

Proof by induction AS

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Read first: [Summation of series](#), [Matrices](#)

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What you need to know

By the end of this topic you should be able to:

- 1. Prove a given result by mathematical induction.** The result is a statement about every positive integer n (or every n from some starting value), and it can be:
 - a formula for a sum, such as $\sum_{r=1}^n r(r+3)$;
 - a formula for the n th term of a sequence defined by a recurrence relation (u_{n+1} in terms of u_n), or an inequality every term satisfies;
 - a formula for the n th power of a matrix;
 - a statement that an expression such as $3^{2n+1} + 5 \times 13^n - 8$ is divisible by a given number;
 - a formula for the n th derivative of a function;
 - an inequality between two expressions in n .
- 2. Guess, then prove.** Recognise when working out the first few cases (for example the first few derivatives, or the first few sums) shows a pattern, write the pattern as a conjecture for general n , and then prove it by induction. You only need to do this in simple cases.

Understand it

A chain of statements

Induction proves infinitely many statements at once: one for $n = 1$, one for $n = 2$, one for $n = 3$, and so on. Call the statement for a particular n " H_n ". Two facts are enough:

- 1. H_1 is true** (check it directly).
- 2. Whenever H_k is true, H_{k+1} is true as well**, for any positive integer k .

Then H_1 is true by fact 1. Using fact 2 with $k = 1$, H_2 is true. Using it with $k = 2$, H_3 is true, and so on for ever. It is like a line of dominoes: the first one falls, and each one knocks over the next, so they all fall.

Fact 2 is a statement about **one link in the chain**, not about H_k on its own. You do not prove H_k ; you **assume** it and show that H_{k+1} follows. That is why assuming it is not cheating: fact 1 is what gets the chain started.

The four parts of every proof

Every induction proof in 9231 has the same four parts, and each earns its own mark:

1. **Base case.** Show H_1 is true: work out **both** sides for $n = 1$ and show they agree (or show the number is divisible, or the inequality holds). Write "so H_1 is true".
2. **Inductive hypothesis.** "Assume H_k is true **for some** positive integer k ", and write out what H_k says as an equation, with k in place of n .
3. **Inductive step.** Using the hypothesis, show that H_{k+1} is true. It helps to write down first what H_{k+1} says (replace every n by $k + 1$), so you know the target.
4. **Conclusion.** " H_k true implies H_{k+1} true, and H_1 is true, so by induction H_n is true for all positive integers n ."

A complete example

Prove that $\sum_{r=1}^n r(r+3) = \frac{1}{3}n(n+1)(n+5)$ for all positive integers n .

Base case. For $n = 1$: the left side is $1 \times 4 = 4$ and the right side is $\frac{1}{3} \times 1 \times 2 \times 6 = 4$. They agree, so H_1 is true.

Hypothesis. Assume that, for some positive integer k , $\sum_{r=1}^k r(r+3) = \frac{1}{3}k(k+1)(k+5)$.

Target. H_{k+1} says $\sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}(k+1)(k+2)(k+6)$.

Step. The sum to $k + 1$ terms is the sum to k terms plus the $(k + 1)$ th term, $(k + 1)(k + 4)$:

$$\sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4) = \frac{1}{3}(k+1)(k^2 + 5k + 3k + 12) = \frac{1}{3}(k+1)(k+2)(k+6).$$

Taking out the common factor $(k + 1)$ straight away keeps the algebra short. This is the target, so H_{k+1} is true.

Conclusion. H_k true implies H_{k+1} true, and H_1 is true, so by induction the result is true for all positive integers n .

Guess first, then prove

Induction can only prove a formula you already have. When a question asks you to **find** a formula, work out the first few cases, spot the pattern and then prove it. For example, the sums of odd numbers are $1 = 1$, $1 + 3 = 4$, $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$. They are the square numbers, so the conjecture is $\sum_{r=1}^n (2r - 1) = n^2$. The step is quick: $k^2 + (2k + 1) = (k + 1)^2$. Trying a few cases never proves the formula on its own; the induction does.

