

Rational functions and graphs

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What you need to know

By the end of this topic you should be able to:

1. **Sketch the graph of a simple rational function**, a fraction whose top and bottom are polynomials of degree at most 2, such as $\frac{2x+1}{x-3}$, $\frac{x^2-x-2}{x-3}$ or $\frac{x^2+3}{x^2+1}$. Your sketch must show the important features: the **asymptotes** (vertical, horizontal or oblique, that is, slanting), the **turning points** and where the curve **meets the axes**. You don't plot points in detail.
2. **Find the set of values that y can take** (or cannot take), usually by turning $y = f(x)$ into a quadratic in x and using its discriminant.
3. **Use the links between the graph of $y = f(x)$ and the graphs of $y = |f(x)|$, $y = f(|x|)$, $y = \frac{1}{f(x)}$ and $y^2 = f(x)$** : sketch each one from the sketch of $y = f(x)$.
4. **Use these sketches to solve equations and inequalities**, such as $|f(x)| < 4$, or to decide how many solutions $f(|x|) = k$ has.

Only numerators and denominators of degree 0, 1 or 2 are examined, so the only asymptotes you meet are vertical lines, horizontal lines and oblique straight lines.

Understand it

What a rational function looks like

A rational function is $y = \frac{P(x)}{Q(x)}$ with P and Q polynomials. Two things make its graph different from a polynomial's:

- where the **denominator is zero** the function is undefined, and close to such an x the value of y becomes huge (positive or negative);
- when x is very large, positive or negative, the curve settles down next to a straight line.

These lines the curve gets closer and closer to are the **asymptotes**. Almost every question in this topic starts by finding them, because once you have them, the stationary points and the intercepts, the sketch almost draws itself.

All of this note's diagrams except the last two use one function:

$$f(x) = \frac{x^2 - x - 2}{x - 3} = \frac{(x + 1)(x - 2)}{x - 3}.$$

Vertical asymptotes

Put the denominator equal to zero. Here $x - 3 = 0$, so $x = 3$ is a **vertical asymptote**. Just to the right of 3 (say $x = 3.01$) the top is about 4 and the bottom is a tiny positive number, so y is very large and positive; just to the left, the bottom is tiny and negative, so y is very large and negative. Checking the sign either side tells you which way each branch goes.

If the denominator can never be zero (for example $x^2 + 1$, or $x^2 + x + 1$, whose discriminant is $1 - 4 = -3 < 0$), the curve has **no vertical asymptotes** and is one unbroken curve.

Horizontal and oblique asymptotes

What happens for large $|x|$ depends on the degrees of the top and bottom.

- **Top degree smaller than bottom** (for example $\frac{x-1}{x^2-4}$): the bottom grows faster, so $y \rightarrow 0$. The asymptote is $y = 0$, the x -axis.
- **Same degree** (for example $\frac{3x^2+x}{x^2-5x+6}$): the x^2 terms dominate, so y tends to the ratio of the leading coefficients, here $\frac{3}{1}$. The asymptote is $y = 3$.
- **Top degree one more than bottom** (quadratic over linear): divide the top by the bottom to write $f(x)$ as a straight line plus a remainder fraction.

For our function, $x^2 - x - 2 = (x - 3)(x + 2) + 4$, so

$$f(x) = x + 2 + \frac{4}{x - 3}.$$

As $|x|$ grows, $\frac{4}{x-3} \rightarrow 0$, so the curve gets closer and closer to the **oblique asymptote** $y = x + 2$. The sign of the leftover fraction also tells you which side the curve is on: for large positive x it is positive, so the curve is **above** the line; for large negative x it is negative, so the curve is **below** it.

To do the division, write $x^2 - x - 2 = (x - 3)(x + c) + r$ and compare coefficients: the x terms give $c - 3 = -1$, so $c = 2$, and the constants give $-6 + r = -2$, so $r = 4$. Always find the **constant** c in the line; forgetting it is the classic slip.

Crossing an asymptote

A curve never crosses a **vertical** asymptote, but it can cross a horizontal or oblique one. To check, set $f(x)$ equal to the asymptote. For $y = \frac{x^2-x+1}{x^2+x+1}$ and its asymptote $y = 1$: $x^2 - x + 1 = x^2 + x + 1$ gives $x = 0$, so the curve crosses $y = 1$ at $(0, 1)$. For an oblique asymptote the equation has no solution (the remainder fraction is never 0), so a quadratic-over-linear curve never meets its oblique asymptote.

Turning points

Differentiate and set $\frac{dy}{dx} = 0$. The divided form is usually easiest:

$$\frac{dy}{dx} = 1 - \frac{4}{(x-3)^2} = 0 \Rightarrow (x-3)^2 = 4 \Rightarrow x = 1 \text{ or } x = 5.$$

Then $f(1) = 1$ and $f(5) = 9$. The point $(1, 1)$ is a **maximum** and $(5, 9)$ is a **minimum**. That looks odd (the maximum is lower than the minimum) until you see the graph: they are on different branches.

