

Polar coordinates

AS

Read online at pastopic.com/cambridge/further-mathematics-9231/notes/polar-coordinates

Last checked 30 September 2026

© 2026 Pastopic. Free for personal study. Please share the link, not the file.

What you need to know

By the end of this topic you should be able to:

1. **Move between Cartesian and polar coordinates.** A point can be given by (x, y) or by (r, θ) : its distance r from the pole and the angle θ from the initial line. Know how the two are linked, and use the links to turn a curve's Cartesian equation into a polar one and back again. In this course r is never negative ($r \geq 0$).
2. **Sketch simple polar curves** for $0 \leq \theta < 2\pi$, for $-\pi < \theta \leq \pi$, or for part of one of these ranges. You don't plot points carefully, but your sketch must show the important features: any **symmetry**, where the curve **meets the initial line**, how the curve **approaches the pole**, and the **greatest and least values of r** .
3. **Recall that the area of a sector of a polar curve is $\frac{1}{2} \int r^2 d\theta$** , and use it to find areas.

Exam questions on this topic also ask for the point of a curve furthest from the pole or from the initial line.

These use differentiation from Pure Mathematics applied to r , $y = r \sin \theta$ or $x = r \cos \theta$, and this note shows how.

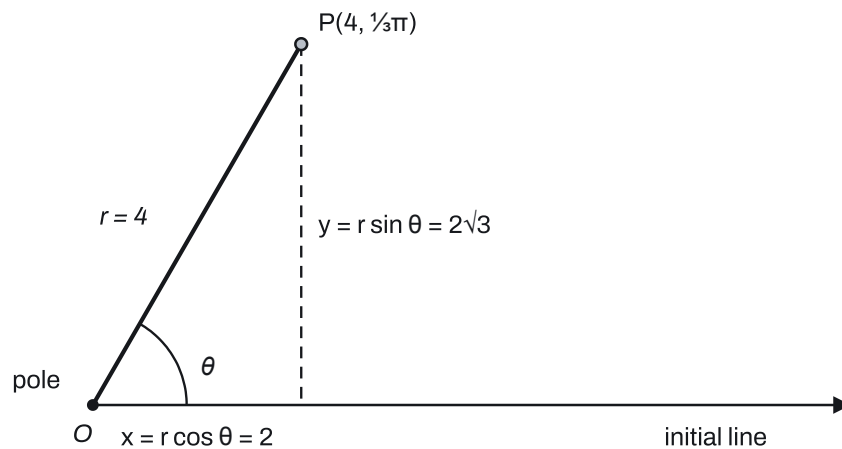
Understand it

Polar coordinates

Fix a point O , the **pole**, and a half-line from it going to the right, the **initial line**. Any point P is then described by

- r , the distance OP , and
- θ , the angle from the initial line to OP , measured **anticlockwise** (clockwise angles are negative).

We write P as (r, θ) , always r **first**. Angles are in radians.



Put the pole at the origin and the initial line along the positive x -axis. The right-angled triangle in the diagram gives

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}.$$

So $(4, \frac{1}{3}\pi)$ is the Cartesian point $(4 \cos \frac{1}{3}\pi, 4 \sin \frac{1}{3}\pi) = (2, 2\sqrt{3})$. Going the other way, take care with $\tan \theta = \frac{y}{x}$: your calculator only gives angles between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, so check which quadrant the point is in first. The point $(-1, -\sqrt{3})$ has $r = 2$ and is in the third quadrant, so $\theta = -\frac{2}{3}\pi$, not the $\frac{1}{3}\pi$ that $\tan^{-1} \sqrt{3}$ gives.

Changing an equation

Cartesian to polar: replace x by $r \cos \theta$ and y by $r \sin \theta$ (and $x^2 + y^2$ by r^2), then simplify, usually to $r = \dots$. For example, $(x^2 + y^2)^2 = 4x^3$ becomes $r^4 = 4r^3 \cos^3 \theta$. Dividing by r^3 (the pole, $r = 0$, is on both forms anyway) gives $r = 4 \cos^3 \theta$.

Polar to Cartesian: multiply through by r (or $r^2, r^3, \dots$) so that every $\cos \theta$ and $\sin \theta$ comes with an r , then replace $r \cos \theta$ by x , $r \sin \theta$ by y and r^2 by $x^2 + y^2$. For example, $r = 2 \sin \theta \tan \theta$ gives $r \cos \theta = 2 \sin^2 \theta$. Multiply by r^2 : $r^2 \cdot r \cos \theta = 2(r \sin \theta)^2$, so $x(x^2 + y^2) = 2y^2$. Identities such as $\sin 2\theta = 2 \sin \theta \cos \theta$ and $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ often come in first.

