

Probability generating functions

A2

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What you need to know

By the end of this topic you should be able to:

1. **Say what a probability generating function (PGF) is and build one.** For a discrete random variable X that takes whole-number values, $G_X(t) = E(t^X)$: a polynomial or power series in which the coefficient of t^r is $P(X = r)$. Build it from a table, a formula, or a situation (coins, dice, counters drawn from a bag), and know the PGFs of the four standard distributions: **discrete uniform, binomial, geometric and Poisson**.
2. **Use a PGF to find the mean and variance**, with $E(X) = G'_X(1)$ and $\text{Var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2$, for a given distribution or a given PGF.
3. **Use the result that the PGF of a sum of independent random variables is the product of their PGFs**, $G_{X+Y}(t) = G_X(t) G_Y(t)$, and read probabilities of the sum off the product.

That is all of it. Moment generating functions and sums of a random number of random variables are not in the syllabus.

Understand it

The idea: probabilities stored as coefficients

Take a spinner whose score X is 1, 2 or 3 with probabilities 0.5, 0.3 and 0.2. Write each probability as the coefficient of t raised to the value it belongs to:

$$G_X(t) = 0.5t + 0.3t^2 + 0.2t^3.$$

This is the **probability generating function** of X . The letter t is just a "tag": the power says which value, the coefficient says how likely it is. Nothing is lost, so the PGF describes the distribution completely, and two variables with the same PGF have the same distribution.

In general, for X taking values $0, 1, 2, \dots$,

$$G_X(t) = \sum_r P(X = r) t^r = E(t^X).$$

The second form is the definition in the formula list: t^X is a random quantity, and its expectation is "each possible value of t^X times its probability", which is exactly the sum.

Three facts follow at once:

- $G_X(1) = \sum P(X = r) = 1$. Every PGF equals 1 at $t = 1$. Use this to find an unknown constant and to check your work.
- $G_X(0) = P(X = 0)$, the constant term. It is 0 when X cannot be 0, and then every term has a factor t .
- $P(X = r)$ is the coefficient of t^r . If the PGF is not a polynomial, expand it as a series to find the coefficient.

Mean and variance from the PGF

Differentiate $G_X(t) = \sum p_r t^r$ term by term:

$$G'_X(t) = \sum r p_r t^{r-1}, \quad G''_X(t) = \sum r(r-1) p_r t^{r-2}.$$

At $t = 1$ every power of t becomes 1, so

$$G'_X(1) = \sum r p_r = E(X), \quad G''_X(1) = \sum r(r-1) p_r = E(X^2) - E(X).$$

Rearranging, $E(X^2) = G''_X(1) + G'_X(1)$, and $\text{Var}(X) = E(X^2) - (E(X))^2$ gives

$$\text{Var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2.$$

For the spinner: $G'_X(t) = 0.5 + 0.6t + 0.6t^2$, so $E(X) = 1.7$; $G''_X(t) = 0.6 + 1.2t$, so $G''_X(1) = 1.8$ and $\text{Var}(X) = 1.8 + 1.7 - 1.7^2 = 0.61$. (Check: $E(X^2) = 0.5 + 1.2 + 1.8 = 3.5$ and $3.5 - 2.89 = 0.61$.)

Why a sum means a product

If X and Y are **independent**, so are t^X and t^Y , and the expectation of a product of independent quantities is the product of their expectations:

$$G_{X+Y}(t) = E(t^{X+Y}) = E(t^X t^Y) = E(t^X) E(t^Y) = G_X(t) G_Y(t).$$

You can see it happen. Let $G_A(t) = 0.6 + 0.4t$ and $G_B(t) = 0.7 + 0.3t^2$. Multiplying the brackets pairs every value of A with every value of B : the powers **add** (the values add) and the coefficients **multiply** (independent probabilities multiply).

	0.7 (B = 0)	0.3t ² (B = 2)
0.6 (A = 0)	0.42	0.18t ²
0.4t (A = 1)	0.28t	0.12t ³

So $G_{A+B}(t) = 0.42 + 0.28t + 0.18t^2 + 0.12t^3$, and for instance $P(A + B = 2) = 0.18$. When several products give the same power, add them: that is the "or" rule for the different ways of getting the same total.

For n independent observations of the same X , $G_{X_1+\dots+X_n}(t) = (G_X(t))^n$.