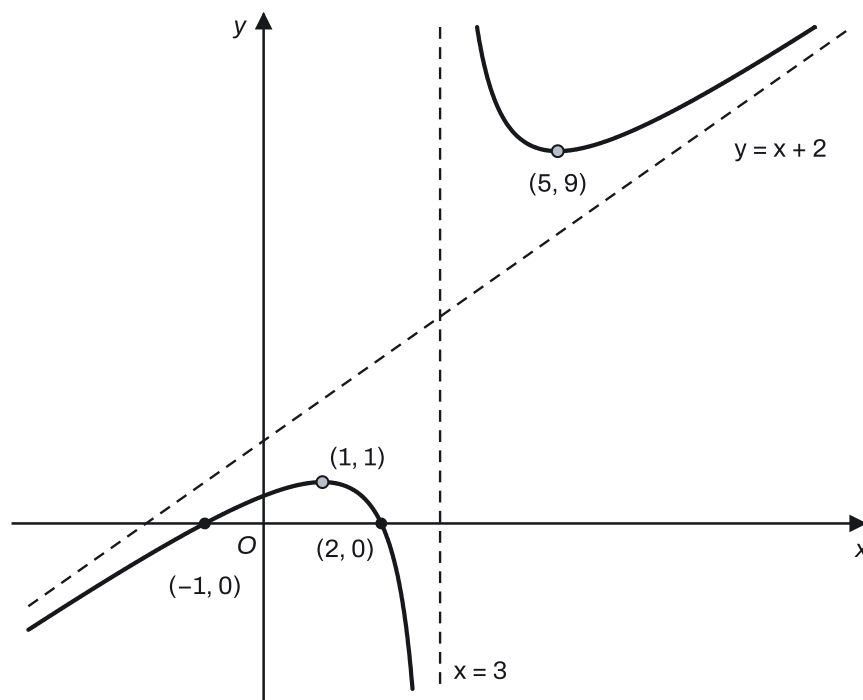
For a quadratic over a quadratic, use the quotient rule, $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{vu' - uv'}{v^2}$. Only the **numerator** can be zero, so set $vu' - uv' = 0$; the x^2 terms often cancel and leave a quadratic or even a linear equation.

To **show there are no stationary points**, show the equation $\frac{dy}{dx} = 0$ has no real roots (its discriminant is negative), or show $\frac{dy}{dx}$ always has one sign, for example $\frac{dy}{dx} = 1 + \frac{3}{(x+4)^2} > 0$ for every $x \neq -4$.

Intercepts

- **y-axis:** put $x = 0$. Here $y = \frac{-2}{-3} = \frac{2}{3}$, so $(0, \frac{2}{3})$.
- **x-axis:** put the **numerator** equal to 0. Here $(x+1)(x-2) = 0$, so $(-1, 0)$ and $(2, 0)$. If the numerator has no real roots, the curve doesn't meet the x -axis.

Putting it together gives the sketch.



The set of values y takes

The sketch suggests y never lies strictly between 1 and 9. To prove it without calculus, ask: **for which y does the equation $y = f(x)$ have a real solution x ?** Multiply up and collect terms as a quadratic in x :

$$x^2 - x - 2 = y(x - 3) \Rightarrow x^2 - (1 + y)x + (3y - 2) = 0.$$

A real x exists exactly when the discriminant is ≥ 0 :

$$(1 + y)^2 - 4(3y - 2) = y^2 - 10y + 9 = (y - 1)(y - 9) \geq 0,$$

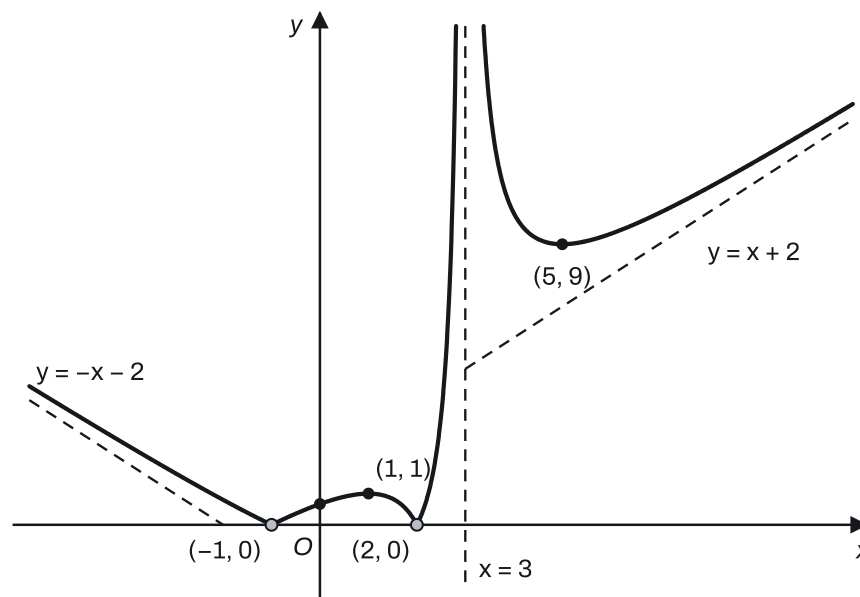
so $y \leq 1$ or $y \geq 9$. The curve takes every value except those with $1 < y < 9$. The boundary values 1 and 9 are the y -coordinates of the turning points: a boundary value gives a **repeated** root x , which is where the horizontal line touches the curve.