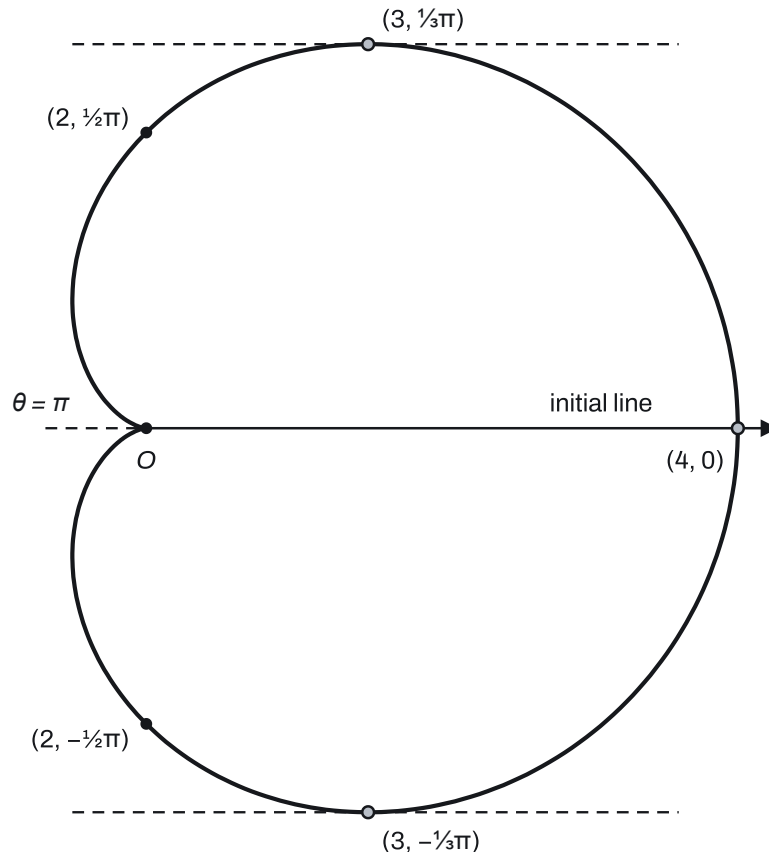
Some polar equations are simple in Cartesian form. $r \sin \theta = 5$ is just the line $y = 5$, and $r \cos \theta = 3$ is the line $x = 3$.

Sketching a polar curve

Think of a ray from the pole turning anticlockwise. For each θ you mark the point at distance r along the ray. To sketch, don't plot lots of points; find these features instead.

1. **Where r is biggest and smallest.** At its greatest value the curve is furthest from the pole.

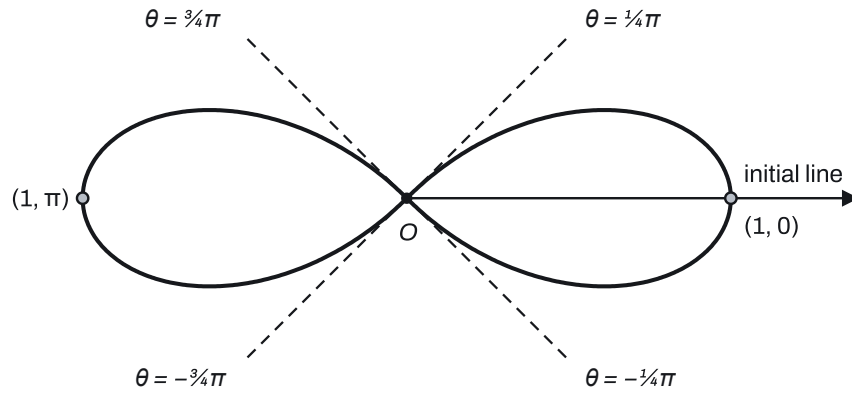
2. **Where $r = 0$.** The curve passes through the pole there. If $r = 0$ when $\theta = \alpha$, the curve arrives at the pole **along the half-line $\theta = \alpha$** (that half-line is a tangent at the pole).
3. **Where the curve meets the initial line** ($\theta = 0$) and other easy angles such as $\theta = \pm \frac{1}{2}\pi$ or π .
4. **Symmetry.** If r is unchanged when θ is replaced by $-\theta$ (for example, anything built from $\cos \theta$), the curve is symmetrical about the initial line. If r is unchanged when θ is replaced by $\pi - \theta$ (for example, anything built from $\sin \theta$), it is symmetrical about the line $\theta = \frac{1}{2}\pi$.
5. **Whether r is increasing or decreasing** as θ increases. A spiral such as $r = \theta^2$ has r increasing all the way.



For $r = 2(1 + \cos \theta)$, $-\pi < \theta \leq \pi$: r is greatest, 4, at $\theta = 0$, and 0 at $\theta = \pi$. It is 2 at $\theta = \pm \frac{1}{2}\pi$. Replacing θ by $-\theta$ changes nothing, so the curve is symmetrical about the initial line. As θ goes from 0 to π , r falls from 4 to 0, and the curve arrives at the pole along the half-line $\theta = \pi$. This heart shape is called a **cardioid**.

Why $r \geq 0$ matters

In this course a point with negative r is not drawn. So when r is given by a formula that can go negative, **the curve only exists for the angles where the formula is at least 0**. Before sketching, solve $r \geq 0$ (or $r^2 \geq 0$ for an equation such as $r^2 = \dots$) to find where the curve lives.