Starting somewhere else

If a result is claimed "for all integers $n \geq 2$ " (or $n \geq 7$), the base case is that first value, not $n = 1$, and the hypothesis is "assume true for some integer $k \geq 2$ ". Everything else is the same. Some results really do fail for small n : $n! > 3^n$ is false for $n = 6$ ($720 < 729$) but true from $n = 7$ onwards.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\sum_{r=1}^n r = \frac{1}{2}n(n+1), \sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1), \sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$	Given
Product rule: $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$	Given
$\frac{d}{dx}x^n = nx^{n-1}, \frac{d}{dx}\ln x = \frac{1}{x}, \frac{d}{dx}e^x = e^x, \frac{d}{dx}\sin x = \cos x, \frac{d}{dx}\cos x = -\sin x$	Given
$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$	Given
The four parts: base case, hypothesis "for some k ", step to $k+1$, conclusion "for all n "	Learn it
Sum to $k+1$ terms = sum to k terms + the $(k+1)$ th term	Learn it
Chain rule: $\frac{d}{dx}(ax+b)^m = am(ax+b)^{m-1}$	Learn it
$a^{k+1} = a \times a^k, (k+1)! = (k+1) \times k!, (-1)^{k+1} = -(-1)^k$	Learn it
$\cos \theta = \sin(\theta + \frac{1}{2}\pi)$ and $-\sin \theta = \cos(\theta + \frac{1}{2}\pi)$ (from the given $\sin(A+B)$ and $\cos(A+B)$)	Learn it
$a^{2n} - 1 = (a^n - 1)(a^n + 1)$ (difference of two squares)	Learn it
Inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}; \det(\mathbf{M}^n) = (\det \mathbf{M})^n$	Learn it

How to do it

In the Paper 1 questions we have tagged for this topic (32 questions, 2021–2025; 22 different ones, because the same question often appears on two papers of a session), the types came up this often. Each question is one type, so the counts add up to 32; five of them also have parts on another topic (summation of series or matrices).

Question type	Questions
n th derivatives	8
Divisibility	8
Inequalities, including bounds on a sequence	6
Sums of series	4
n th term of a sequence from a recurrence relation	3
Powers of a matrix	3

1. n th derivatives (8 questions)

1. **Base case.** Differentiate once (usually with the product rule) and show the result matches the formula with $n =$

1. Put $n = 1$ into the formula visibly, for example " $(-1)^1(1-1)!\dots$ ", so the examiner sees the link.

2. **Hypothesis.** Assume $\frac{d^k y}{dx^k} = \dots$ (the formula with k) for some positive integer k .

3. **Step.** Differentiate the hypothesis once more. Everything that does not contain x ($(-1)^k$, $k!$, 2^k) is a constant: leave it outside and differentiate only the part with x .
4. **Tidy up** to the formula with $n = k + 1$. Use $(-1)^k \times (-1) = (-1)^{k+1}$, $k! \times (k + 1) = (k + 1)!$ and $2^k \times 2 = 2^{k+1}$.
5. **Conclude.**

Variations you will meet:

- **Odd-numbered derivatives**, such as $\frac{d^{2n-1}y}{dx^{2n-1}}$. When n goes up by 1, $2n - 1$ goes up by 2, so the step needs **two** differentiations. Do them one at a time and write each result.
- **Starting at $n = 2$** , for example when the first derivative has an extra term. The base case is then the second derivative: differentiate twice.
- **A general function f** with a given property (for example a rule linking f'' to f). Use the product rule with f , f' and f'' as letters, and use the given property to replace higher derivatives whenever they appear.
- **"... where $P_n(x)$ is a polynomial of degree ..."**. You do not know P_k exactly, so differentiate the general form ($P_k(x)$ times a power of a bracket) with the product rule, take out the lowest power of the bracket, and show that what is left, $P_{k+1}(x)$, is a polynomial of the right degree (look at the highest power of x in each part).
- **Trigonometric patterns.** $\frac{d}{dx} \sin(ax + \theta) = a \cos(ax + \theta) = a \sin(ax + \theta + \frac{1}{2}\pi)$, so each derivative adds $\frac{1}{2}\pi$ inside the bracket.
- **"Find the n th derivative"** with no formula given: find the first three or four, spot the pattern (see Example 3), then prove it.