Watch the x^2 coefficient. For a quadratic over a quadratic, the equation in x has x^2 coefficient containing y , for example $(y - 1)x^2 + \dots$. For the value of y that makes it zero (usually the horizontal asymptote) the equation is not a quadratic, so check that value separately. For $y = \frac{x^2+2}{x^2+1}$ the equation is $(y - 1)x^2 + (y - 2) = 0$; the discriminant condition gives $1 \leq y \leq 2$, but $y = 1$ turns the equation into $-1 = 0$, which is impossible. So $y = 1$ is not taken, and the answer is $1 < y \leq 2$, with one strict sign.

The graph of $y = |f(x)|$

$|f(x)|$ is never negative. Keep every part of the curve that is on or above the x -axis, and **reflect every part below the x -axis in the x -axis**. Asymptotes below the axis are reflected too: here the left end followed $y = x + 2$ below the axis, so $|f(x)|$ follows $y = -x - 2$ there.

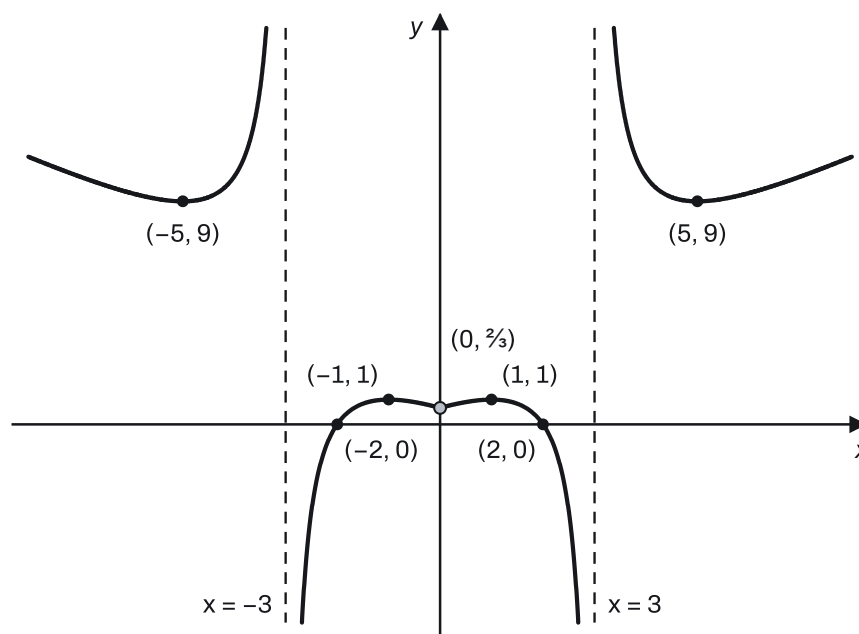
Where the curve crossed the x -axis, the new graph has a **sharp point** (a cusp) on the axis, not a smooth turning point.



The graph of $y = f(|x|)$

For $x \geq 0$, $|x| = x$, so the graph is the same as $y = f(x)$. For $x < 0$, $f(|x|) = f(-x)$, so the left half is the **mirror image of the right half in the y -axis**. So: throw away the part of the curve left of the y -axis, keep the right part, and reflect it in the y -axis. The graph is symmetrical about the y -axis, the vertical asymptote $x = 3$

gets a partner $x = -3$, and there is usually a **sharp point** on the y -axis (here at $(0, \frac{2}{3})$), because the gradient there is $f'(0) = \frac{5}{9}$, not 0).