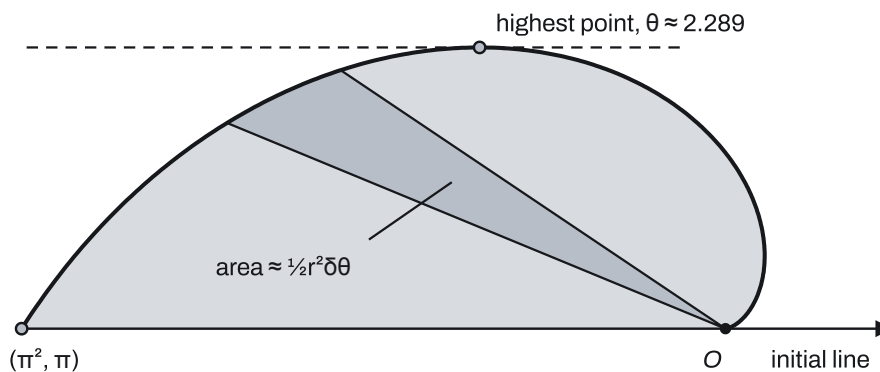
$r = \cos 2\theta$ for $-\pi < \theta \leq \pi$ is only drawn where $\cos 2\theta \geq 0$: for $-\frac{1}{4}\pi \leq \theta \leq \frac{1}{4}\pi$ and for $\frac{3}{4}\pi \leq \theta \leq \pi$ or $-\pi < \theta \leq -\frac{3}{4}\pi$. It gives two loops, not the four you might expect. At the ends of each range $r = 0$, so each loop meets the pole along the half-lines $\theta = \pm\frac{1}{4}\pi$ and $\theta = \pm\frac{3}{4}\pi$.

For an equation $r^2 = f(\theta)$, $r = \sqrt{f(\theta)}$ is the positive root, and the curve exists only where $f(\theta) \geq 0$.

Area of a polar region

A sector of a circle of radius r with angle $\delta\theta$ has area $\frac{1}{2}r^2 \delta\theta$. A region swept out by a polar curve between $\theta = \alpha$ and $\theta = \beta$ can be cut into many thin sectors from the pole. Each is almost a circular sector, with area about $\frac{1}{2}r^2 \delta\theta$. Adding them and letting $\delta\theta \rightarrow 0$ gives

$$\text{Area} = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta.$$



This is the area swept by the line OP as P moves along the curve from $\theta = \alpha$ to $\theta = \beta$. So:

- "The region bounded by the curve and the half-lines $\theta = \alpha$ and $\theta = \beta$ " uses exactly those limits.
- "The region enclosed by a loop" uses the two angles where $r = 0$ at the ends of the loop.
- "The region bounded by the curve and the initial line", for a spiral like $r = \theta^2$ from 0 to π , uses 0 to π , because the half-lines $\theta = 0$ and $\theta = \pi$ together make up the whole initial line through the pole.

For the spiral in the diagram, the area is $\frac{1}{2} \int_0^\pi \theta^4 d\theta = \frac{\pi^5}{10}$.

Furthest from the pole, or from the initial line

- The point furthest from the **pole** has the greatest r . Sometimes you can see it (for example $\cos \theta \leq 1$), otherwise solve $\frac{dr}{d\theta} = 0$.
- The point furthest from the **initial line** has the greatest $|y|$, where $y = r \sin \theta$. Solve $\frac{dy}{d\theta} = 0$. There the tangent to the curve is **parallel to the initial line** (the dashed lines on the cardioid above).
- The point furthest from the **line** $\theta = \frac{1}{2}\pi$ has the greatest $|x|$, where $x = r \cos \theta$. Solve $\frac{dx}{d\theta} = 0$. There the tangent is **perpendicular to the initial line**.

Squaring doesn't move a maximum of a positive quantity, so for $r^2 = f(\theta)$ it is easier to differentiate r^2 , $y^2 = r^2 \sin^2 \theta$ or $x^2 = r^2 \cos^2 \theta$.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$x = r \cos \theta, y = r \sin \theta$	Learn it
$r^2 = x^2 + y^2, \tan \theta = \frac{y}{x}$ (check the quadrant)	Learn it
Convention: $r \geq 0$; points are written (r, θ)	Learn it
Area = $\frac{1}{2} \int_\alpha^\beta r^2 d\theta$	Learn it
Sector of a circle: area $\frac{1}{2}r^2\theta$; triangle: area $\frac{1}{2}ab \sin C$	Learn it
$\cos 2A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A \equiv \cos^2 A - \sin^2 A$	Given
So $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ and $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$	Learn it
$\sin 2A \equiv 2 \sin A \cos A$	Given
$1 + \tan^2 \theta \equiv \sec^2 \theta, \cot^2 \theta + 1 \equiv \operatorname{cosec}^2 \theta$	Given
$\int \sec x dx = \ln \sec x + \tan x $ and other standard integrals	Given
$\int \tan x dx = \ln \sec x = -\ln \cos x $	Learn it
Furthest from the pole: $\frac{dr}{d\theta} = 0$	Learn it
Furthest from the initial line: $\frac{d}{d\theta}(r \sin \theta) = 0$; from $\theta = \frac{1}{2}\pi$: $\frac{d}{d\theta}(r \cos \theta) = 0$	Learn it
Useful integrals: $\int \cos^2 k\theta d\theta = \frac{1}{2}\theta + \frac{1}{4k} \sin 2k\theta, \int \tan^2 \theta d\theta = \tan \theta - \theta, \int \sec^2 \theta d\theta = \tan \theta,$ $\int f'(\theta)e^{f(\theta)} d\theta = e^{f(\theta)}.$	

How to do it

In the Paper 1 questions we have tagged for this topic (32 questions, 2021–2025), the types came up this often. Every question is a multi-part question that mixes several types, so the counts add up to far more than 32.

