

Motion of a projectile

A2

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What you need to know

By the end of this topic you should be able to:

1. **Model a projectile as a particle** moving with constant acceleration g vertically downwards, and say what the model leaves out: air resistance, the size and shape of the object, and any spin. Because of these, real objects fall a little short of the model's range and height. Vector methods are not needed.
2. **Use the horizontal and vertical equations of motion** separately to solve problems: find the speed and direction of the velocity at a given time or position, the range on horizontal ground, and the greatest height.
3. **Derive and use the Cartesian equation of the trajectory** $y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$, including problems where the speed of projection, the angle of projection, or both, are unknown.

The "bounding parabola" (the boundary of all the points a projectile can reach) is **not** in the syllabus.

Understand it

The model

A ball in flight is pulled down by gravity and slowed a little by the air. The model keeps only gravity: the ball is a **particle** (a point, so no size and no spin) moving **freely under gravity**, so its only acceleration is g downwards, and there is **no horizontal acceleration at all**. In 9231, take $g = 10 \text{ m s}^{-2}$.

Limitations to quote when asked: air resistance is ignored (so the model over-estimates range and height), the object is treated as a point (so spin and shape have no effect), and g is taken as constant.

Split the motion into two directions

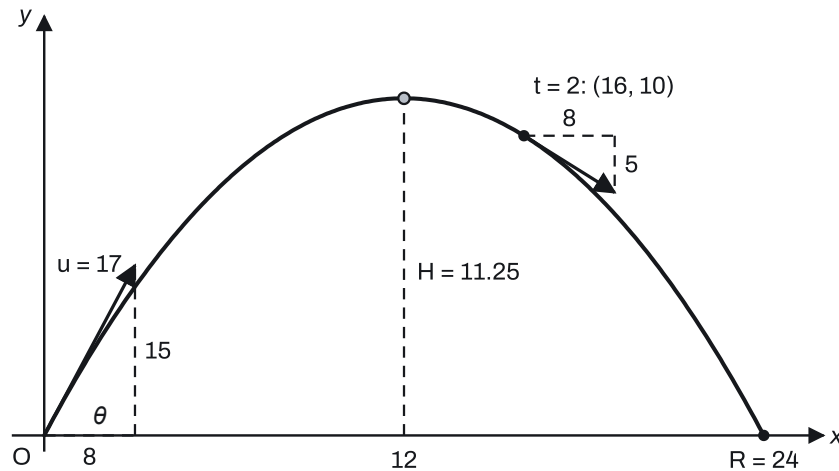
The key idea: the horizontal and vertical motions happen **at the same time but independently**. Time t is the only thing they share.

Project a particle from O with speed u at angle θ above the horizontal. Split the velocity into components:

- **horizontal:** $u \cos \theta$, which never changes (no horizontal force);
- **vertical:** $u \sin \theta$ upwards, which changes at g downwards, like a ball thrown straight up.

Taking x horizontally and y vertically upwards from O , the constant-acceleration equations give, at time t ,

$$\begin{aligned}x &= (u \cos \theta) t, & y &= (u \sin \theta) t - \frac{1}{2}gt^2, \\v_x &= u \cos \theta, & v_y &= u \sin \theta - gt.\end{aligned}$$



For the diagram, $u = 17 \text{ m s}^{-1}$ and $\tan \theta = \frac{15}{8}$, so $\sin \theta = \frac{15}{17}$, $\cos \theta = \frac{8}{17}$: the components are 8 m s^{-1} across and 15 m s^{-1} up.

Speed and direction at any moment

The velocity at time t has components v_x and v_y , so

$$\text{speed} = \sqrt{v_x^2 + v_y^2}, \quad \tan \alpha = \frac{v_y}{v_x},$$

where α is the angle the direction of motion makes with the horizontal (above it if $v_y > 0$, below if $v_y < 0$). The direction of motion is always along the tangent to the path.

In the diagram, at $t = 2$: $v_x = 8$ and $v_y = 15 - 20 = -5$. The speed is $\sqrt{64 + 25} = \sqrt{89} = 9.43 \text{ m s}^{-1}$, moving at $\tan^{-1} \frac{5}{8} = 32.0^\circ$ **below** the horizontal. The position is $x = 16$, $y = 30 - 20 = 10$.