If X and Y are **not** independent (for example, both are worked out from the same draw), the product rule fails. Then find the distribution of the sum directly, case by case.

The four standard PGFs

Write $q = 1 - p$.

Discrete uniform on $1, 2, \dots, n$: each value has probability $\frac{1}{n}$, so

$$G(t) = \frac{1}{n} (t + t^2 + \dots + t^n) = \frac{t(1 - t^n)}{n(1 - t)}.$$

The second form is the sum of a geometric series. For the mean and variance, the first form is easier to differentiate.

Binomial $B(n, p)$: one trial has PGF $q + pt$ (value 0 with probability q , value 1 with probability p), and X is the sum of n independent trials, so

$$G(t) = (q + pt)^n.$$

Then $G'(t) = np(q + pt)^{n-1}$ and $G''(t) = n(n-1)p^2(q + pt)^{n-2}$. Since $q + p = 1$, $E(X) = np$ and $\text{Var}(X) = n(n-1)p^2 + np - n^2p^2 = np(1-p) = npq$.

Geometric $\text{Geo}(p)$, the number of trials up to and including the first success ($X = 1, 2, 3, \dots$), has $P(X = r) = pq^{r-1}$. So

$$G(t) = \sum_{r=1}^{\infty} pq^{r-1}t^r = pt \sum_{r=1}^{\infty} (qt)^{r-1} = pt(1 + qt + (qt)^2 + \dots) = \frac{pt}{1 - qt},$$

the sum to infinity of a geometric series with first term pt and common ratio qt (valid for $|qt| < 1$).

Differentiating with the quotient rule gives $G'(t) = \frac{p}{(1 - qt)^2}$ and $G''(t) = \frac{2pq}{(1 - qt)^3}$. At $t = 1$, $1 - q = p$, so $E(X) = \frac{1}{p}$, $G''(1) = \frac{2q}{p^2}$ and

$$\text{Var}(X) = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2q + p - 1}{p^2} = \frac{q}{p^2}.$$

Poisson $\text{Po}(\lambda)$: $P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}$, so, using the series for e^x ,

$$G(t) = e^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda t)^r}{r!} = e^{-\lambda} e^{\lambda t} = e^{\lambda(t-1)}.$$

Worked Example 4 uses it to find the mean and variance.

Because a PGF fixes the distribution, a product that has one of these shapes tells you the distribution of the sum. For example, if $X \sim B(3, 0.25)$ and $Y \sim B(5, 0.25)$ are independent, $G_{X+Y}(t) = (0.75 + 0.25t)^3(0.75 + 0.25t)^5 = (0.75 + 0.25t)^8$, so $X + Y \sim B(8, 0.25)$. This only works when the two binomials have the **same** p .

PGFs that are not polynomials

A variable with infinitely many values has a PGF such as $\frac{pt}{1-qt}$. To find a probability, expand with the binomial series

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots, \quad (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

after taking out a factor so the bracket starts with 1: for example $\frac{1}{5-2t} = \frac{1}{5} \left(1 - \frac{2t}{5}\right)^{-1}$. Then pick out the coefficient of the power you need.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$G_X(t) = E(t^X) = \sum P(X=r) t^r$	Given
$E(X) = G'_X(1)$	Given
$\text{Var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2$	Given
$G_X(1) = 1; G_X(0) = P(X=0)$	Learn it
$P(X=r) = \text{coefficient of } t^r = \frac{G_X^{(r)}(0)}{r!}$	Learn it
$X, Y \text{ independent: } G_{X+Y}(t) = G_X(t) G_Y(t)$	Learn it
Sum of n independent observations of X : $(G_X(t))^n$	Learn it
Discrete uniform on $1, \dots, n$: $\frac{1}{n}(t + t^2 + \dots + t^n) = \frac{t(1-t^n)}{n(1-t)}$	Learn it
$B(n, p): (q + pt)^n$	Learn it
$\text{Geo}(p): \frac{pt}{1-qt}$	Learn it
$\text{Po}(\lambda): e^{\lambda(t-1)}$	Learn it
$P(X=r)$ and the mean for $B(n, p)$, $\text{Geo}(p)$, $\text{Po}(\lambda)$; variance for $B(n, p)$ and $\text{Po}(\lambda)$	Given
$\text{Geo}(p)$ variance $\frac{q}{p^2}$	Learn it
Binomial series $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$ for x	x
Sum to infinity of a geometric series $\frac{a}{1-r}$, x	r
$e^x = 1 + x + \frac{x^2}{2!} + \dots$	Given

How to do it

In the Paper 4 questions we have tagged for this topic (32 questions, 2021–2025; 10 repeat a question from another paper of the same session, so there are 22 different questions), the types came up this often. Every question is a chain of parts, so all 32 mix two or more types and the counts add up to 104.