Question type	Questions
Sketching a polar curve	32
Area of a region bounded by one curve	28
Greatest (or least) distance from the pole	21
Greatest distance from the initial line or the line $\theta = \frac{1}{2}\pi$	19
Changing between Cartesian and polar equations	16
Where two curves meet, and the area between them	4

1. Sketching a polar curve (32 questions)

1. **Find the range of θ** where the curve exists. Use the range given, and remove any angles where $r < 0$ (or $r^2 < 0$).
2. **Work out r at the ends of the range and at easy angles** ($0, \frac{1}{4}\pi, \frac{1}{2}\pi, \dots$). Note where $r = 0$ (the pole) and where r is greatest.
3. **Decide how r changes**: strictly increasing, strictly decreasing, or up then down.
4. **Look for symmetry** ($\theta \rightarrow -\theta$ for the initial line; $\theta \rightarrow \pi - \theta$ for $\theta = \frac{1}{2}\pi$; for $\sin k\theta$ loops, the middle angle of the loop).
5. **Draw** the pole, the initial line (label it $\theta = 0$) and the curve. Make it meet the pole along the right half-line, and cross the initial line or other half-lines at the right angle. Label the key points with polar coordinates, for example $(4, 0)$.

Variations you will meet:

- **A single loop** such as $r = \sin 4\theta$ for $0 \leq \theta \leq \frac{1}{4}\pi$: $r = 0$ at both ends, greatest in the middle ($\theta = \frac{1}{8}\pi$), and symmetrical about that middle half-line. Say so if asked for the line of symmetry, as an equation $\theta = \dots$
- **A spiral** (r strictly increasing, such as $r = \theta^2$): it starts at the pole, leaving along the initial line, and winds outwards. Where it meets the initial line again it should cross it at a sensible angle, not along it.
- **A curve that is never at the pole** (such as $r = \frac{1}{1+\cos\theta}$ in Worked example 2). Say that the least distance from the pole is the least r , and give its value.
- **"State the greatest distance from the pole"**: after the sketch, write the greatest r as a sentence or label it on the sketch.
- **Two curves on one diagram**: draw both, label each, and mark their intersection points.
- **Completing a given diagram**: if part of the curve is drawn, use the symmetry to finish it.

2. Area of a region bounded by one curve (28 questions)

1. **Write the integral with its limits:** $\frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$. The limits are the angles of the half-lines bounding the region, or the angles where the loop starts and ends at the pole.
2. **Expand r^2** and turn it into something you can integrate:
 - $\cos^2 k\theta$ or $\sin^2 k\theta$: use $\cos 2A$ to write $\frac{1}{2}(1 \pm \cos 2k\theta)$;
 - $\tan k\theta = \frac{\sin k\theta}{\cos k\theta}$: the top is a multiple of the derivative of the bottom, so $\int \tan k\theta d\theta = -\frac{1}{k} \ln|\cos k\theta| = \frac{1}{k} \ln|\sec k\theta|$ (this comes up when $r^2 = \tan k\theta$);
 - $\tan^2 \theta = \sec^2 \theta - 1$, $\cot^2 \theta = \operatorname{cosec}^2 \theta - 1$;
 - $\sec^4 \theta = \sec^2 \theta(1 + \tan^2 \theta)$, which integrates to $\tan \theta + \frac{1}{3} \tan^3 \theta$;
 - a product such as $\sin \theta \cos^2 \theta$: spot the reverse chain rule, $-\frac{1}{3} \cos^3 \theta$;
 - θ^2 , $\theta e^{k\theta}$ or $\tan^{-1} \theta$: integrate **by parts** (twice if needed);
 - if the question gives a substitution, change **everything**: the function, $d\theta$ and the limits.
3. **Integrate and substitute the limits**, giving an exact answer if asked.

For example, the cardioid $r = 2(1 + \cos \theta)$ encloses area $\frac{1}{2} \int_{-\pi}^{\pi} 4(1 + \cos \theta)^2 d\theta = 6\pi$ (Worked example 1).

Variations you will meet:

- **Use symmetry**: for a curve symmetrical about the initial line, find the area for $0 \leq \theta \leq \beta$ and double it, but don't forget to double.
- **A region bounded by the curve and a straight line that is not a half-line from the pole**, such as the x -axis and the line $x = c$. Split it into a polar area and a triangle with a vertex at the pole. If the curve's end point is (R, α) , the triangle with vertices O , $(R \cos \alpha, 0)$ and that end point has area $\frac{1}{2} R^2 \sin \alpha \cos \alpha$. Add or subtract the polar area to get the region the question describes; a sketch shows which.
- **"Show that the area is ..."** with a given answer: show the integral with limits, each integration step, the substituted limits and any log or surd simplification.

3. Greatest (or least) distance from the pole (21 questions)

1. If the greatest r is obvious (for example $r = 2(1 + \cos \theta)$ is greatest when $\cos \theta = 1$), just state it, or read it from the end of a spiral's range.
2. Otherwise **differentiate** r (or r^2 for an $r^2 = \dots$ curve) and set $\frac{dr}{d\theta} = 0$.
3. **Solve for θ** . It is often a quadratic in $\sin \theta$ or $\cos \theta$ after using $\cos^2 \theta = 1 - \sin^2 \theta$ or a double angle. Keep the root that lies in the range (and a valid sine or cosine between -1 and 1).
4. **Find r** , and give the point as (r, θ) , to the accuracy asked.