Greatest height, time of flight and range

At the **highest point** the particle is moving horizontally, so $v_y = 0$:

$$t = \frac{u \sin \theta}{g}, \quad H = \frac{u^2 \sin^2 \theta}{2g} \quad (\text{from } v_y^2 = (u \sin \theta)^2 - 2gy).$$

On **horizontal ground** it lands when $y = 0$ again: $t(u \sin \theta - \frac{1}{2}gt) = 0$, so the **time of flight** is $\frac{2u \sin \theta}{g}$, twice the time to the top, because the path is symmetrical. The **range** is the horizontal distance covered in that time:

$$R = u \cos \theta \times \frac{2u \sin \theta}{g} = \frac{u^2 \sin 2\theta}{g}.$$

For the diagram: time to the top $\frac{15}{10} = 1.5$ s, $H = \frac{15^2}{20} = 11.25$ m, time of flight 3 s, $R = 8 \times 3 = 24$ m.

$\sin 2\theta$ is largest at $\theta = 45^\circ$, so that angle gives the longest range; angles equally far either side of 45° (such as 20° and 70°) give the same range.

The symmetry is useful: the particle is at any height below H at **two** times, equally spaced either side of the time to the top, and it has the same speed at both.

The equation of the trajectory

Remove t from the two displacement equations. From the horizontal one, $t = \frac{x}{u \cos \theta}$; put this into the vertical one:

$$y = u \sin \theta \cdot \frac{x}{u \cos \theta} - \frac{1}{2}g \left(\frac{x}{u \cos \theta} \right)^2 = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}.$$

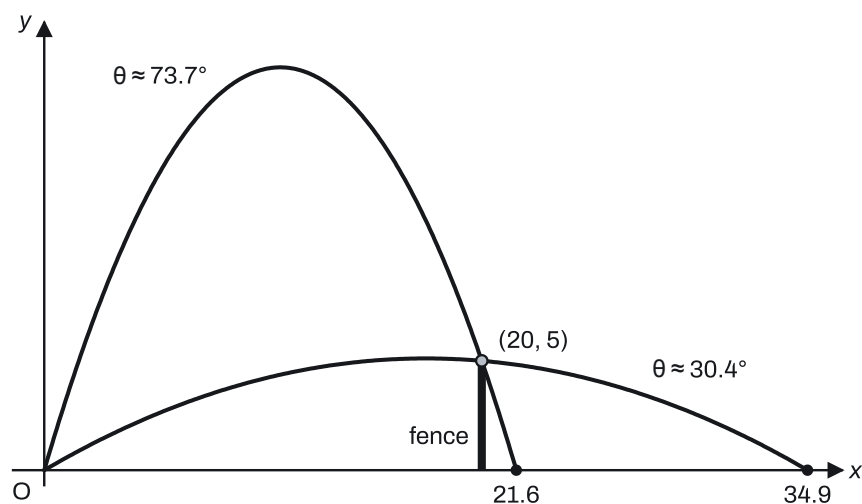
This is the **Cartesian equation of the trajectory**: a parabola, and it is in MF19 (written there with V for the speed). Using $\frac{1}{\cos^2 \theta} = \sec^2 \theta = 1 + \tan^2 \theta$ gives the form you need when the angle is unknown:

$$y = x \tan \theta - \frac{gx^2}{2u^2} (1 + \tan^2 \theta).$$

For the diagram: $y = \frac{15}{8}x - \frac{10x^2}{2 \times 64} = \frac{15}{8}x - \frac{5}{64}x^2$. Check: $x = 16$ gives $30 - 20 = 10$, the same point as before.

Two uses:

- **A known point on the path** gives an equation linking u and θ . With u known, the second form is a **quadratic in $\tan \theta$** , so there are usually **two** angles that send the particle through the point: a steep one and a flat one.
- **Differentiating** gives the direction: $\frac{dy}{dx} = \tan \theta - \frac{gx}{u^2 \cos^2 \theta}$ is the gradient of the path, which equals $\frac{v_y}{v_x}$.
For the diagram, at $x = 16$: $\frac{15}{8} - \frac{10}{64} \times 16 = -\frac{5}{8}$, matching $-\frac{5}{8}$.



The second diagram shows the two angles for one point, the top of a fence (Example 2 works it out). Angles between the two send the particle over the fence; angles outside them hit it.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember or derive it. Take $g = 10 \text{ m s}^{-2}$ unless told otherwise.

