printed. Writing only the answer scores nothing.
- **Answer every part of the demand.** If a part asks for B's speed **and** A's component, give both there, not later.
- **Kinetic energy equations** need all the terms: every sphere, both components, the correct masses. The method mark is for an equation with everything present, so write it before substituting.
- **Exact answers** ($e = \frac{7}{8}$, $\sqrt{\frac{250}{17}}$, $\frac{32}{125}mu^2$) are expected where the data are exact. Use surds or fractions for $\sin$ and $\cos$ from a given $\tan$, not decimals. Angles to 0.1° or 3 significant figures otherwise.
- **Conditions on e** come from $0 \leq e \leq 1$; a range question expects both ends stated (strict or not).
- **Interpret your answers.** A negative velocity means the other direction; say so. "Perpendicular to the line of centres" follows from a zero along-component; state the reason.
- **Method marks follow through**, so if an early value is wrong, keep going with it.

Quick check

1. A ball hits a smooth floor vertically at 6 m/s and rebounds at 4.2 m/s. Find the coefficient of restitution. What would the rebound speed be if the impact were perfectly elastic, and if it were inelastic?
2. Sphere P, of mass 5 kg, moving at 4 m/s, catches up with sphere Q, of mass 3 kg, moving at 2 m/s in the same direction along the same line. The coefficient of restitution is 0.6. Find their velocities after the impact.
3. A particle moving at 13 m/s on a smooth floor strikes a smooth wall. Its path makes an angle α with the wall, where $\tan \alpha = \frac{12}{5}$, and $e = \frac{1}{4}$. Find its speed and the angle its path makes with the wall after the impact, and the fraction of its kinetic energy it keeps.
4. Two identical smooth spheres collide. One is at rest; the other is moving at an angle θ to the line of centres. The collision is perfectly elastic. Show that after the collision the spheres move at right angles to each other.
5. A ball is projected from a point on horizontal ground with speed 20 m/s at 30° above the horizontal. It bounces on the ground, where $e = 0.5$. Taking $g = 10 \text{ m/s}^2$, find the distance between the first and second places where it hits the ground.

Answers

- $e = \frac{4.2}{6} = 0.7$. Perfectly elastic: 6 m/s. Inelastic: 0 (it doesn't bounce).
- Momentum: $5(4) + 3(2) = 5v_P + 3v_Q$, so $5v_P + 3v_Q = 26$. Restitution: $v_Q - v_P = 0.6(4 - 2) = 1.2$. Then $8v_P + 3.6 = 26$, so $v_P = 2.8$ m/s and $v_Q = 4$ m/s, both in the original direction.
- $\sin \alpha = \frac{12}{13}$, $\cos \alpha = \frac{5}{13}$. Parallel: $13 \times \frac{5}{13} = 5$. Perpendicular: $\frac{1}{4} \times 13 \times \frac{12}{13} = 3$. Speed = $\sqrt{5^2 + 3^2} = \sqrt{34} \approx 5.83$ m/s at $\tan^{-1} \frac{3}{5} \approx 31.0^\circ$ to the wall (check: $\tan \beta = \frac{1}{4} \times \frac{12}{5} = \frac{3}{5}$). Fraction kept = $\frac{34}{169}$.
- Let each mass be m and the moving sphere's speed be u . Along the line: $mu \cos \theta = mv_A + mv_B$ and $v_B - v_A = u \cos \theta$. Adding gives $v_B = u \cos \theta$, so $v_A = 0$. The moving sphere keeps only its across component $u \sin \theta$, so it moves perpendicular to the line of centres, while the other moves along the line of centres. So their paths are at right angles.
- Before the bounce: $u_x = 20 \cos 30^\circ = 10\sqrt{3}$, $u_y = 10$; it lands after 2 s with vertical speed 10 downwards. After the bounce: vertical $0.5 \times 10 = 5$ upwards, horizontal $10\sqrt{3}$ unchanged. Time to the next bounce = $\frac{2 \times 5}{10} = 1$ s, so the distance is $10\sqrt{3} \approx 17.3$ m.

Past questions to try

- 9231/33/M/J/23 Q4**: an oblique impact where the spheres then move in the same direction, and the kinetic energy lost.
- 9231/32/O/N/24 Q3**: both spheres moving, finding an angle, the kinetic energy lost and the angle of deflection.
- 9231/33/M/J/22 Q6**: a particle hitting two barriers in turn, with a kinetic energy condition.
- 9231/33/O/N/22 Q7**: a projectile that rebounds from a vertical barrier.