For example, $r = \theta(\pi - \theta)$ for $0 \leq \theta \leq \pi$ has $\frac{dr}{d\theta} = \pi - 2\theta = 0$, so $\theta = \frac{1}{2}\pi$ and the point furthest from the pole is $(\frac{1}{4}\pi^2, \frac{1}{2}\pi)$.

Variations you will meet:

- **A circle through the pole**, such as $r = 2 \cos \theta + 4 \sin \theta$: the greatest r is the diameter. Its Cartesian form (Section 5) is $x^2 + y^2 = 2x + 4y$, that is $(x - 1)^2 + (y - 2)^2 = 5$, so the greatest distance from the pole is $2\sqrt{5}$.
- **A given greatest distance** that fixes a constant: set r at the right angle equal to it and solve for the constant.
- **Least distance** for a curve that never reaches the pole: the smallest r .

4. Greatest distance from the initial line or the line $\theta = \frac{1}{2}\pi$ (19 questions)

1. **Write** $y = r \sin \theta$ in terms of θ (or $x = r \cos \theta$ for the distance from $\theta = \frac{1}{2}\pi$).
2. **Differentiate** (product rule; the chain rule for inner functions) and set it equal to 0.
3. **Solve** using identities to get one trig function: divide by $\cos \theta$ to get $\tan \theta$, or turn everything into $\sin \theta$ or $\cos \theta$.
4. **Find y (or x) at that angle**, not r . The question asks for a distance from a line.

For the cardioid, $y = 2(1 + \cos \theta) \sin \theta$, so $\frac{dy}{d\theta} = 2(\cos \theta + \cos^2 \theta - \sin^2 \theta) = 2(2 \cos \theta - 1)(\cos \theta + 1)$. This is 0 when $\cos \theta = \frac{1}{2}$ (the other root, $\theta = \pi$, is the pole), so $\theta = \pm \frac{1}{3}\pi$, $r = 3$, and the greatest distance from the initial line is $3 \sin \frac{1}{3}\pi = \frac{3\sqrt{3}}{2}$.

Variations you will meet:

- **"Show that, at the point furthest from the initial line, [equation] = 0, and verify that it has a root between a and b ."** Differentiate y , set it to 0, and rearrange into the printed form, showing each step (dividing by a factor that isn't 0 in the range, such as $e^{k\theta}$ or $\cos \theta$). Then substitute a and b into the **printed** expression, give both values, and say that the sign changes, so there is a root between them (the function is continuous there). You can do this last part even if you can't do the "show that".
- **$r^2 = f(\theta)$ curves**: maximise $y^2 = f(\theta) \sin^2 \theta$ (or $x^2 = f(\theta) \cos^2 \theta$), then take the square root at the end.
- **Given identities** such as $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$: use them to write y or x as a polynomial in one trig function before differentiating.
- **Exact form** such as $p\sqrt{q}$: keep the sine and cosine values exact, for example $\sin \theta = \frac{1}{3}$ gives $\cos \theta = \frac{2\sqrt{2}}{3}$.

5. Changing between Cartesian and polar equations (16 questions)

Cartesian to polar ("show that the curve has polar equation ..."):

1. Replace $x^2 + y^2$ by r^2 , x by $r \cos \theta$ and y by $r \sin \theta$.
2. Use identities to combine: $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$, $2 \sin \theta \cos \theta = \sin 2\theta$.
3. Divide by the common power of r (say $r \neq 0$) and reach the printed form.

Polar to Cartesian ("find a Cartesian equation"):

1. If needed, expand with identities first, for example $\sin 2\theta = 2 \sin \theta \cos \theta$ or $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ (and any identity the question gives).
2. Multiply by powers of r so each trig term has its r : $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, $\tan \theta = \frac{y}{x}$.

3. Replace r^2 by $x^2 + y^2$ and tidy up so there are **no fractions and no square roots**, or put it in the form asked, such as $y = f(x)$.

Variations you will meet:

- **Show that the Cartesian equation is a circle:** complete the square to get $(x - p)^2 + (y - q)^2 = c^2$, then state the centre (p, q) and the radius c .
- **A line in polar form:** $r \sin \theta = k$ is $y = k$; $r \cos \theta = k$ is $x = k$.
- **Squared forms:** $r = \sqrt{x^2 + y^2}$, so equations such as $r = x + 1$ square to $x^2 + y^2 = (x + 1)^2$.

6. Where two curves meet, and the area between them (4 questions)

1. **Intersection:** set the two r expressions equal and solve for θ . Also check whether both curves pass through the pole. If the equation can't be solved exactly, you may be asked to show a sign change between two given values.
2. **Sketch both curves** and shade the region, so you can see which curve is the outer edge for each range of θ .
3. **Split the region at the intersection angle:** for each range of θ , the region's boundary is one curve. Area $= \frac{1}{2} \int (\text{that curve})^2 d\theta$ for each piece, added.
4. **For a region between two curves over the same angles**, area $= \frac{1}{2} \int (r_{\text{outer}}^2 - r_{\text{inner}}^2) d\theta$.
5. **For a region cut off by a straight line**, add or subtract a triangle from the pole.