2. Divisibility (8 questions)

Call the expression $f(n)$. To show $f(n)$ is divisible by d :

1. **Base case.** Work out $f(1)$ as a number and write it as a multiple of d , for example $84 = 7 \times 12$.
2. **Hypothesis.** Assume $f(k)$ is divisible by d for some positive integer k . Writing $f(k) = dm$ for an integer m keeps the step clear.
3. **Step.** Write $f(k + 1)$ and split each power: $a^{k+1} = a \times a^k$ and $a^{2(k+1)} = a^2 \times a^{2k}$. Then either
 - **rewrite each multiplier as "a multiple of d , plus the rest"**, for example $13 = 12 + 1$, so that $f(k + 1) = f(k) + (\text{terms that are clearly multiples of } d)$; or
 - **work out $f(k + 1) - f(k)$** (or $f(k + 1) - af(k)$ for a suitable a) and show it is a multiple of d . Then **say** why this finishes the job: $f(k + 1) = f(k) + (\text{a multiple of } d)$, and $f(k)$ is a multiple of d by the hypothesis.
4. **Conclude**, repeating what was proved: " $f(n)$ is divisible by d for all positive integers n ".

The bases of the powers in these questions are usually chosen to be one more than a multiple of d (or of a factor of d), as $13 = 12 + 1$ is in Example 1. Look for that first.

Variations you will meet: powers like a^{3n} or a^{4n} (then $a^{3(k+1)} = a^3 \times a^{3k}$, so the multiplier is a^3).

An extension (not yet in a past paper): a term such as $10n$ in the expression. Subtract a multiple of $f(k)$ so that the powers cancel, then check that what is left is a multiple of d (Quick check 1).

3. Inequalities, including bounds on a sequence (6 questions)

An inequality in n , such as $2^n > 2n + 1$ for $n \geq 3$:

1. **Base case** at the first allowed n : $2^3 = 8 > 7$.
2. **Hypothesis**: assume $2^k > 2k + 1$ for some integer $k \geq 3$.
3. **Step**: start from the left side of H_{k+1} , write it using the left side of H_k , and use the hypothesis: $2^{k+1} = 2 \times 2^k > 2(2k + 1) = 4k + 2$. Only multiply an inequality by something **positive**.
4. **Compare with the target**: $4k + 2 = (2k + 3) + (2k - 1)$, and $2k - 1 > 0$, so $2^{k+1} > 2k + 3 = 2(k + 1) + 1$. Throwing away a positive extra piece keeps the inequality true.
5. **Conclude**.

A bound on a sequence, such as "prove $u_n < c$ for all n " when u_{n+1} is given in terms of u_n :

1. **Base case**: it is usually **given** (for example " $u_1 < c$ ", or $u_1 = 1$). Say so. Don't work out u_2 instead.
2. **Hypothesis**: assume $u_k < c$ for some positive integer k .
3. **Step**: find the sign of $c - u_{k+1}$ (the question often tells you to "consider" it). Substitute the recurrence, put everything over one denominator and factorise, so that the top contains $(c - u_k)$. Then use the hypothesis and the fact that the terms are positive to decide the sign. If the recurrence has a square root, you can instead use " $u_k < c \Rightarrow u_k + b < c + b \Rightarrow \sqrt{u_k + b} < \sqrt{c + b}$ ", because $\sqrt{}$ is an increasing function.
4. **Conclude**.
 - **A lower bound**, "prove $u_n > c$ ": the same steps, but find the sign of $u_{k+1} - c$ instead. Put it over one denominator, factorise so that $(u_k - c)$ appears, and use the hypothesis $u_k > c$ and the positive terms to show it is positive.

A follow-on part usually asks you to show the sequence is increasing (or decreasing). Find $u_{n+1} - u_n$ over one denominator, factorise, and use the bound you have just proved to fix its sign. This part is not induction.

Sometimes the inequality is between logarithms. Use the laws of logarithms, such as $\ln(x^m) = m \ln x$, and any inequality the question gives you, to get from $\ln u_{k+1}$ to an expression in $\ln u_k$; **then** use the hypothesis.

4. Sums of series (4 questions)

1. **Base case**: work out the first term, and the formula with $n = 1$, and show they are equal.
2. **Hypothesis**: assume $\sum_{r=1}^k (\dots) = (\text{formula with } k)$.
3. **Step**: $\sum_{r=1}^{k+1} = \sum_{r=1}^k + (\text{term with } r = k + 1)$. Replace the sum to k by the hypothesis, and add the new term.
4. **Simplify to the target**, the formula with $n = k + 1$. Take out common factors first (see the complete example in "Understand it"); if the formula has a fraction such as $\frac{\dots}{(1-x)^2}$, put everything over that denominator. If you do expand everything, expand the target too and compare.
5. **Conclude**.