Formula	Status
$v = u + at, s = ut + \frac{1}{2}at^2, v^2 = u^2 + 2as, s = \frac{1}{2}(u + v)t$	Given
Components at time t : $v_x = u \cos \theta, v_y = u \sin \theta - gt$	Learn it
Displacements: $x = (u \cos \theta)t, y = (u \sin \theta)t - \frac{1}{2}gt^2$	Learn it
Trajectory: $y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$	Given
$1 + \tan^2 \theta \equiv \sec^2 \theta$	Given
Trajectory in $\tan \theta$: $y = x \tan \theta - \frac{gx^2}{2u^2}(1 + \tan^2 \theta)$	Learn it
Speed = $\sqrt{v_x^2 + v_y^2}$; direction $\tan \alpha = \frac{v_y}{v_x}$	Learn it
Gradient of the path $\frac{dy}{dx} = \frac{v_y}{v_x}$	Learn it
Time to greatest height $\frac{u \sin \theta}{g}$; greatest height $H = \frac{u^2 \sin^2 \theta}{2g}$	Learn it
Time of flight on level ground $\frac{2u \sin \theta}{g}$; range $R = \frac{u^2 \sin 2\theta}{g}$	Learn it
Two directions are perpendicular when $v_{x1}v_{x2} + v_{y1}v_{y2} = 0$ (gradients multiply to -1)	Learn it

Derive H , R and the time of flight in a line or two rather than just quoting them: examiners have seen memorised formulas mixed up, such as the time to the top used as the time of flight.

How to do it

In the Paper 3 questions we have tagged for this topic (41 questions, 2021–2025; 13 of them repeat a question from another paper of the same session, so there are 28 different questions), the types came up this often. 23 of the 41 questions mix two types, so the counts add up to more than 41. Some start with a particle moving in a circle or end with an impact; those parts belong to Circular motion and Momentum, and only the projectile part is counted here.

Question type	Questions
Velocity at a time or point: speed and direction	16
Using the trajectory: unknown speed or angle through given points	12

Question type	Questions
Greatest height, time of flight and range	12
Two projectiles that meet or land together	7
Projectile after leaving a circular path	7
A new flight after an impact	7
Deriving the equation of the trajectory	3

1. Velocity at a time or point: speed and direction (16 questions)

- Write both components** at time t : $v_x = u \cos \theta$ and $v_y = u \sin \theta - gt$. Draw a small sketch with the directions marked.
- Speed** $= \sqrt{v_x^2 + v_y^2}$. **Direction**: $\tan \alpha = \frac{|v_y|}{v_x}$, then say "above" or "below the horizontal" from the sign of v_y .
- Given a position instead of a time**, first find t from $x = (u \cos \theta)t$, or use $\frac{dy}{dx}$ from the trajectory equation.

Variations you will meet:

- "The speed after t seconds is k ."** Square and add, set equal to k^2 : $(u \cos \theta)^2 + (u \sin \theta - gt)^2 = k^2$ simplifies to $u^2 - 2gtu \sin \theta + g^2 t^2 = k^2$, an equation in u and $\sin \theta$.
- "Moving at angle α to the horizontal"** (for example "at 50° below"): write $\frac{v_y}{v_x} = -\tan 50^\circ$, with the **minus** sign because it is moving downwards. This gives t , or an unknown speed.
- Direction at a given angle with the vertical.** "Makes 40° with the downward vertical" is the same as 50° below the horizontal; draw it before writing the ratio.
- Two directions perpendicular** (at two times, or a later direction perpendicular to the direction of projection): the scalar product of the two velocities is zero,

$$v_{x1}v_{x2} + v_{y1}v_{y2} = 0.$$

Since v_x never changes, this is $(u \cos \theta)^2 + (u \sin \theta - gt_1)(u \sin \theta - gt_2) = 0$. For a later velocity perpendicular to the **initial** one ($t_1 = 0$), it simplifies to $u^2 - gt_2 u \sin \theta = 0$, so $t = \frac{u}{g \sin \theta}$.

- "Moving at less than 45° to the horizontal"** (a set of x or t): the path is steeper than 45° near the start and near the end, so solve $\frac{dy}{dx} = 1$ and $\frac{dy}{dx} = -1$ and take the x -values **between** them. For the path in the first diagram, $\frac{dy}{dx} = \frac{15}{8} - \frac{5}{32}x = \pm 1$ gives $x = 5.6$ and $x = 18.4$, so the direction makes less than 45° with the horizontal for $5.6 < x < 18.4$.
- Before hitting the ground** ("a given time before it lands"): find the time of flight first, subtract, then find the components.

2. Using the trajectory: unknown speed or angle through given points (12 questions)

- Write the trajectory equation** in the form that suits: the $\cos^2 \theta$ form when θ is known, the $(1 + \tan^2 \theta)$ form when θ is unknown.

2. **Substitute the point** (x, y) it passes through, and $g = 10$.