"The larger region": if two curves divide a closed curve into two parts, the larger is the whole area (for example $\pi \times \text{radius}^2$ for a circle) minus the smaller.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
-

Example 1

The curve C has polar equation $r = 2(1 + \cos \theta)$, for $-\pi < \theta \leq \pi$.

- (a) Sketch C and state the greatest distance of a point on C from the pole. **[3]**
- (b) Find the exact area of the region enclosed by C . **[4]**

(a) The sketch is the second diagram in "Understand it": a closed curve symmetrical about the initial line, through the pole at $\theta = \pi$ (meeting the pole along $\theta = \pi$), crossing the initial line at $(4, 0)$. **B1** for a closed curve through the pole with the right symmetry, **B1** for the right shape at the pole and at $(4, 0)$.

Since $\cos \theta \leq 1$, the greatest value of r is $2(1 + 1) = 4$. **B1**

(b)

$$\text{Area} = \frac{1}{2} \int_{-\pi}^{\pi} 4(1 + \cos \theta)^2 d\theta = 2 \int_{-\pi}^{\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta.$$

M1 for $\frac{1}{2} \int r^2 d\theta$ with the correct limits.

Using $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$:

$$2 \int_{-\pi}^{\pi} \left(\frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta = 2 \left[\frac{3}{2}\theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_{-\pi}^{\pi}.$$

M1 for the double angle, **A1** for the correct integral.

$$= 2 \left(\frac{3}{2}\pi + \frac{3}{2}\pi \right) = 6\pi. \text{ **A1**}$$

Example 2

The curve C has polar equation $r = \frac{1}{1 + \cos \theta}$, for $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$.

(a) Show that a Cartesian equation of C is $y^2 = 1 - 2x$. **[3]**

(b) Sketch C . **[2]**

(c) Using the identity $1 + \cos \theta \equiv 2 \cos^2 \frac{1}{2}\theta$, find the exact area of the region bounded by C and the half-lines $\theta = \frac{1}{2}\pi$ and $\theta = -\frac{1}{2}\pi$. **[5]**

(a) $r(1 + \cos \theta) = 1$, so $r + r \cos \theta = 1$, that is $r = 1 - x$. **M1** for using $x = r \cos \theta$.

Square: $x^2 + y^2 = (1 - x)^2 = 1 - 2x + x^2$. **M1** for $r^2 = x^2 + y^2$.

So $y^2 = 1 - 2x$. **A1**

(b) $r = \frac{1}{2}$ at $\theta = 0$, which is the least value; r increases to 1 at $\theta = \pm \frac{1}{2}\pi$, giving the points $(0, 1)$ and $(0, -1)$ in Cartesian form. The curve is an arc of the parabola, symmetrical about the initial line, crossing it at $(\frac{1}{2}, 0)$ at right angles, and never passing through the pole. **B1** for the arc through $(1, \pm \frac{1}{2}\pi)$ and $(\frac{1}{2}, 0)$, **B1** for the symmetry and shape.

(c) The two half-lines together make the line $x = 0$ through the pole, so the limits are $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$:

$$\text{Area} = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{(1 + \cos \theta)^2} = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{4 \cos^4 \frac{1}{2}\theta} = \frac{1}{8} \int_{-\pi/2}^{\pi/2} \sec^4 \frac{1}{2}\theta d\theta.$$

M1 for the area formula with correct limits, **M1** for using the identity.

$\sec^4 u = \sec^2 u (1 + \tan^2 u)$, which integrates to $\tan u + \frac{1}{3} \tan^3 u$. With $u = \frac{1}{2}\theta$, the integral of $\sec^4 \frac{1}{2}\theta$ is $2 \left(\tan \frac{1}{2}\theta + \frac{1}{3} \tan^3 \frac{1}{2}\theta \right)$. **M1 A1**

$$\frac{1}{8} \times 2 \left[\tan \frac{1}{2}\theta + \frac{1}{3} \tan^3 \frac{1}{2}\theta \right]_{-\pi/2}^{\pi/2} = \frac{1}{4} \left(\left(1 + \frac{1}{3}\right) - \left(-1 - \frac{1}{3}\right) \right) = \frac{2}{3}.$$

A1. Check with the Cartesian form: the region between the parabola and the y -axis has area $\int_{-1}^1 \frac{1-y^2}{2} dy = \frac{2}{3}$.

Example 3

The curve C has polar equation $r = \theta^2$, for $0 \leq \theta \leq \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. **[3]**

(b) Find the area of the region bounded by C and the initial line. **[2]**

(c) Show that, at the point of C furthest from the initial line, $\theta + 2 \tan \theta = 0$, and verify that this equation has a root between 2.28 and 2.29. **[5]**

(a) The sketch is the third diagram in "Understand it": a spiral with r strictly increasing, starting at the pole along the initial line and ending on the initial line on the other side of the pole. **B1** for the spiral with r increasing, **B1** for the shape at the pole and at $\theta = \pi$.