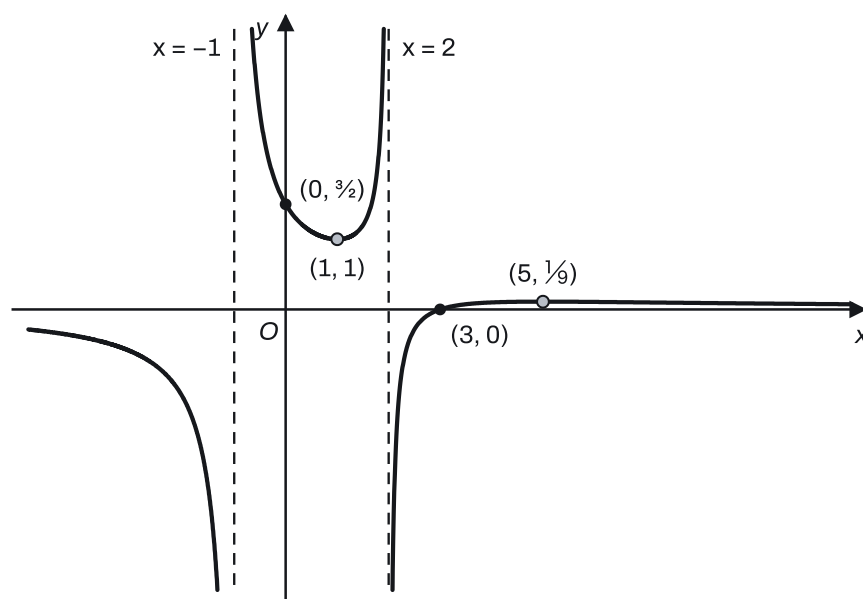
Don't mix the two up: $|f(x)|$ reflects in the x -axis (parts below), $f(|x|)$ reflects in the y -axis (the right half).

The graph of $y = \frac{1}{f(x)}$

Work from these rules, point by point:

- **Same sign.** Where f is positive, so is $\frac{1}{f}$; where f is negative, so is $\frac{1}{f}$.
- **Big becomes small.** Where f is large, $\frac{1}{f}$ is close to 0; where f is close to 0, $\frac{1}{f}$ is large.
- So **zeros of f become vertical asymptotes** of $\frac{1}{f}$, and **vertical asymptotes of f become zeros** of $\frac{1}{f}$.
- A **horizontal asymptote** $y = k$ ($k \neq 0$) becomes $y = \frac{1}{k}$. An **oblique asymptote** means f grows without limit, so $\frac{1}{f} \rightarrow 0$: the new asymptote is $y = 0$.
- **Stationary points keep their x -coordinate**, because $\frac{d}{dx} \left(\frac{1}{f} \right) = -\frac{f'(x)}{f(x)^2}$ is zero exactly where $f'(x) = 0$. The y -coordinate becomes $\frac{1}{y}$, and a maximum becomes a minimum (and the other way round).
- Points where $f = 1$ or $f = -1$ stay where they are.

For our f : $\frac{1}{f(x)} = \frac{x-3}{(x+1)(x-2)}$ has vertical asymptotes $x = -1$ and $x = 2$, horizontal asymptote $y = 0$, meets the x -axis at $(3, 0)$ and the y -axis at $(0, -\frac{3}{2})$; the maximum $(1, 1)$ becomes a minimum $(1, 1)$ and the minimum $(5, 9)$ becomes a maximum $(5, \frac{1}{9})$.

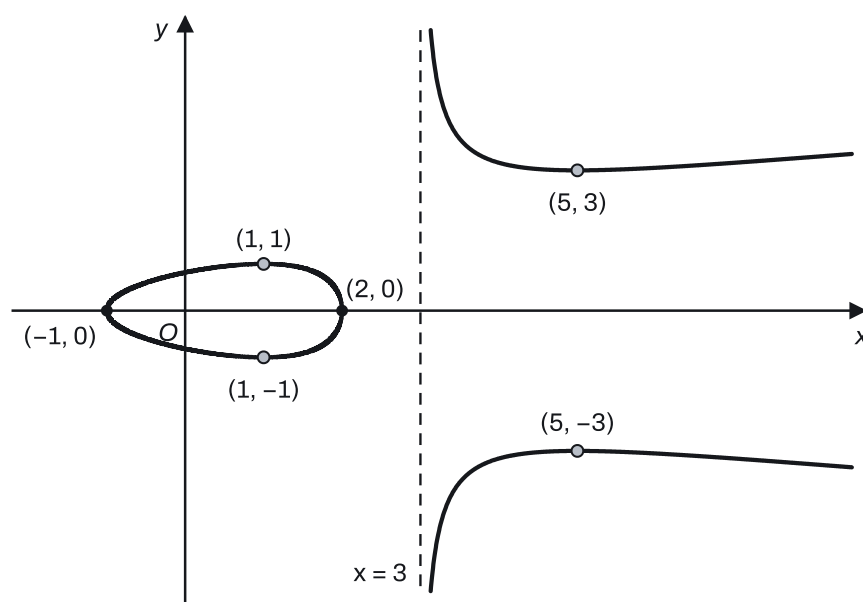


The graph of $y^2 = f(x)$

$y^2 = f(x)$ means $y = \pm\sqrt{f(x)}$, so:

- there is **no curve where** $f(x) < 0$ (you can't square-root a negative);
- the graph is **symmetrical about the x-axis**: draw $y = \sqrt{f(x)}$, then reflect it;
- where $f(x) = 0$ the curve crosses the x-axis **vertically** (its tangent is vertical there), so the two halves join smoothly, with no sharp point;
- a maximum or minimum of f at height $k > 0$ gives points at height $\pm\sqrt{k}$ with the same x ;
- where $f > 1$, $\sqrt{f}$ is smaller than f ; where $0 < f < 1$, $\sqrt{f}$ is bigger. Vertical asymptotes stay.