3. **Solve:**

- θ known, u unknown: a simple equation for u^2 ;
- u known, θ unknown: a **three-term quadratic in $\tan \theta$** . Give **both** roots; each is a possible angle.

Variations you will meet:

- **A tan-ratio angle** such as $\tan \theta = \frac{12}{5}$: draw the right-angled triangle to get $\sin \theta = \frac{12}{13}$, $\cos \theta = \frac{5}{13}$ exactly, rather than using a calculator angle.
- **Two points, both u and θ unknown.** Write the trajectory as $y = x \tan \theta - kx^2$ with $k = \frac{g}{2u^2 \cos^2 \theta}$. Each point gives a linear equation in $\tan \theta$ and k ; solve them, then find u from k (Example 3).
- **An obstacle or wall** of height h at distance d :
 - "just clears" means the path goes through (d, h) : two values of $\tan \theta$, so two angles;
 - "clears" means $y > h$ at $x = d$: the angles **between** those two;
 - to find the **least range** for angles that clear it, note that $R = \frac{u^2 \sin 2\theta}{g}$ gets smaller the further θ is from 45° , so work out R at **both** end angles and take the smaller.
- **A barrier the particle must hit** at unknown distance D : it hits a barrier of height h if $0 \leq y \leq h$ at $x = D$. Solve $y = h$ (two roots, one going up and one coming down) and $y = 0$ (the range). If the barrier is beyond a point the particle is known to pass on its way, only the coming-down part counts, so D runs from the larger root of $y = h$ to the range.
- **Two speeds for the same obstacle** ("with speed u_1 it just clears; with speed u_2 it reaches only part of the height"): write one trajectory equation for each speed at the same x , link the y -values, and solve for $\tan \theta$ first.
- **The particle starts elsewhere** (after leaving a string, or from a cliff): put the origin at the starting point and measure x and y from there. A point below the start has negative y .

3. Greatest height, time of flight and range (12 questions)

1. **Greatest height:** set $v_y = 0$ to get the time, $t = \frac{u \sin \theta}{g}$, and use $v_y^2 = (u \sin \theta)^2 - 2gy$ with $v_y = 0$ for H .
2. **Time of flight on level ground:** set $y = 0$ and take the non-zero root, $t = \frac{2u \sin \theta}{g}$.
3. **Range:** horizontal speed $\times$ time of flight, or set $y = 0$ in the trajectory equation: $x \left(\tan \theta - \frac{gx}{2u^2 \cos^2 \theta} \right) = 0$ gives $x = \frac{2u^2 \sin \theta \cos \theta}{g} = \frac{u^2 \sin 2\theta}{g}$.

Variations you will meet:

- **Two times at the same height h** that differ by Δt : by symmetry they sit $\frac{1}{2}\Delta t$ either side of the time to the top. Find the time to the top, add $\frac{1}{2}\Delta t$, and substitute into $y = (u \sin \theta)t - \frac{1}{2}gt^2$.
- **Falling a distance from the top:** from the highest point the particle falls from rest vertically, so in time τ it drops $\frac{1}{2}g\tau^2$.
- **Greatest height from the trajectory:** the top is halfway along the range, so put $x = \frac{R}{2}$ into the trajectory equation.

- **Answers in terms of θ :** leave H and T as expressions such as $H = 45 \sin^2 \theta$ when $u = 30$, then use them in later parts.
- **From a cliff or above the ground:** the landing point has $y = -(\text{height of cliff})$, so the time comes from a quadratic; take the positive root.
- **Motion from a start moving horizontally** (thrown horizontally, or leaving the top of a curve): $v_y = 0$ at the start, so the vertical fall in time t is $\frac{1}{2}gt^2$.

4. Two projectiles that meet or land together (7 questions)

1. **Write each particle's x and y** at time t , with its own start point, speed, angle and start time. If one starts a second later, its time is $t - 1$ (or the other's is $t + 1$). Check which particle has which time.
2. **They collide** when both coordinates are equal at the same instant: set the x -expressions equal, and the y -expressions equal.
3. Solve the two equations together for the unknowns (a time, a speed, a distance).