A later part may ask you to use the proved sum together with the MF19 sums to find another sum; that uses the summation of series methods.

5. n th term of a sequence from a recurrence relation (3 questions)

For example, $u_1 = 3$ and $u_{n+1} = 5u_n - 8$; prove that $u_n = 2 + 5^{n-1}$.

1. **Base case:** $u_1 = 3$ and $2 + 5^0 = 3$, so H_1 is true.
2. **Hypothesis:** assume $u_k = 2 + 5^{k-1}$ for some positive integer k .
3. **Step:** put the hypothesis into the recurrence: $u_{k+1} = 5(2 + 5^{k-1}) - 8 = 2 + 5^k$, which is the formula with $n = k + 1$.
4. **Conclude.**

A follow-on part may say "deduce": use the formula you have proved rather than the recurrence. Write the terms the question mentions with that formula, then look for a factorisation (such as a difference of two squares) or a simplification that shows the property asked for.

6. Powers of a matrix (3 questions)

1. **Base case:** put $n = 1$ into the claimed $\mathbf{M}^n$ and show it simplifies to $\mathbf{M}$.
2. **Hypothesis:** write out $\mathbf{M}^k$ as a matrix.
3. **Step:** $\mathbf{M}^{k+1} = \mathbf{M}^k \mathbf{M}$ (or $\mathbf{M} \mathbf{M}^k$). **Multiply** the matrices, never add them, and show each entry before simplifying it to the formula with $n = k + 1$.
4. **Conclude.**

If the claim is about a multiple, such as $3^n \mathbf{M}^n$, multiply $3^k \mathbf{M}^k$ by $3\mathbf{M}$. A follow-on part may ask for $(\mathbf{M}^n)^{-1}$: use the 2×2 inverse on the proved formula, with $\det(\mathbf{M}^n) = (\det \mathbf{M})^n$ as a check.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
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Example 1

Prove by mathematical induction that $3^{2n+1} + 5 \times 13^n - 8$ is divisible by 12 for all positive integers n . **[6]**

Let $f(n) = 3^{2n+1} + 5 \times 13^n - 8$.

Base case: $f(1) = 27 + 65 - 8 = 84 = 7 \times 12$, which is divisible by 12. **B1**

Hypothesis: assume $f(k)$ is divisible by 12 for some positive integer k . **B1**

Step: $3^{2(k+1)+1} = 3^2 \times 3^{2k+1}$ and $13^{k+1} = 13 \times 13^k$, so

$$f(k+1) = 9 \times 3^{2k+1} + 13 \times 5 \times 13^k - 8 = (8+1)3^{2k+1} + (12+1)5 \times 13^k - 8.$$

M1 for splitting the powers (or for working out $f(k+1) - f(k)$).

$$f(k+1) = \underbrace{3^{2k+1} + 5 \times 13^k - 8}_{f(k)} + 8 \times 3^{2k+1} + 60 \times 13^k = f(k) + 12(2 \times 3^{2k} + 5 \times 13^k).$$

A1. The second part is a multiple of 12, since $8 \times 3^{2k+1} = 24 \times 3^{2k}$, and $f(k)$ is divisible by 12 by the hypothesis, so $f(k+1)$ is divisible by 12. **A1**

Conclusion: $f(1)$ is divisible by 12, and if $f(k)$ is divisible by 12 then so is $f(k+1)$. So by induction $3^{2n+1} + 5 \times 13^n - 8$ is divisible by 12 for all positive integers n . **A1**

Example 2

Let $\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.

(a) Prove by mathematical induction that, for all positive integers n , $\mathbf{M}^n = \begin{pmatrix} 2^n & 3^n - 2^n \\ 0 & 3^n \end{pmatrix}$. [5]

(b) Find $(\mathbf{M}^n)^{-1}$ in terms of n . [2]

(a) Base case: with $n = 1$ the formula gives $\begin{pmatrix} 2 & 3-2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \mathbf{M}$, so it is true for $n = 1$. **B1**

Hypothesis: assume $\mathbf{M}^k = \begin{pmatrix} 2^k & 3^k - 2^k \\ 0 & 3^k \end{pmatrix}$ for some positive integer k . **B1**

Step:

$$\mathbf{M}^{k+1} = \mathbf{M}^k \mathbf{M} = \begin{pmatrix} 2^k & 3^k - 2^k \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 2^k + 3^{k+1} - 3 \times 2^k \\ 0 & 3^{k+1} \end{pmatrix}.$$

M1 for multiplying $\mathbf{M}^k$ by $\mathbf{M}$.