For our f : $f(x) \geq 0$ for $-1 \leq x \leq 2$ and for $x > 3$. The first piece becomes a closed oval through $(-1, 0)$ and $(2, 0)$, highest at $(1, 1)$ and lowest at $(1, -1)$; the second becomes two branches with $|y| \geq 3$, closest to the axis at $(5, \pm 3)$.



Using a sketch to solve an inequality

To solve, say, $|f(x)| < 4$ or $f(|x|) < 3$:

1. find the **critical values**: solve the **equations** $f(x) = 4$ and $f(x) = -4$ (for a modulus inequality you need both);
2. mark them on the sketch, together with the vertical asymptotes;
3. read off the x -intervals where the graph is below (or above) the line.

Working with equations and then the graph is much safer than manipulating inequalities, because multiplying an inequality by an expression such as $x - 3$ reverses it when that expression is negative.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
Quotient rule: $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$	Given
Vertical asymptotes: solve (denominator) = 0	Learn it
Degree of top < degree of bottom: asymptote $y = 0$	Learn it
Same degree: $y = \frac{\text{leading coefficient of top}}{\text{leading coefficient of bottom}}$	Learn it
Quadratic over linear: $\frac{ax^2 + bx + c}{px + q} = mx + k + \frac{r}{px + q}$, oblique asymptote $y = mx + k$	Learn it
Stationary points: $\frac{dy}{dx} = 0$, that is $vu' - uv' = 0$	Learn it
$ax^2 + bx + c = 0$ has real roots $\iff b^2 - 4ac \geq 0$	Learn it
Values taken: rewrite $y = f(x)$ as a quadratic in x and use discriminant ≥ 0	Learn it
$\frac{d}{dx} \left(\frac{1}{f(x)} \right) = -\frac{f'(x)}{f(x)^2}$	Learn it
$ f(x) < k \iff -k < f(x) < k$	Learn it

How to do it

In the Paper 1 questions we have tagged for this topic (33 questions, 2021–2025; many appear on two papers of the same session), the types came up this often. Every one is a multi-part question that mixes several types, so the counts add up to far more than 33.

Question type	Questions
Sketch the curve, with intersections with the axes	33

Question type	Questions
Find the asymptotes	32
Stationary points (find them, or show there are none)	30
Sketch $y = f(x) $ or $y = f(x)$	26
Solve an inequality using the sketch	24
Number of solutions of an equation (values of k or c)	6
Sketch $y = \frac{1}{f(x)}$	4
Show which values y can take (discriminant)	4
Sketch $y^2 = f(x)$	2

1. Sketch the curve, with intersections with the axes (33 questions)

- 1. Draw and label the axes and every asymptote** (dashed, with its equation written on it).
- 2. Mark the intercepts:** $(0, f(0))$ and the roots of the numerator. Write the coordinates on the sketch.
- 3. Mark the stationary points** you found, with coordinates.
- 4. Join up branch by branch.** Between two vertical asymptotes (or beyond the last one) there is one branch. Use the signs near each vertical asymptote (up or down) and the side from which the curve approaches the horizontal or oblique asymptote.
- 5. Check consistency:** a branch with no stationary point must be increasing or decreasing throughout; a curve can cross a horizontal asymptote only where $f(x) = k$ has a solution.

Variations you will meet:

- **"Sketch C "** with no intercepts asked: still show the asymptotes, and the stationary points if you found them.
- **A constant a in the equation** (for example $a > 1$): the asymptotes and points are in terms of a ; the sign facts you are given decide where the branches go.
- **The curve crosses its horizontal asymptote:** show it clearly crossing at the right point and then approaching from the other side, as in Example 2.

2. Find the asymptotes (32 questions)

- 1. Vertical:** solve (denominator) = 0 and write each as $x = \dots$. If the denominator is quadratic, factorise it or use the formula (exact surds are fine: $x^2 + 4x + 1 = 0$ gives $x = -2 \pm \sqrt{3}$).
- 2. Horizontal:** by the degree rule, $y = 0$ or $y =$ (ratio of leading coefficients).
- 3. Oblique** (quadratic over linear): divide, or write the numerator as (denominator)($mx + k$) + r and compare coefficients. The asymptote is $y = mx + k$. For example, $\frac{2x^2+3x-1}{x+2} = 2x - 1 + \frac{1}{x+2}$, so $y = 2x - 1$.