Variations you will meet:

- **Same vertical start and same start time.** If both leave the same height together, equal heights at the meeting time t give $(u \sin \theta)t - \frac{1}{2}gt^2 = (w \sin \beta)t - \frac{1}{2}gt^2$, so $u \sin \theta = w \sin \beta$: they have the same vertical velocity component all the way.
- **One above the other**, a distance d apart vertically: the x -equations force equal horizontal components; the difference of the y -equations gives d . A particle projected at angle β **below** the horizontal starts with vertical component $-w \sin \beta$, so its height is $d - (w \sin \beta)t - \frac{1}{2}gt^2$.
- **"Land at the same point at the same time"** where the second is launched later: its range equals the first's range, minus any horizontal distance between the start points, and its time of flight is what is left of the first's.
- **Moving target** (a model plane flying horizontally at constant speed): its x is start + speed $\times$ time and its y stays fixed.

5. Projectile after leaving a circular path (7 questions)

The circular-motion part (energy and Newton's second law to find where and how fast it leaves) belongs to the Circular motion topic. The projectile part starts at the point where contact is lost:

1. **The starting velocity is along the tangent** to the circle at that point, so perpendicular to the radius. If the radius makes angle α with the vertical, the velocity makes angle α with the horizontal.
2. **Split it:** $v \cos \alpha$ horizontal and $v \sin \alpha$ vertical (up or down, depending on which side of the circle).
3. **Continue as a projectile:** greatest height $\frac{(v \sin \alpha)^2}{2g}$ above the leaving point if it is moving upwards; the vertical speed on reaching the ground from $v_y^2 = (v \sin \alpha)^2 + 2g \times (\text{drop})$ if moving downwards.
4. **Heights** are measured from wherever the question asks: add the height of the leaving point above that level (for example $a \cos \alpha$ above the centre).
5. **Angle of impact** with the ground: $\tan \beta = \frac{|v_y|}{v_x}$, where $v_x = v \cos \alpha$ is unchanged.

A second approach, useful when a speed is given at the ground, is energy for the whole motion; the horizontal component still stays at $v \cos \alpha$.

6. A new flight after an impact (7 questions)

The restitution rule belongs to the Momentum topic; the projectile side is:

1. **Find the velocity components just before the impact**, from the motion so far.
2. **Apply the impact**, with e the coefficient of restitution:
 - smooth **vertical wall**: horizontal component reverses and becomes e times as big; vertical component unchanged;
 - smooth **horizontal ground**: vertical component reverses and becomes e times as big; horizontal component unchanged;
 - smooth **inclined surface** (a barrier or plane at an angle): resolve the velocity **along** and **perpendicular to** the surface instead. The component along the surface is unchanged; the perpendicular component reverses and becomes e times as big.
3. **Start a new projectile** from the impact point with the new components.

Variations you will meet:

- **"Explain why the time of flight is unchanged"** when a vertical wall is added: the wall doesn't change the vertical component, and the vertical motion alone decides when it lands.
- **Time to the wall and back**: distance $\div$ horizontal speed each way ($d \div u \cos \theta$ there, $d \div eu \cos \theta$ back); their sum equals the time of flight $\frac{2u \sin \theta}{g}$.
- **Striking a surface when moving horizontally**: at that moment $v_y = 0$, so it is at its highest point, $H = \frac{u^2 \sin^2 \theta}{2g}$ above the start.
- **Striking an inclined surface at right angles**. If the direction of motion is perpendicular to the surface (for example moving at 40° below the horizontal onto a barrier inclined at 50° to the horizontal), there is no component along the surface, so it rebounds **back along the same line** with e times the speed. Then split that speed into horizontal and vertical components (here at 40° **above** the horizontal, heading back the way it came) and start the new flight from the impact point, whose height you found earlier.
- **Striking an inclined surface at another angle**. Resolve along and perpendicular to the surface, apply the rule, then **recombine** the two new components into horizontal and vertical ones (or use the angle they make with the surface) before starting the new flight. If the question says the rebound is vertical, the ratio of the new components fixes e or the angle of the surface.
- **Speed and height after bouncing on the ground**: new speed $\sqrt{v_x^2 + (ev_y)^2}$; next greatest height $\frac{(ev_y)^2}{2g}$.

7. Deriving the equation of the trajectory (3 questions)

1. Write $x = (u \cos \theta)t$ and $y = (u \sin \theta)t - \frac{1}{2}gt^2$, each on its own line.
2. Make t the subject of the first: $t = \frac{x}{u \cos \theta}$.
3. Substitute, **showing the substitution**, and simplify: $y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$.

4. If the answer is asked with $\sec^2 \theta$ or $(1 + \tan^2 \theta)$, replace $\frac{1}{\cos^2 \theta}$ with $\sec^2 \theta = 1 + \tan^2 \theta$ as a separate line.