Question type	Questions
Mean and variance from a PGF	32
PGF of a sum of independent variables	24
Writing down a PGF from a situation	19
Probabilities from a PGF	11
Finding unknown constants	9
Standard distributions: derive or recognise the PGF	4
Working back from the PGF of a sum	4
A sum of variables that are not independent	1

1. Mean and variance from a PGF (32 questions)

1. **Differentiate** $G(t)$ once and twice. For a polynomial, go term by term. For a product or a fraction, use the product, quotient or chain rule, and simplify $G'(t)$ before differentiating again.
2. **Substitute** $t = 1$ into both, and write the values: $E(X) = G'(1)$.
3. **Use the variance formula** with the numbers in: $\text{Var}(X) = G''(1) + G'(1) - (G'(1))^2$.

Variations you will meet:

- **A sum** $Z = X + Y$. Either expand G_Z as a polynomial and differentiate it, or use the product rule on $G_X G_Y$ at $t = 1$, where $G_X(1) = G_Y(1) = 1$: $G'_Z(1) = G'_X(1) + G'_Y(1)$ and $G''_Z(1) = G''_X(1) + 2G'_X(1)G'_Y(1) + G''_Y(1)$. If the question says "use the PGF of Z ", you must differentiate G_Z ; adding $E(X) + E(Y)$ is then only a check.
- **"Show that $E(Y) = nE(X)$ "** for Y the sum of n independent observations: $G_Y = (G_X)^n$, so $G'_Y(t) = n(G_X(t))^{n-1}G'_X(t)$. At $t = 1$, say why $G_X(1) = 1$ (the probabilities add to 1, written in the question's letters), so $G'_Y(1) = nG'_X(1)$. The alternative, $\text{Var}(Y) = n\text{Var}(X)$, is a quick way to a variance when the PGF is not demanded.
- **Rational PGFs** such as $\frac{k}{(a - bt)^2}$: write them as $k(a - bt)^{-2}$ and use the chain rule; G' is $2bk(a - bt)^{-3}$, and G'' is $6b^2k(a - bt)^{-4}$. If there is a t on top as well, use the product rule and don't treat the numerator as a constant.
- **Letters instead of numbers** ("show that $\text{Var}(X) = \dots$ in terms of a and b "): the same three steps, then rearrange to the printed form. Use the fact that the probabilities add to 1 when you need to.
- **PGFs with a power**, such as $ct^k(a + bt)^n$: product rule and chain rule, or expand first if the power is small.

2. PGF of a sum of independent variables (24 questions)

1. **Find each PGF** (type 3 below), or read them from the question.
2. **Multiply:** $G_Z(t) = G_X(t) G_Y(t)$. Never add them.
3. **Expand and collect** powers of t when the question says "as a polynomial". Pull out common denominators first: $\frac{1}{20}(\dots) \times \frac{1}{7}(\dots) = \frac{1}{140}(\dots)(\dots)$ keeps the arithmetic in whole numbers.
4. **Check** that the coefficients add to 1 (put $t = 1$).

Variations you will meet:

- **n observations of the same variable:** $(G_X(t))^n$. If every value of X is at least k , take out t^k first: $G_X(t) = t^k(a + bt)$ gives $(G_X(t))^n = t^{nk}(a + bt)^n$, and the binomial theorem expands the bracket. Questions sometimes ask for exactly this " $t^m(\dots)^n$ " form.
- **Two different situations** (coins and dice, a bag and a spinner): find each PGF separately. A second PGF is not the same as the first just because the story looks alike; work it out.
- **Standard distributions:** the product of two PGFs of the same standard type can be another standard PGF (same- p binomials, any two Poissons). Say so, giving the parameters.