Variations you will meet:

- **"Show that C has no vertical asymptotes"**: show the denominator is never zero, for example $x^2 + 1 = 0$ has no real roots, or the discriminant of $x^2 + x + 1$ is $-3 < 0$.
- **A denominator like $5x - 2$** : the vertical asymptote is $x = 0.4$, and dividing gives fractions in the asymptote: $\frac{x^2+x}{5x-2} = \frac{1}{5}x + \frac{7}{25} + \frac{14/25}{5x-2}$, so $y = \frac{1}{5}x + \frac{7}{25}$. Keep going until you have the constant.
- **A constant a** : the same steps give, for example, $x = -a$ and $y = x - a$.

3. Stationary points (30 questions)

1. **Differentiate**. For quadratic over linear, use the divided form: $y = x + 2 + \frac{4}{x-3}$ gives $\frac{dy}{dx} = 1 - \frac{4}{(x-3)^2}$. Otherwise use the quotient rule.
2. **Set the numerator of $\frac{dy}{dx}$ to zero** and simplify to a quadratic (or linear) equation.
3. **Solve**, then substitute each x into the **curve's equation** to get y . Give both coordinates, exactly unless told otherwise (for example "correct to 1 decimal place").

Variations you will meet:

- **"Show that C has no stationary points"**: after step 2, show the discriminant is negative and conclude "so $\frac{dy}{dx} \neq 0$ and there are no stationary points". Or show $\frac{dy}{dx}$ is always positive, such as $1 + \frac{5}{(2x+1)^2} > 0$.
- **In terms of a constant**, for example $x^2 + 2ax - 3a^2 = 0$ gives $(x - a)(x + 3a) = 0$, so $x = a$ or $x = -3a$.
- **Stationary points of $\frac{1}{f(x)}$ or $y^2 = f(x)$** : same x -values as for f ; the y -values become $\frac{1}{y}$ or $\pm\sqrt{y}$ (only where $y > 0$).
- **Using the range instead**: if an earlier part showed $1 < y \leq 2$ for $y = \frac{x^2+2}{x^2+1}$, and $y = 2$ gives the repeated root $x = 0$, then $(0, 2)$ is the stationary point.

4. Sketch $y = |f(x)|$ or $y = f(|x|)$ (26 questions)

For $|f(x)|$:

1. Start from your sketch of $y = f(x)$.
2. Keep the parts on or above the x -axis. Reflect the parts below it in the x -axis, including any part of an oblique asymptote (so $y = x - 2$ becomes $y = 2 - x$ where the curve was below).
3. Draw **sharp points** where the curve meets the x -axis, and keep all vertical asymptotes.

For $f(|x|)$:

1. Keep the part of $y = f(x)$ for $x \geq 0$ only.
2. Reflect it in the y -axis, including any vertical asymptote, so the graph is symmetrical about the y -axis.
3. At the y -axis the graph has a sharp point, unless $f'(0) = 0$ (a stationary point of f is on the y -axis), when it is smooth there.

The question writes $f(|x|)$ out in full, for example $y = \frac{|x|^2-3|x|+1}{|x|^2+4}$: spot that every x has become $|x|$.

5. Solve an inequality using the sketch (24 questions)

1. **Find the critical values by solving equations.** For $|f(x)| < k$, solve $f(x) = k$ and $f(x) = -k$. For $f(|x|) < k$, solve $f(x) = k$ for $x \geq 0$ only, then use symmetry.
2. Each equation is usually a quadratic after multiplying up. Give exact roots when asked ("in exact form" means surds).
3. **Put the critical values and the vertical asymptotes on your sketch** and draw the line $y = k$ (and $y = -k$).
4. **Read off** the intervals of x where the graph is on the correct side of the line, using the asymptotes as boundaries. Write strict or non-strict signs to match the question.

For example, for $f(x) = x + 2 + \frac{4}{x-3}$ and $|f(x)| < 3$: $f(x) = 3$ gives $x^2 - 4x + 7 = 0$ (discriminant -12 , no roots) and $f(x) = -3$ gives $x^2 + 2x - 11 = 0$, so $x = -1 \pm 2\sqrt{3}$. The graph lies between $y = -3$ and $y = 3$ only on the left branch, between these two values: $-1 - 2\sqrt{3} < x < -1 + 2\sqrt{3}$.

Variations you will meet:

- **Four critical values**, from two quadratics: the answer is often two separate intervals, one on each side of an asymptote, or three intervals. Examiners note that $x = 0$ is a common one to lose, for example from $3x^2 - 7x = 0$.
- **An inequality between two related functions**, such as $\frac{1}{f(x)} < f(x)$: the graphs of f and $\frac{1}{f}$ meet where $f(x) = \pm 1$, so those are the critical values.
- **Working backwards**: you are given part of the solution set, for example $|f(x)| < 4$ for $a < x < p$ and $b < x < q$, and asked for p and q . Solve both equations; the given values are the roots of one of them, and p and q are roots of the other. Use the sketch to match each root to its interval.
- **A multiple of f** , such as $|2f(x)| < 15$: divide first, so it becomes $|f(x)| < \frac{15}{2}$ for the curve you have sketched.