It is a "show that", so every line matters: quoting the MF19 equation earns nothing here.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
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Example 1

A particle is projected from a point O on horizontal ground with speed 39 m s^{-1} at an angle θ above the horizontal, where $\tan \theta = \frac{12}{5}$. It moves freely under gravity.

- (a) Find the greatest height reached and the range on the ground. **[4]**
- (b) Find the speed and direction of motion of the particle 5 s after projection. **[3]**
- (c) Find the length of time for which the speed of the particle is less than 25 m s^{-1} . **[3]**

(a) $\sin \theta = \frac{12}{13}$, $\cos \theta = \frac{5}{13}$, so the components are $39 \times \frac{5}{13} = 15 \text{ m s}^{-1}$ horizontally and $39 \times \frac{12}{13} = 36 \text{ m s}^{-1}$ vertically. **B1**

At the top $v_y = 0$: $0 = 36^2 - 20H$, so $H = 64.8 \text{ m}$. **M1 A1**

Time of flight: $36t - 5t^2 = 0$ gives $t = 7.2 \text{ s}$, so the range is $15 \times 7.2 = 108 \text{ m}$. **A1**

(b) At $t = 5$: $v_x = 15$ and $v_y = 36 - 50 = -14$. **B1**

Speed = $\sqrt{15^2 + 14^2} = \sqrt{421} = 20.5 \text{ m s}^{-1}$. **M1**

Direction: $\tan^{-1} \frac{14}{15} = 43.0^\circ$ **below** the horizontal. **A1**

(c) Speed less than 25: $15^2 + (36 - 10t)^2 < 25^2$, so $(36 - 10t)^2 < 400$. **M1**

$-20 < 36 - 10t < 20$ gives $1.6 < t < 5.6$. **A1**

So the speed is less than 25 m s^{-1} for $5.6 - 1.6 = 4 \text{ s}$. **A1**

Example 2

A ball is kicked from a point O on horizontal ground with speed 20 m s^{-1} at an angle θ above the horizontal. A fence of height 5 m stands 20 m from O , in the vertical plane of the ball's motion.

(a) Show that, for the ball just to clear the fence, $\tan^2 \theta - 4 \tan \theta + 2 = 0$, and find the two possible values of θ . [5]

(b) Find the set of values of θ for which the ball passes over the fence. [1]

(c) For the smaller angle in (a), find how far beyond the fence the ball lands. [3]

(a) Trajectory with $g = 10$, $u = 20$, and $T = \tan \theta$:

$$y = xT - \frac{10x^2}{2 \times 400}(1 + T^2) = xT - \frac{x^2}{80}(1 + T^2).$$

M1 for the trajectory equation with $\sec^2 \theta = 1 + \tan^2 \theta$.

At $(20, 5)$: $5 = 20T - 5(1 + T^2)$. **A1** Rearranging, $5T^2 - 20T + 10 = 0$, so $T^2 - 4T + 2 = 0$. **A1** (the given answer, every step shown)

$T = 2 \pm \sqrt{2}$, so $\theta = \tan^{-1}(2 + \sqrt{2}) = 73.7^\circ$ or $\theta = \tan^{-1}(2 - \sqrt{2}) = 30.4^\circ$. **M1 A1**

(b) At $x = 20$, $y - 5 = -5(T^2 - 4T + 2)$, which is positive between the roots, so $30.4^\circ < \theta < 73.7^\circ$. **B1** (The second diagram in "Understand it" shows both paths.)

(c) Put $y = 0$: $x = \frac{80T}{1 + T^2}$. **M1** With $T = 2 - \sqrt{2}$, $1 + T^2 = 7 - 4\sqrt{2}$, so

$$x = \frac{80(2 - \sqrt{2})}{7 - 4\sqrt{2}} = \frac{80(6 + \sqrt{2})}{17} = 34.89 \dots \text{ m. } \textbf{A1}$$

It lands $34.89 - 20 = 14.9 \text{ m}$ beyond the fence. **A1**

Example 3

A particle P is projected from O with speed u at angle θ above the horizontal and moves freely under gravity. At time t its horizontal and vertical displacements from O are x and y .

(a) Derive the equation of the trajectory in the form $y = x \tan \theta - \frac{gx^2}{2u^2} \sec^2 \theta$. [3]

P passes through the points $(2, 5)$ and $(8, 8)$, in metres.