The furthest point is the end, (π^2, π) . **B1**

(b) $\frac{1}{2} \int_0^\pi \theta^4 d\theta = \frac{1}{2} \left[\frac{\theta^5}{5} \right]_0^\pi = \frac{\pi^5}{10} (= 30.6 \text{ to } 3 \text{ s.f.}).$ **M1 A1**

(c) $y = r \sin \theta = \theta^2 \sin \theta$. **B1**

$\frac{dy}{d\theta} = 2\theta \sin \theta + \theta^2 \cos \theta$. **M1 A1** for differentiating by the product rule and setting it to 0.

At the furthest point $\theta \neq 0$ and $\cos \theta \neq 0$, so divide $2\theta \sin \theta + \theta^2 \cos \theta = 0$ by $\theta \cos \theta$: $2 \tan \theta + \theta = 0$, that is $\theta + 2 \tan \theta = 0$. **A1** (the given answer, with the division shown)

Let $f(\theta) = \theta + 2 \tan \theta$. $f(2.28) = -0.0506$ and $f(2.29) = 0.0060$. The sign changes and f is continuous on $[2.28, 2.29]$, so there is a root between 2.28 and 2.29. **B1**

(The root is $\theta = 2.289$, so the greatest distance from the initial line is $2.289^2 \sin 2.289 = 3.95$ to 3 s.f.)

Example 4

The curves C_1 and C_2 have polar equations $r = 2(1 + \cos \theta)$ and $r = 3$, for $-\pi < \theta \leq \pi$.

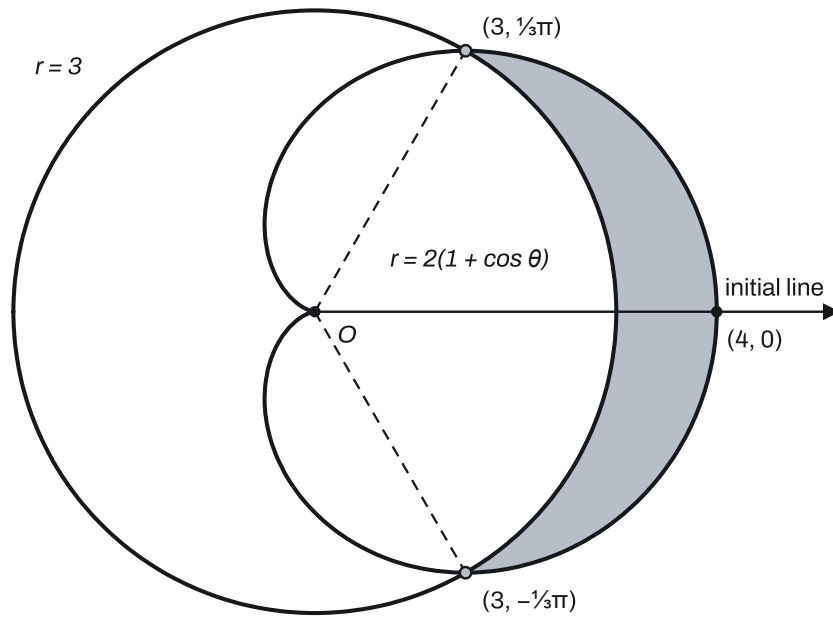
(a) Find the polar coordinates of the points where C_1 and C_2 meet. **[2]**

(b) Find the exact area of the region that lies inside C_1 and outside C_2 . **[5]**

(a) $2(1 + \cos \theta) = 3 \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{1}{3}\pi$. **M1**

The points are $(3, \frac{1}{3}\pi)$ and $(3, -\frac{1}{3}\pi)$. **A1**

(b) The region lies between $\theta = -\frac{1}{3}\pi$ and $\theta = \frac{1}{3}\pi$, with the cardioid outside and the circle inside.



$$\text{Area} = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (4(1 + \cos \theta)^2 - 9) d\theta.$$

M1 for $\frac{1}{2} \int (r_1^2 - r_2^2) d\theta$, **A1** for the correct limits.

$4(1 + 2\cos \theta + \cos^2 \theta) - 9 = -5 + 8\cos \theta + 2(1 + \cos 2\theta) = -3 + 8\cos \theta + 2\cos 2\theta$. **M1** for the double angle.

$$\frac{1}{2} \left[-3\theta + 8\sin \theta + \sin 2\theta \right]_{-\pi/3}^{\pi/3} = -\pi + 8 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{2} - \pi.$$

A1 for the integral, **A1** for $\frac{9\sqrt{3}}{2} - \pi$ (= 4.65 to 3 s.f.).

Common mistakes

- **No pole or initial line on the sketch.** Examiners want the pole clearly marked and the initial line drawn. Drawing x - and y -axes instead is marked wrong.
- **Wrong shape at the ends.** A curve with $r = 0$ at $\theta = \alpha$ must arrive at the pole along the half-line $\theta = \alpha$. Many sketches lose a mark for the wrong direction at the pole or the wrong gradient where the curve crosses the initial line.
- **Ignoring the range of θ** or drawing points where $r < 0$. Draw only what the given range and $r \geq 0$ allow.
- **Coordinates in the wrong order.** A polar point is (r, θ) . Writing (θ, r) , or giving r and θ separately, loses the mark. Remember any constant: if $r = 3a$ there, the point is $(3a, \dots)$, not $(3, \dots)$.
- **Forgetting to state the greatest or least distance** after sketching, when the question asks for it.
- **Leaving out the $\frac{1}{2}$** in $\frac{1}{2} \int r^2 d\theta$, or forgetting to square r .