6. Number of solutions of an equation (6 questions)

1. The solutions of $g(x) = k$ are where the horizontal line $y = k$ meets the graph of $y = g(x)$.
2. Slide the line up and down your sketch and count the intersections. The number changes at the y -values of turning points, sharp points and horizontal asymptotes.
3. State the set of k with the right strict or non-strict signs.

Variations you will meet:

- **" $|f(x)| = k$ has 4 distinct real solutions"**: sketch $|f(x)|$ first, then find the band of heights where the line crosses four times. Its ends are heights of turning points, of the sharp points on the x -axis (height 0) or of a horizontal asymptote, and they are usually excluded.
- **Two real solutions, with a constant a** : form the equations ($f(x) = a$ and $f(x) = -a$), and use discriminant > 0 (strict, since the roots must be distinct) on the one that can have roots.
- **A slanting line**, such as $y = x + c$: find the values of c where the line passes through a key point or is parallel to an asymptote, then read off.

7. Sketch $y = \frac{1}{f(x)}$ (4 questions)

1. Zeros of f become vertical asymptotes; vertical asymptotes of f become zeros.
2. Horizontal asymptote $y = k$ becomes $y = \frac{1}{k}$; an oblique asymptote (or $f \rightarrow \pm\infty$) gives $y = 0$; if $f \rightarrow 0$ then $\frac{1}{f}$ grows without limit.
3. Stationary points: same x , $y \rightarrow \frac{1}{y}$, maximum $\leftrightarrow$ minimum. Keep every sign.
4. y -intercept becomes $\frac{1}{f(0)}$.

To find where $\frac{1}{f(x)} < k$, solve $\frac{1}{f(x)} = k$, that is $f(x) = \frac{1}{k}$, and read off from the sketch.

8. Show which values y can take (4 questions)

1. Write $y = f(x)$ and multiply by the denominator.
2. Collect everything on one side as a quadratic in x : $Ax^2 + Bx + C = 0$, where A, B, C involve y .
3. **Real x exists** $\iff B^2 - 4AC \geq 0$. Expand, factorise the quadratic in y , and solve the inequality.
4. **Check the value of y that makes $A = 0$** , and say whether it is taken.
5. Conclude clearly, in the form the question asks: "there is no point on C with $1 < y < 9$ " or " $1 < y \leq 2$ for all real x ".

Stationary points alone do **not** prove a range: you would also need to show which is a maximum and which a minimum and use the asymptotes and branches, so the discriminant method is the one to use.

9. Sketch $y^2 = f(x)$ (2 questions)

1. Draw $y = \sqrt{f(x)}$ only where $f(x) \geq 0$; it keeps the vertical asymptotes and the x -values of turning points (heights become $\sqrt{\text{height}}$).
2. Reflect it in the x -axis.
3. Where $f = 0$, the curve crosses the x -axis vertically and smoothly.
4. If asked to draw it with $y = f(x)$ on one diagram, label **every branch** of each curve clearly.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
-

Example 1

The curve C has equation $y = \frac{x^2 + 2x + 5}{x + 1}$.

- (a) Find the equations of the asymptotes of C . [3]
- (b) Show that there is no point on C for which $-4 < y < 4$. [4]
- (c) Hence find the coordinates of the turning points of C . [2]
- (d) Sketch C . [3]

(a) Denominator zero: $x = -1$. B1

$x^2 + 2x + 5 = (x + 1)(x + 1) + 4$, so

$$y = x + 1 + \frac{4}{x + 1}.$$

M1 for dividing and finding the constant. The oblique asymptote is $y = x + 1$. **A1**

(b) $x^2 + 2x + 5 = y(x + 1) \Rightarrow x^2 + (2 - y)x + (5 - y) = 0$. M1 A1 for a correct quadratic in x .

For real x : $(2 - y)^2 - 4(5 - y) \geq 0 \Rightarrow y^2 - 16 \geq 0$. **M1** for using the discriminant.

So $(y - 4)(y + 4) \geq 0$, giving $y \leq -4$ or $y \geq 4$. There is no point on C with $-4 < y < 4$. **A1**

(c) The boundary values give repeated roots. $y = 4$: $x^2 - 2x + 1 = 0$, so $x = 1$. $y = -4$: $x^2 + 6x + 9 = 0$, so $x = -3$. M1

Turning points $(1, 4)$ and $(-3, -4)$. **A1** (Check: $\frac{dy}{dx} = 1 - \frac{4}{(x+1)^2} = 0$ gives $x = 1$ or -3 .)

(d) The sketch has the same shape as the first diagram in "Understand it": dashed asymptotes $x = -1$ and $y = x + 1$; a left branch below $y = x + 1$ with a maximum at $(-3, -4)$, falling to $-\infty$ as $x \rightarrow -1^-$; a right branch coming down from $+\infty$, through $(0, 5)$, to a minimum at $(1, 4)$, then rising above $y = x + 1$. It doesn't meet the x -axis (the discriminant of $x^2 + 2x + 5$ is -16). B1 axes and asymptotes, B1 left branch, B1 right branch.