(b) Find $\tan \theta$ and the exact value of u . [5]

(c) Find the greatest height of P above O . [2]

(a) $x = (u \cos \theta)t$ and $y = (u \sin \theta)t - \frac{1}{2}gt^2$. **B1**

From the first, $t = \frac{x}{u \cos \theta}$. Substituting: **M1**

$$y = u \sin \theta \cdot \frac{x}{u \cos \theta} - \frac{1}{2}g \frac{x^2}{u^2 \cos^2 \theta} = x \tan \theta - \frac{gx^2}{2u^2} \sec^2 \theta. \quad \mathbf{A1}$$

(b) Write $y = x \tan \theta - kx^2$, where $k = \frac{g \sec^2 \theta}{2u^2}$. **M1**

(2, 5): $5 = 2 \tan \theta - 4k$. (8, 8): $8 = 8 \tan \theta - 64k$. **A1** for both.

Multiply the first by 4: $20 = 8 \tan \theta - 16k$. Subtract the second: $12 = 48k$, so $k = \frac{1}{4}$ and $\tan \theta = \frac{5+1}{2} = 3$. **M1 A1**

$\sec^2 \theta = 1 + 9 = 10$, so $\frac{1}{4} = \frac{10 \times 10}{2u^2}$, giving $u^2 = 200$ and $u = 10\sqrt{2} \text{ m s}^{-1}$. **A1**

(c) The trajectory is $y = 3x - \frac{1}{4}x^2$, whose top is where $\frac{dy}{dx} = 3 - \frac{1}{2}x = 0$, so $x = 6$. **M1**

$H = 18 - 9 = 9 \text{ m}$. **A1** (Check: $\frac{u^2 \sin^2 \theta}{2g} = \frac{200 \times \frac{9}{10}}{20} = 9$.)

Example 4

Points O and C are on horizontal ground, 52 m apart. A particle A is projected from O towards C with speed 29 m s^{-1} at angle α above the horizontal, where $\tan \alpha = \frac{21}{20}$. At the same instant a particle B is projected from C towards O with speed $w \text{ m s}^{-1}$ at angle β above the horizontal. Both move freely under gravity in the same vertical plane and collide 1.6 s later.

(a) Explain why $w \sin \beta = 21$. **[2]**

(b) Find w and β . **[4]**

(c) Find the height at which they collide, and state whether they are rising or falling at that moment. **[3]**

(a) $\sin \alpha = \frac{21}{29}$, $\cos \alpha = \frac{20}{29}$, so A has components 20 across and 21 up. **B1**

Both start at ground level at the same time, so at the collision $21t - 5t^2 = (w \sin \beta)t - 5t^2$ with $t = 1.6$, which gives $w \sin \beta = 21$. **B1**

(b) Horizontally, the two distances add up to 52: $20 \times 1.6 + (w \cos \beta) \times 1.6 = 52$. **M1**

$32 + 1.6w \cos \beta = 52$, so $w \cos \beta = 12.5$. **A1**

$w = \sqrt{12.5^2 + 21^2} = \sqrt{597.25} = 24.4 \text{ m s}^{-1}$ and $\beta = \tan^{-1} \frac{21}{12.5} = 59.2^\circ$. **M1 A1**

(c) Height = $21 \times 1.6 - 5 \times 1.6^2 = 33.6 - 12.8 = 20.8 \text{ m}$. **M1 A1**

Vertical velocity of each: $21 - 10 \times 1.6 = 5 > 0$, so both are still rising. **B1**

Common mistakes

- **Mixing up time to the top and time of flight.** Candidates who quote memorised formulas often use $\frac{u \sin \theta}{g}$ for the whole flight. Derive it: $y = 0$ for landing, $v_y = 0$ for the top.

- **Losing the sign of a downward velocity.** A particle moving downwards has $v_y < 0$. Writing +14 instead of -14, with no sketch, gives a wrong direction. Draw the velocity and say "below the horizontal".
- **Using the displacement instead of the velocity for the direction of motion.** The direction comes from $\frac{v_y}{v_x}$ (or $\frac{dy}{dx}$), not from $\frac{y}{x}$. Examiners report this most for the "perpendicular directions" questions.
- **Setting the gradient product to +1 for perpendicular directions.** It is -1, or equivalently the scalar product of the velocities is 0.
- **Quoting the trajectory in a "derive" question.** When the question says derive or show, start from $x = (u \cos \theta)t$ and $y = (u \sin \theta)t - \frac{1}{2}gt^2$ and show the substitution for t .
- **Keeping only one root of the quadratic in $\tan \theta$.** Both angles usually work; give both unless the question rules one out.
- **Using the vertical equations when the horizontal ones are needed** (and the other way round) when two particles collide: equal x and equal y are two separate facts; each part of the question usually uses one of them.
- **Giving the wrong particle the extra second** when one is projected later. Write down which time goes with which particle before forming equations.
- **Wrong starting point after leaving a circle.** Find the coordinates of the leaving point relative to the origin you chose; examiners suggest a diagram to get these right.
- **Not taking θ between the two roots** for "clears the wall". The steep and flat angles are the limits; the answer is the interval between them.