The top-right entry is $3^{k+1} - 2 \times 2^k = 3^{k+1} - 2^{k+1}$, so $\mathbf{M}^{k+1} = \begin{pmatrix} 2^{k+1} & 3^{k+1} - 2^{k+1} \\ 0 & 3^{k+1} \end{pmatrix}$, the formula with $n = k + 1$. **A1**

Conclusion: true for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so by induction it is true for all positive integers n . **A1**

(b) $\det(\mathbf{M}^n) = 2^n \times 3^n - 0 = 6^n$ (as a check, $\det \mathbf{M} = 6$). **B1**

$$(\mathbf{M}^n)^{-1} = \frac{1}{6^n} \begin{pmatrix} 3^n & 2^n - 3^n \\ 0 & 2^n \end{pmatrix} = \begin{pmatrix} 2^{-n} & 3^{-n} - 2^{-n} \\ 0 & 3^{-n} \end{pmatrix}.$$

B1. It is the same formula with $-n$ in place of n , as it should be.

Example 3

It is given that $y = \frac{1}{1+2x}$.

(a) Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$, and hence suggest a formula for $\frac{d^ny}{dx^n}$. [3]

(b) Prove your formula by mathematical induction. **[5]**

(a) $y = (1 + 2x)^{-1}$, so, by the chain rule each time,

$$\frac{dy}{dx} = -2(1 + 2x)^{-2}, \quad \frac{d^2y}{dx^2} = 8(1 + 2x)^{-3}, \quad \frac{d^3y}{dx^3} = -48(1 + 2x)^{-4}.$$

B1 B1 (first two; third). Write the numbers so the pattern shows: $-2 = (-2)^1 \times 1!$, $8 = (-2)^2 \times 2!$ and $-48 = (-2)^3 \times 3!$, and the power of $(1 + 2x)$ is always one more than the number of differentiations. So the conjecture is

$$\frac{d^n y}{dx^n} = (-2)^n n! (1 + 2x)^{-(n+1)}.$$

B1

(b) **Base case:** with $n = 1$ the formula gives $(-2)^1 \times 1! \times (1 + 2x)^{-2} = -2(1 + 2x)^{-2}$, which is $\frac{dy}{dx}$. So it is true for $n = 1$. **B1**

Hypothesis: assume $\frac{d^k y}{dx^k} = (-2)^k k! (1 + 2x)^{-(k+1)}$ for some positive integer k . **B1**

Step: differentiate once more; $(-2)^k k!$ is a constant.

$$\frac{d^{k+1} y}{dx^{k+1}} = (-2)^k k! \times (-(k+1))(1 + 2x)^{-(k+2)} \times 2$$

M1 for differentiating the hypothesis, with the chain rule.

$$= (-2)^{k+1} (k+1)! (1 + 2x)^{-(k+2)},$$

because $(-2)^k \times (-2) = (-2)^{k+1}$ and $k! \times (k+1) = (k+1)!$. This is the formula with $n = k + 1$. **A1**

Conclusion: true for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so by induction $\frac{d^n y}{dx^n} = (-2)^n n! (1 + 2x)^{-(n+1)}$ for all positive integers n . **A1**

Example 4

The sequence $u_1, u_2, u_3, \dots$ is such that $u_1 = 1$ and $u_{n+1} = \sqrt{u_n + 6}$ for $n \geq 1$.

(a) Prove by mathematical induction that $0 < u_n < 3$ for all positive integers n . **[5]**

(b) Show that $u_{n+1} > u_n$ for all $n \geq 1$. **[3]**

(a) **Base case:** $u_1 = 1$, and $0 < 1 < 3$, so it is true for $n = 1$. **B1**

Hypothesis: assume $0 < u_k < 3$ for some positive integer k . **B1**

Step: consider $3 - u_{k+1}$. Multiply top and bottom by $3 + \sqrt{u_k + 6}$:

$$3 - u_{k+1} = 3 - \sqrt{u_k + 6} = \frac{9 - (u_k + 6)}{3 + \sqrt{u_k + 6}} = \frac{3 - u_k}{3 + u_{k+1}}.$$

M1. The top is positive by the hypothesis and the bottom is positive, so $3 - u_{k+1} > 0$, that is $u_{k+1} < 3$. Also $u_{k+1} = \sqrt{u_k + 6} \geq \sqrt{6} > 0$. **A1**

(Another way: $u_k < 3 \Rightarrow u_k + 6 < 9 \Rightarrow \sqrt{u_k + 6} < 3$, because the square root is an increasing function.)