3. Writing down a PGF from a situation (19 questions)

1. **List the possible values** of the variable, including 0 if it can happen.
2. **Find each probability:** a tree, combinations for drawing without replacement ($P(X = r) = \frac{\binom{K}{r} \binom{N-K}{n-r}}{\binom{N}{n}}$), or the formula given.
3. **Write the PGF** with each probability on the matching power of t , and check the coefficients add to 1.

Variations you will meet:

- **Several independent coins with different probabilities of a head:** each coin alone has PGF $(1 - p) + pt$, so the number of heads has PGF equal to the product. For coins with probabilities 0.1 and 0.6: $(0.9 + 0.1t)(0.4 + 0.6t) = 0.36 + 0.58t + 0.06t^2$. This is quicker and safer than listing outcomes.
- **Identical independent trials** (the same coin thrown three times, several fair dice counting sixes): the count is binomial, so the PGF is $(q + pt)^n$. Expand it if the question wants a polynomial.
- **A formula for the probabilities**, such as $P(X = r) = k(5 - r)$ for $r = 1, 2, 3, 4$: add the probabilities and set the total to 1 to find k (here $10k = 1$, so $k = \frac{1}{10}$), then $G(t) = \frac{1}{10}(4t + 3t^2 + 2t^3 + t^4)$.
- **A table with letters:** write $G(t)$ straight from the table, for example $a + bt + (1 - a - b)t^2$ for probabilities a, b and $1 - a - b$ on 0, 1, 2.
- **A variable defined some other way than a simple count** (for example, how many different kinds appear in a selection): work out which draws give each value before writing probabilities. A value that cannot happen simply has no term.

4. Probabilities from a PGF (11 questions)

1. **Get the PGF into a form you can read:** expanded as a polynomial, or as a factor times a bracket to a power, or as a series.
2. **Pick out the coefficient** of t^r for $P(X = r)$.
3. For "at least" or "more than", use the complement: $P(X > r) = 1 - P(X = 0) - \dots - P(X = r)$.

Variations you will meet:

- **A bracket to a power**, such as $P(Y = 7)$ when $G_Y(t) = ct^3(a + bt)^6$: you need the t^4 term of $(a + bt)^6$, which is $\binom{6}{4}a^2b^4t^4$. Multiply by c .
- **Infinite PGFs**, such as $\frac{k}{(a - bt)^n}$ or $\frac{kt}{a - bt}$: write the bracket as $a^{-n} \left(1 - \frac{b}{a}t\right)^{-n}$ and expand with the binomial series as far as the power you need. This is far shorter than repeated differentiation. If you do use derivatives, remember the $r!$: $P(X = r) = G^{(r)}(0)/r!$.
- **Only even values** (a PGF containing only even powers of t): odd values have probability 0. $P(X = 2)$ is the coefficient of t^2 , which is the **first** term of the series after the constant.
- **"State the most probable value"**: the power of t whose coefficient is largest in the expanded polynomial. Give the power (the value), not the coefficient, and not the mean.
- **A probability for a sum from two expansions**: you can list the pairs of values that give the total and multiply their probabilities, which is the same as picking the coefficient from the product.

5. Finding unknown constants (9 questions)

1. **Use** $G(1) = 1$. This is always the first equation.
2. **Use any other fact given**: a mean ($G'(1) = \dots$), a known coefficient, $P(X = 0) = G(0)$.
3. **Solve** the equations together, then carry on with the numbers.

For example, $G(t) = \frac{k}{m - t}$ with $P(X = 0) = 0.6$: $G(0) = \frac{k}{m} = 0.6$ and $G(1) = \frac{k}{m - 1} = 1$, so $k = m - 1$ and $\frac{m - 1}{m} = 0.6$, giving $m = 2.5$ and $k = 1.5$. So $G(t) = \frac{3}{5 - 2t}$, and then $G'(t) = \frac{6}{(5 - 2t)^2}$ and $G''(t) = \frac{24}{(5 - 2t)^3}$ give $E(X) = \frac{2}{3}$ and $\text{Var}(X) = \frac{8}{9} + \frac{2}{3} - \frac{4}{9} = \frac{10}{9}$.

Variations you will meet:

- **A PGF built from unknown probabilities**, for example two coins with unknown chances a and b of a head and given coefficients: the coefficient of t^2 is ab , the coefficient of t is $a(1 - b) + (1 - a)b$. Solve for a and b . Then the constant term is $(1 - a)(1 - b)$; don't leave it out.
- **Constants in an infinite PGF**: put $t = 1$ as usual; for example $\frac{k}{m - nt}$ needs $k = m - n$.
- **Keep letters when asked** ("in terms of a "), and use the constant you found in every later part.