How to get the marks

- **Show both components** before using them: $v_x = u \cos \theta$ and $v_y = u \sin \theta - gt$ each earn a mark in many schemes, and later method marks need them.
- **"Show that" and "derive":** the answer is printed, so every step must be visible, including the substitution for t and the step $\frac{1}{\cos^2 \theta} = 1 + \tan^2 \theta$. The last mark goes only when every earlier step is there.
- **"Use the equation of the trajectory":** start from the MF19 equation with the numbers substituted. A method that avoids it can lose the marks when the question names the method.
- **A quadratic in $\tan \theta$:** mark schemes want a three-term quadratic set equal to 0 before solving, and both roots.
- **Directions:** give the angle **and** say above or below the horizontal. An angle alone is often marked incomplete.
- **"In either order"** means you may find the two unknowns in any order; both are needed.
- **"Exact"** means surds and fractions (such as $u = 10\sqrt{2}$); otherwise give 3 significant figures, or 1 decimal place for angles in degrees.
- **Intervals** from "clears" or "hits the barrier" can usually be given with $<$ or $\leq$; state both ends.
- **Draw a diagram:** examiner reports repeatedly say the best answers had a labelled sketch of the path, which prevents sign errors and wrong starting points.
- **Method marks follow through:** an earlier wrong value used correctly still earns later method marks, so keep going.

Quick check

1. A stone is thrown horizontally with speed 12 m s^{-1} from the top of a vertical cliff 45 m above the sea. Find how far from the foot of the cliff it hits the sea, and its speed and direction just before it does.
2. A particle is projected from the top of a high cliff with speed 26 m s^{-1} at angle θ above the horizontal, where $\tan \theta = \frac{5}{12}$. Find the time after projection when its direction of motion is perpendicular to its initial direction.
3. A particle projected from O on horizontal ground has trajectory $y = \frac{1}{3}x - \frac{1}{90}x^2$ (metres, $g = 10$). Find its speed and angle of projection, its range and its greatest height.
4. A ball hits horizontal ground moving with velocity components 12 m s^{-1} horizontally and 16 m s^{-1} downwards. The coefficient of restitution is 0.5 . Find its speed just after the bounce and the greatest height of its next flight.

Answers

1. Vertically from rest: $45 = 5t^2$, so $t = 3$ s. Distance = $12 \times 3 = 36$ m. At the sea $v_y = -30$, so speed = $\sqrt{12^2 + 30^2} = \sqrt{1044} = 32.3 \text{ m s}^{-1}$, at $\tan^{-1} \frac{30}{12} = 68.2^\circ$ below the horizontal.
2. Components $26 \times \frac{12}{13} = 24$ and $26 \times \frac{5}{13} = 10$. Perpendicular: $24 \times 24 + 10(10 - 10t) = 0$, so $676 = 100t$ and $t = 6.76$ s. (Or $t = \frac{u}{g \sin \theta} = \frac{26}{10 \times \frac{5}{13}} = 6.76$.)
3. $\tan \theta = \frac{1}{3}$, so $\theta = 18.4^\circ$ and $\cos^2 \theta = \frac{9}{10} \cdot \frac{10}{2u^2 \times \frac{9}{10}} = \frac{1}{90}$ gives $u^2 = 500$, $u = 10\sqrt{5} = 22.4 \text{ m s}^{-1}$. Range: $y = 0$ gives $x = 30$ m. Greatest height at $x = 15$: $5 - 2.5 = 2.5$ m.
4. After the bounce the components are 12 and $0.5 \times 16 = 8$ upwards. Speed = $\sqrt{144 + 64} = \sqrt{208} = 14.4 \text{ m s}^{-1}$. Greatest height = $\frac{8^2}{20} = 3.2$ m.

Past questions to try

- [9231/33/M/J/24 Q6](#): speed after a given time and perpendicular velocities, both unknowns found.
- [9231/33/O/N/24 Q1](#): unknown speed from two points at the same height on the trajectory.
- [9231/32/M/J/23 Q7](#): greatest height and time in terms of θ , then speed at a later time.
- [9231/32/M/J/22 Q7](#): two particles projected from a cliff one second apart that collide.