Conclusion: true for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so by induction $0 < u_n < 3$ for all positive integers n . **A1**

(b) Compare the squares, since both terms are positive:

$$u_{n+1}^2 - u_n^2 = u_n + 6 - u_n^2 = (3 - u_n)(2 + u_n).$$

M1 A1. By part (a), $3 - u_n > 0$ and $2 + u_n > 0$, so $u_{n+1}^2 > u_n^2$. As u_{n+1} and u_n are both positive, $u_{n+1} > u_n$. **A1**

(The first few terms are 1, $\sqrt{7} = 2.646$, 2.940, 2.990 to 3 d.p.: rising, and staying below 3.)

Common mistakes

- **Assuming the result for every k .** The hypothesis is "assume it is true for **some** positive integer k ". Assuming it for all k assumes what you are trying to prove, and loses the mark.
- **A hypothesis with no algebra.** "Assume true for $n = k$ " on its own is not enough, and neither is "assume $3^{2k+1} + 5 \times 13^k - 8$ is true". Write the statement: "assume $3^{2k+1} + 5 \times 13^k - 8$ is divisible by 12", or the equation with k in it.
- **A base case with no link.** Show both sides for $n = 1$ and say they are equal, or show the number and a multiple of the divisor, and then say "so it is true for $n = 1$ ". In a sequence question the base case is u_1 (usually given); working out u_2 is not a base case.
- **Starting at the wrong value.** If the result is for $n \geq 2$, the base case is $n = 2$.
- **Showing the difference, then stopping.** $f(k+1) - f(k)$ being a multiple of d proves nothing on its own. Say that $f(k+1) = f(k) + (\text{that multiple})$ and that $f(k)$ is a multiple of d by the hypothesis.
- **Stopping when the algebra is done.** The final conclusion earns a mark: "true for $n = 1$; true for $n = k$ implies true for $n = k + 1$; so true for all positive integers n ". It only earns it if the earlier steps are all there.
- **Wrong inequality steps.** Multiplying or dividing by a negative reverses an inequality, and you cannot subtract one inequality from another. Before multiplying through by u_n (or any expression), say why it is positive. Working with the **sign** of one expression, such as $c - u_{k+1}$ written as a single factorised fraction, is safer than juggling several inequalities.
- **Matrix slips.** $\mathbf{M}^{k+1}$ is $\mathbf{M}^k$ **times** $\mathbf{M}$, not plus, and you cannot divide a matrix equation by a matrix. If the claim is about $3^n \mathbf{M}^n$, multiply by $3\mathbf{M}$, not by $\mathbf{M}$.
- **Differentiation errors in the step.** Most lost marks in derivative questions are in the differentiation, not the induction. Treat $(-1)^k$, $k!$ and similar factors as constants, use the product rule carefully, and when two derivatives are needed do them one at a time.
- **Jumping to the answer.** In the step, show the intermediate line (the multiplied-out matrix, the expression before factorising). A "show that" answer that appears without it loses the accuracy mark.

How to get the marks

- **The mark scheme follows the four parts.** A typical 6-mark proof is B1 (base case), B1 (hypothesis), M1 A1 (the step, as far as the target), A1 (why it gives the $k + 1$ case) and A1 (conclusion). Write each part as its own labelled line.
- **Write the target.** Replacing n by $k + 1$ in the given result before you start the step shows where you are going and makes the last line easy to check.

- **"Prove by (mathematical) induction"** means the proof must be by induction; a direct proof, however neat, earns nothing.
- **"Hence" or "deduce"** in a later part means use the result you have just proved, such as the formula for u_n or M^n .
- **"By considering $c - u_{n+1}$ "** tells you the method: find that expression as one fraction and decide its sign.
- **"Show that"** results are given, so every step must be shown and your last line must match the printed form exactly, with $k + 1$ in place of n .
- **"Suggest" or "find" the formula** (a conjecture): show the first few cases you worked out and state the general formula clearly before proving it.
- **The conclusion** should restate the result and the range: "for all positive integers n " (or "for all integers $n \geq 2$ ").