6. Standard distributions: derive or recognise the PGF (4 questions)

1. **Derive:** write $G(t) = \sum P(X = r)t^r$ with the distribution's formula, take out the common factor, and recognise the sum (a geometric series for $\text{Geo}(p)$, the e^x series for $\text{Po}(\lambda)$, the binomial theorem for $B(n, p)$). Name the series and its common ratio.
2. **Mean and variance from it:** differentiate twice, put $t = 1$, and replace $1 - q$ by p where it appears. For a "show that", keep every line.
3. **Recognise:** a PGF of the form $(q + pt)^n$ with $p + q = 1$ is $B(n, p)$; $e^{\lambda(t-1)}$ is $\text{Po}(\lambda)$; $\frac{pe^{pt}}{1-qt}$ is $\text{Geo}(p)$. "Define fully" needs the name **and** the parameters.

For a sum of independent geometric variables (for example the total number of throws needed to get two separate successes), just multiply: $\left(\frac{pt}{1-qt}\right)^2$.

7. Working back from the PGF of a sum (4 questions)

You are given $G_Y(t)$, where Y is the sum of two (or n) independent observations of X , and must find $G_X(t)$. It is the square root (or n th root) of the **whole** PGF, not of each term.

1. **If G_Y is factorised**, such as $\frac{t^2}{(a - bt)^2}$ or $\frac{1}{c}(\dots)^2(\dots)^2$, halve the powers.
2. **If it is an expanded polynomial**, suppose $G_X(t) = a + bt + ct^2$, square it and match coefficients: a^2 is the constant term, c^2 is the top term, $2ab$ is the t term. Take positive roots (they are probabilities) and check the middle term and that $a + b + c = 1$.

For example, $G_Y(t) = \frac{1}{16}t^2(1 + 2t + 5t^2 + 4t^3 + 4t^4)$. Every value of Y is at least 2, so each X is at least 1: take out t^2 now and put t back into G_X at the end. For the bracket, $a^2 = \frac{1}{16}$ and $c^2 = \frac{4}{16}$ give $a = \frac{1}{4}$ and $c = \frac{1}{2}$; then $2ab = \frac{2}{16}$ gives $b = \frac{1}{4}$. The t^2 term checks: $b^2 + 2ac = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$. So $G_X(t) = t\left(\frac{1}{4} + \frac{1}{4}t + \frac{1}{2}t^2\right) = \frac{1}{4}t + \frac{1}{4}t^2 + \frac{1}{2}t^3$, and its coefficients add to 1.

Then answer the rest from G_X (for example a probability, by expanding).

8. A sum of variables that are not independent (1 question)

When X and Y come from the **same** experiment, $G_X G_Y$ is **not** the PGF of $X + Y$: the reason to give is that X and Y are not independent.

1. List every outcome of the experiment with its probability.
2. For each, write the values of X and Y , and their sum.
3. Collect the probabilities for each value of the sum and write the PGF.

For example, throw a fair coin twice; X is the number of heads and $Y = 1$ if the first throw is a head, 0 if not. The outcomes HH, HT, TH, TT give $X + Y = 3, 2, 1, 0$, each with probability $\frac{1}{4}$, so $G_{X+Y}(t) = \frac{1}{4}(1 + t + t^2 + t^3)$, which is not $G_X(t)G_Y(t) = \left(\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2\right)\left(\frac{1}{2} + \frac{1}{2}t\right)$.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

A bag contains 5 green counters and 3 yellow counters. Two counters are taken at random, without replacement. The random variable X is the number of yellow counters taken.

- (a) Find the probability generating function of X . **[3]**
- (b) Use the probability generating function to find $E(X)$ and $\text{Var}(X)$. **[4]**

(a) There are $\binom{8}{2} = 28$ equally likely pairs.

$$P(X = 0) = \frac{\binom{5}{2}}{28} = \frac{10}{28}, \quad P(X = 1) = \frac{5 \times 3}{28} = \frac{15}{28}, \quad P(X = 2) = \frac{\binom{3}{2}}{28} = \frac{3}{28}.$$

B1 for two correct probabilities.

$$G_X(t) = \frac{1}{28} (10 + 15t + 3t^2).$$

M1 for a quadratic in t with the probabilities on the right powers, **A1**. The coefficients add to $\frac{28}{28} = 1$.