- **Limits that don't match the region.** Draw the region first and read the angles off it, especially for regions that include a triangle or are between two curves.
- **Finding the wrong maximum.** "Furthest from the initial line" means maximise $y = r \sin \theta$, not r or x . And after finding θ , work out y at that angle, not r .
- **Integration going wrong** for $\tan^{-1}$, $\sec^4 \theta$ or $\cot^2$ terms. Use $\sec^2 = 1 + \tan^2$ or $\operatorname{cosec}^2 = 1 + \cot^2$ to reach a standard form, and take the hint when a question points to MF19. Don't reach for a logarithm unless the top really is the derivative of the bottom.
- **Missing a factor with a substitution.** When using $t = \tan \theta$, change $d\theta$ to $\frac{dt}{1+t^2}$ and change the limits too.
- **Messy Cartesian algebra.** Take out common factors before multiplying out $(x^2 + y^2)^2$, and remember to square the constant when squaring both sides.
- **Too little working on a "show that".** Show the substitution of limits and how the log or surd terms combine; the answer is given, so the marks are for the steps.

How to get the marks

- **Sketches** earn marks for the pole, the initial line, the right range of θ , the right behaviour of r (increasing, decreasing, loop) and the right shape where the curve meets the pole and the initial line. Label key points with polar coordinates.
- **Area:** the first mark is for writing $\frac{1}{2} \int r^2 d\theta$ with **correct limits**. The next is for turning r^2 into an integrable form (double angle, $\sec^2 = 1 + \tan^2$, by parts), then for the integration, then the answer.
- **Greatest distance from a line:** there is a mark just for writing $y = r \sin \theta$ (or $x = r \cos \theta$). Then marks for differentiating correctly and setting it to 0, solving, and finally the distance.
- **"Show that"** means the answer is printed. Every step must be visible, and your last line must match the printed answer exactly.
- **"Verify that there is a root between a and b "** means substitute both values into the expression, give the values (1 s.f. or better), and state that the sign changes.
- **"Exact"** means surds, π , e and logs, no decimals. Otherwise give answers to 3 significant figures, or to the number of decimal places asked (for example "correct to 3 decimal places" for polar coordinates).
- **"Hence" or "using your answer to (b)"** means build on the earlier part, for example subtract its area from a triangle.
- **Give polar coordinates as (r, θ)** with θ in radians, inside the given range.

Quick check

1. (a) Write the Cartesian point $(-3, 3)$ in polar form. (b) Write the polar point $(2, -\frac{1}{6}\pi)$ in Cartesian form.
2. Find a polar equation, in the form $r = f(\theta)$, for the curve $y = x^2$.
3. Find the exact area of the region bounded by the curve $r = 1 + \theta$, for $0 \leq \theta \leq \pi$, and the initial line.
4. Find the exact area enclosed by one loop of the curve $r = \cos 2\theta$.
5. The curve C has polar equation $r = \theta e^{-\theta}$, for $0 \leq \theta \leq 2\pi$. Find the polar coordinates of the point of C furthest from the pole, giving r correct to 3 significant figures.

Answers

- (a) $r = \sqrt{9+9} = 3\sqrt{2}$. The point is in the second quadrant, so $\theta = \pi - \frac{1}{4}\pi = \frac{3}{4}\pi$: $(3\sqrt{2}, \frac{3}{4}\pi)$. (b) $x = 2\cos(-\frac{1}{6}\pi) = \sqrt{3}$ and $y = 2\sin(-\frac{1}{6}\pi) = -1$: $(\sqrt{3}, -1)$.
- $r \sin \theta = r^2 \cos^2 \theta$. Divide by $r \cos^2 \theta$: $r = \frac{\sin \theta}{\cos^2 \theta} = \tan \theta \sec \theta$.
- Area = $\frac{1}{2} \int_0^\pi (1 + \theta)^2 d\theta = \frac{1}{2} \left[\frac{(1 + \theta)^3}{3} \right]_0^\pi = \frac{(1 + \pi)^3 - 1}{6}$ (= 11.7 to 3 s.f.).
- One loop runs from $\theta = -\frac{1}{4}\pi$ to $\frac{1}{4}\pi$. Area = $\frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta d\theta = \frac{1}{4} \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta) d\theta = \frac{1}{4} \left[\theta + \frac{1}{4} \sin 4\theta \right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$.
- $\frac{dr}{d\theta} = e^{-\theta} - \theta e^{-\theta} = e^{-\theta}(1 - \theta) = 0$, so $\theta = 1$. $r = e^{-1} = 0.368$. The point is $(0.368, 1)$ (at the ends, $r = 0$ and $r = 2\pi e^{-2\pi} = 0.0117$, both smaller).

Past questions to try

- [9231/12/M/J/25 Q5](#): sketching a spiral, an area by parts twice, and the point furthest from the initial line.
- [9231/13/O/N/23 Q6](#): Cartesian to polar, where two curves meet, and the area between them.
- [9231/12/O/N/21 Q5](#): completing a sketch, meeting a straight line, and an area with a triangle.
- [9231/13/M/J/22 Q6](#): Cartesian to polar and an area with the substitution $t = \tan \theta$.