Quick check

1. Prove by mathematical induction that $4^{2n} + 10n - 1$ is divisible by 25 for all positive integers n .
2. Work out $\sum_{r=1}^n \frac{1}{(2r-1)(2r+1)}$ for $n = 1, 2$ and 3 , suggest a formula for the sum, and prove it by induction.
3. Prove by mathematical induction that $n! > 3^n$ for all integers $n \geq 7$.
4. Prove by mathematical induction that $\frac{d^n}{dx^n}(\sin 2x) = 2^n \sin\left(2x + \frac{1}{2}n\pi\right)$ for all positive integers n .
5. The sequence $u_1, u_2, \dots$ has $u_1 = 1$ and $u_{n+1} = 2u_n + n$. Prove by induction that $u_n = 3 \times 2^{n-1} - n - 1$ for all positive integers n .

Answers

1. Let $f(n) = 4^{2n} + 10n - 1 = 16^n + 10n - 1$. Base: $f(1) = 16 + 10 - 1 = 25$. Assume $f(k)$ is divisible by 25 for some positive integer k . Then $f(k+1) = 16 \times 16^k + 10k + 9$, and $f(k+1) - 16f(k) = 10k + 9 - 160k + 16 = 25 - 150k = 25(1 - 6k)$. So $f(k+1) = 16f(k) + 25(1 - 6k)$, which is divisible by 25 because $f(k)$ is. So by induction $f(n)$ is divisible by 25 for all positive integers n .
2. The sums are $\frac{1}{3}$, $\frac{1}{3} + \frac{1}{15} = \frac{2}{5}$ and $\frac{2}{5} + \frac{1}{35} = \frac{3}{7}$, which suggests $\frac{n}{2n+1}$. Base: $\frac{1}{1 \times 3} = \frac{1}{3} = \frac{1}{2+1}$. Assume the sum to k terms is $\frac{k}{2k+1}$. Then the sum to $k+1$ terms is $\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2(k+1)+1}$. So by induction the sum is $\frac{n}{2n+1}$ for all positive integers n .
3. Base: $7! = 5040 > 3^7 = 2187$. Assume $k! > 3^k$ for some integer $k \geq 7$. Then $(k+1)! = (k+1) \times k! > (k+1) \times 3^k > 3 \times 3^k = 3^{k+1}$, because $k+1 \geq 8 > 3$ and $3^k > 0$. So by induction $n! > 3^n$ for all $n \geq 7$. (For $n = 6$, $720 < 729$, which is why the result starts at 7.)
4. Base: $\frac{d}{dx} \sin 2x = 2 \cos 2x = 2 \sin(2x + \frac{1}{2}\pi)$. Assume $\frac{d^k}{dx^k} \sin 2x = 2^k \sin(2x + \frac{1}{2}k\pi)$ for some positive integer k . Differentiating, $\frac{d^{k+1}}{dx^{k+1}} \sin 2x = 2^k \times 2 \cos(2x + \frac{1}{2}k\pi) = 2^{k+1} \sin(2x + \frac{1}{2}k\pi + \frac{1}{2}\pi) = 2^{k+1} \sin(2x + \frac{1}{2}(k+1)\pi)$. So by induction it is true for all positive integers n .
5. Base: $3 \times 2^0 - 1 - 1 = 1 = u_1$. Assume $u_k = 3 \times 2^{k-1} - k - 1$ for some positive integer k . Then $u_{k+1} = 2(3 \times 2^{k-1} - k - 1) + k = 3 \times 2^k - k - 2 = 3 \times 2^{(k+1)-1} - (k+1) - 1$. So by induction the formula holds for all positive integers n . (Check: $u_2 = 3$, $u_3 = 8$, $u_4 = 19$.)

Past questions to try

- **9231/12/O/N/23 Q2:** the n th derivative of a product, with the product rule in the step.
- **9231/12/M/J/24 Q2:** divisibility with a power of the form a^{4n} .
- **9231/12/M/J/22 Q3:** a lower bound for a sequence by considering $u_{n+1} - c$, then showing it decreases.
- **9231/12/M/J/23 Q1:** a formula for a multiple of a matrix power, then its inverse.