(b) $G'_X(t) = \frac{1}{28}(15 + 6t)$, so $E(X) = G'_X(1) = \frac{21}{28} = \frac{3}{4}$. **M1 A1**

$G''_X(t) = \frac{6}{28}$, so $G''_X(1) = \frac{3}{14}$. **M1** for using the variance formula with these values:

$$\text{Var}(X) = \frac{3}{14} + \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{45}{112} = 0.402 \text{ (3 s.f.)}.$$

A1

Example 2

A fair four-sided spinner has sides numbered 1, 2, 3 and 4. The score when it is spun is U . A biased coin, for which the probability of a head is 0.3, is thrown twice, and V is the number of heads. U and V are independent, and $W = U + V$.

- (a) Write down the probability generating functions of U and V . **[2]**
- (b) Find the probability generating function of W , giving your answer as a polynomial in t . **[3]**

(c) Find $P(W \leq 2)$. **[1]**

(d) Use the probability generating function of W to find $E(W)$ and $\text{Var}(W)$. **[5]**

(a) $G_U(t) = \frac{1}{4}(t + t^2 + t^3 + t^4)$ **B1** (discrete uniform) and $G_V(t) = (0.7 + 0.3t)^2 = 0.49 + 0.42t + 0.09t^2$ **B1** (binomial, $n = 2$).

(b) $G_W(t) = G_U(t) G_V(t)$ because U and V are independent. **M1** for multiplying.

$$G_W(t) = \frac{1}{4}(t + t^2 + t^3 + t^4)(0.49 + 0.42t + 0.09t^2).$$

Collecting powers (for t^3 : $0.09 + 0.42 + 0.49 = 1$, and so on) **M1**:

$$G_W(t) = 0.1225t + 0.2275t^2 + 0.25t^3 + 0.25t^4 + 0.1275t^5 + 0.0225t^6.$$

A1. The coefficients add to 1.

(c) W cannot be 0, so $P(W \leq 2) = 0.1225 + 0.2275 = 0.35$. **B1**

(d) $G'_W(t) = 0.1225 + 0.455t + 0.75t^2 + t^3 + 0.6375t^4 + 0.135t^5$. **M1** for differentiating.

$E(W) = G'_W(1) = 3.1$. **A1** (Check: $E(U) + E(V) = 2.5 + 0.6$.)

$G''_W(t) = 0.455 + 1.5t + 3t^2 + 2.55t^3 + 0.675t^4$, so $G''_W(1) = 8.18$. **M1**

$$\text{Var}(W) = 8.18 + 3.1 - 3.1^2 = 1.67.$$

M1 for the formula with their values, **A1**. (Check: $\text{Var}(U) + \text{Var}(V) = 1.25 + 0.42$.)

Example 3

The discrete random variable X has probability generating function

$$G_X(t) = \frac{k + t}{4 - 2t},$$

where k is a constant.

(a) Find the value of k . **[1]**

(b) Find $E(X)$ and $\text{Var}(X)$. **[5]**

(c) Find $P(X = 3)$ and $P(X > 1)$. **[4]**

(a) $G_X(1) = \frac{k + 1}{2} = 1$, so $k = 1$. **B1**

(b) Quotient rule:

$$G'_X(t) = \frac{(4 - 2t) \times 1 - (1 + t) \times (-2)}{(4 - 2t)^2} = \frac{6}{(4 - 2t)^2} = 6(4 - 2t)^{-2}.$$

M1 for a correct method of differentiating. $E(X) = G'_X(1) = \frac{6}{4} = 1.5$. **A1**

$$G_X''(t) = 6 \times (-2)(4 - 2t)^{-3} \times (-2) = \frac{24}{(4 - 2t)^3}, \text{ so } G_X''(1) = 3. \text{ M1}$$

$$\text{Var}(X) = 3 + 1.5 - 1.5^2 = 2.25.$$

M1 A1

(c) Take out 4 so the bracket starts with 1, then use the series for $(1 - x)^{-1}$: **M1**

$$G_X(t) = \frac{1+t}{4} \left(1 - \frac{t}{2}\right)^{-1} = \frac{1}{4}(1+t) \left(1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} + \dots\right).$$

The t^3 terms come from $1 \times \frac{t^3}{8}$ and $t \times \frac{t^2}{4}$: **M1**

$$P(X = 3) = \frac{1}{4} \left(\frac{1}{8} + \frac{1}{4}\right) = \frac{3}{32}.$$

A1

$$P(X = 0) = \frac{1}{4} \text{ and } P(X = 1) = \frac{1}{4} \left(\frac{1}{2} + 1\right) = \frac{3}{8}, \text{ so } P(X > 1) = 1 - \frac{1}{4} - \frac{3}{8} = \frac{3}{8}. \text{ A1}$$

Example 4

The random variable X has the distribution $\text{Po}(\lambda)$.

(a) Show that the probability generating function of X is $e^{\lambda(t-1)}$. **[2]**

(b) Use the probability generating function to show that $E(X) = \text{Var}(X) = \lambda$. **[4]**

(c) The random variable Y has the distribution $\text{Po}(\mu)$ and is independent of X . Use probability generating functions to show that $X + Y$ has a Poisson distribution, stating its parameter. **[2]**

(d) Given that $\lambda = 1.5$ and $\mu = 2.5$, find $P(X + Y = 2)$. **[2]**

(a)

$$G_X(t) = \sum_{r=0}^{\infty} \frac{e^{-\lambda} \lambda^r}{r!} t^r = e^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda t)^r}{r!} = e^{-\lambda} e^{\lambda t} = e^{\lambda(t-1)}.$$

M1 for the sum with t^r , **A1** for recognising the series for $e^{\lambda t}$.

(b) $G_X'(t) = \lambda e^{\lambda(t-1)}$ and $G_X''(t) = \lambda^2 e^{\lambda(t-1)}$. **M1** for both derivatives.

At $t = 1$, $e^0 = 1$, so $E(X) = G_X'(1) = \lambda$ **A1** and $G_X''(1) = \lambda^2$. **M1**

$$\text{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

A1

(c) By independence, $G_{X+Y}(t) = e^{\lambda(t-1)} e^{\mu(t-1)} = e^{(\lambda+\mu)(t-1)}$. **M1**

This is the PGF of $\text{Po}(\lambda + \mu)$, so $X + Y \sim \text{Po}(\lambda + \mu)$. **A1**

(d) $X + Y \sim \text{Po}(4)$, so **M1**

$$P(X + Y = 2) = \frac{e^{-4}4^2}{2!} = 8e^{-4} = 0.147 \text{ (3 s.f.)}.$$

A1

Common mistakes

- **Adding PGFs instead of multiplying them.** Examiners have seen this in several sessions. The PGF of a sum of independent variables is the **product**.
- **Multiplying when the variables are not independent.** If both variables come from the same draw or throw, you must find the distribution of the sum directly. When asked why a claimed product is wrong, "they are not independent" is the answer.
- **Putting a probability on the wrong power of t .** $P(X = r)$ belongs to t^r . Write the values first, then match each probability to its power.
- **Leaving out the constant term.** If X can be 0, the PGF has a constant term. A PGF whose coefficients don't add to 1 is wrong: always check $G(1) = 1$.
- **Differentiating a fraction as if its top were a constant.** For a fraction with t in the numerator, such as $\frac{t}{(5-t)^3}$, the second derivative needs the product or quotient rule, not just the chain rule on the bottom.
- **Using repeated differentiation to find a far-off probability.** Expanding with the binomial series is shorter and safer. If you do differentiate r times, divide by $r!$.
- **Square-rooting each term** when working back from the PGF of a sum. The square root of $a^2 + 2abt + \dots$ is not the sum of the square roots of the terms. Match coefficients instead.
- **Answering "most probable value" with the wrong thing.** It is the **power** of t with the largest coefficient, not the coefficient itself and not the mean rounded.
- **Assuming a second PGF is the same as the first** because the situations sound alike. Work each one out.
- **Arithmetic slips when expanding products.** Multiply with a common denominator and whole numbers, collect like powers carefully, and check the total is 1.

How to get the marks

- **Show the PGF with its powers.** Probabilities alone, or a list of coefficients, don't earn the PGF mark. Write $G_X(t) = \dots$ as an expression in t .
- **Probabilities first.** The first mark for building a PGF is usually for correct probabilities, so show how you got them (combinations, a tree, a product).
- **"Use the PGF" means differentiate it.** Show $G'(t)$ (and $G''(t)$), substitute $t = 1$, and write the values. Finding the mean another way can lose the marks.
- **Write the variance formula with your numbers in** before simplifying. A method mark is for using $G''(1) + G'(1) - (G'(1))^2$ correctly, even with a wrong earlier value.
- **"Expressing your answer as a polynomial"** means expanded and collected, with numerical coefficients. A product of brackets won't do.

- **"Show that"**: the answer is printed, so every step must be visible, including why $G_X(1) = 1$ or how $1 - q$ becomes p . Name the series you sum (geometric series, the e^x series).
- **"Define fully"** a distribution: its name and all its parameters, for example $B(12, 0.35)$.
- **Probabilities as coefficients**: say which coefficient you are finding ("the coefficient of t^5 "). Exact fractions are best; otherwise 3 significant figures.
- **Follow-through**. A wrong PGF used correctly afterwards usually still earns the method marks, as long as its coefficients add to 1. So check that, and keep going.

Quick check

1. The random variable X has $P(X = 0) = 0.3$, $P(X = 2) = 0.5$ and $P(X = 5) = 0.2$. Write down its PGF and use it to find $E(X)$ and $\text{Var}(X)$.
2. The PGF of X is $(0.9 + 0.1t)^6$. State the distribution of X , and find $E(X)$ from the PGF and $P(X = 2)$.
3. The PGF of X is $G_X(t) = \frac{3}{8 - 5t}$. Find $P(X = 1)$ and $E(X)$.
4. X is equally likely to be 0, 1 or 2, and Y is the sum of three independent observations of X . Find the PGF of Y as a polynomial, $P(Y = 4)$, the most probable value of Y , and $\text{Var}(Y)$.
5. X has PGF $a + bt + ct^2$, $E(X) = 0.8$ and $\text{Var}(X) = 0.56$. Find a , b and c .

Answers

1. $G(t) = 0.3 + 0.5t^2 + 0.2t^5$. $G'(t) = t + t^4$, so $E(X) = 2$. $G''(t) = 1 + 4t^3$, so $G''(1) = 5$ and $\text{Var}(X) = 5 + 2 - 4 = 3$.
2. $X \sim B(6, 0.1)$. $G'(t) = 0.6(0.9 + 0.1t)^5$, so $E(X) = 0.6$. $P(X = 2)$ is the coefficient of t^2 : $\binom{6}{2}(0.1)^2(0.9)^4 = 15 \times 0.01 \times 0.6561 = 0.0984$ (3 s.f.).
3. $G_X(t) = \frac{3}{8} \left(1 - \frac{5}{8}t\right)^{-1} = \frac{3}{8} \left(1 + \frac{5}{8}t + \dots\right)$, so $P(X = 1) = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$. $G'_X(t) = \frac{15}{(8 - 5t)^2}$, so $E(X) = \frac{15}{9} = \frac{5}{3}$.
4. $G_X(t) = \frac{1}{3}(1 + t + t^2)$, so $G_Y(t) = \frac{1}{27}(1 + t + t^2)^3 = \frac{1}{27}(1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + t^6)$. $P(Y = 4) = \frac{6}{27} = \frac{2}{9}$. The largest coefficient is on t^3 , so the most probable value is 3. $G'_Y(1) = \frac{81}{27} = 3$ and $G''_Y(1) = \frac{216}{27} = 8$, so $\text{Var}(Y) = 8 + 3 - 9 = 2$.
5. $G'(1) = b + 2c = 0.8$ and $G''(1) = 2c$, so $0.56 = 2c + 0.8 - 0.64$, giving $c = 0.2$. Then $b = 0.4$, and $a + b + c = 1$ gives $a = 0.4$.

Past questions to try

- **9231/42/M/J/22 Q2**: a variance from a given PGF, then the PGF of X from the PGF of a sum.
- **9231/43/O/N/23 Q5**: derive the geometric PGF and its variance, then a sum of two geometric variables.
- **9231/43/M/J/24 Q4**: a constant, a mean, and a sum of two observations with a probability from a binomial expansion.
- **9231/43/M/J/21 Q4**: mean and variance from a rational PGF, and a probability from its series.