Example 2

The curve C has equation $y = \frac{x^2 - x + 1}{x^2 + x + 1}$.

- (a) Show that C has no vertical asymptotes, and state the equation of the horizontal asymptote. [2]
- (b) Find the coordinates of the stationary points of C . [4]
- (c) Sketch C , stating the coordinates of any intersections with the axes. [3]

(d) Sketch the curve with equation $y = \frac{|x|^2 - |x| + 1}{|x|^2 + |x| + 1}$ and find the set of values of k for which

$\frac{|x|^2 - |x| + 1}{|x|^2 + |x| + 1} = k$ has exactly four real solutions. [3]

(a) $x^2 + x + 1 = 0$ has discriminant $1 - 4 = -3 < 0$, so no real roots and no vertical asymptotes. **B1**
Horizontal asymptote $y = 1$. **B1**

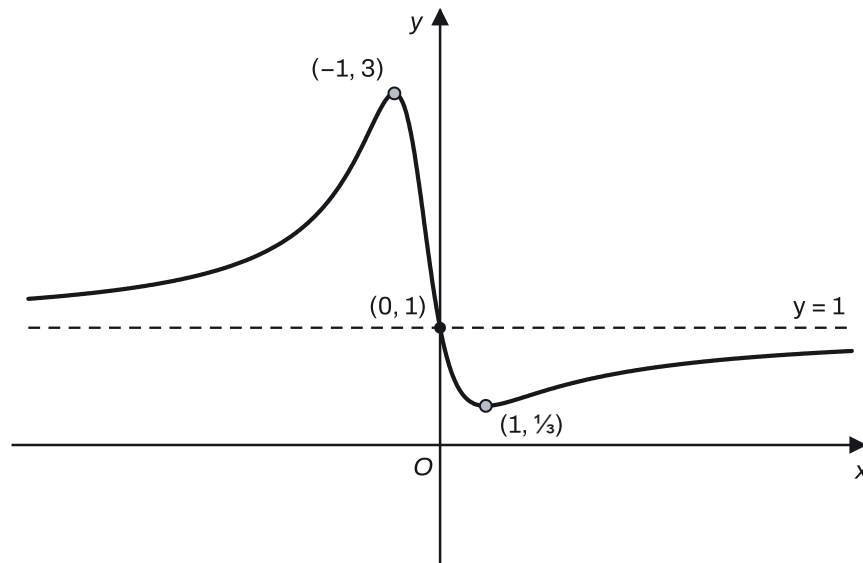
(b)

$$\frac{dy}{dx} = \frac{(x^2 + x + 1)(2x - 1) - (x^2 - x + 1)(2x + 1)}{(x^2 + x + 1)^2}.$$

M1 for the quotient rule. The numerator simplifies to $2x^2 - 2$. **M1** for setting it to 0: $x^2 = 1$, so $x = \pm 1$.

$x = 1: y = \frac{1}{3}$. $x = -1: y = \frac{3}{1} = 3$. Stationary points $(1, \frac{1}{3})$ and $(-1, 3)$. **A1 A1**

(c) y -intercept $(0, 1)$. The numerator $x^2 - x + 1$ has discriminant -3 , so no x -intercepts. The curve crosses its asymptote where $x^2 - x + 1 = x^2 + x + 1$, that is at $x = 0$, the point $(0, 1)$.



B1 axes and asymptote, **B1** correct shape (crossing $y = 1$ at $(0, 1)$, approaching $y = 1$ from above as $x \rightarrow -\infty$ and from below as $x \rightarrow +\infty$), **B1** $(0, 1)$ stated.

(d) Keep the part for $x \geq 0$: it starts at $(0, 1)$, falls to the minimum $(1, \frac{1}{3})$ and rises back towards $y = 1$ from below. Reflect it in the y -axis: a W shape with minima $(\pm 1, \frac{1}{3})$, a sharp point at $(0, 1)$ and asymptote $y = 1$. **B1**

A horizontal line $y = k$ with $\frac{1}{3} < k < 1$ meets each half twice, four times in all; $k = \frac{1}{3}$ gives two, $k = 1$ gives one ($x = 0$), and other k give none. So $\frac{1}{3} < k < 1$. **M1 A1**

Example 3

The curve C has equation $y = \frac{x - 1}{x^2 - 4}$.

(a) Find the equations of the asymptotes of C . [2]

(b) Show that C has no stationary points. [3]

(c) Sketch C , stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left| \frac{x-1}{x^2-4} \right|$. [2]

(e) Find in exact form the set of values of x for which $\left| \frac{x-1}{x^2-4} \right| < \frac{1}{3}$. [4]

(a) $x^2 - 4 = 0$: $x = 2$ and $x = -2$. **B1** The top has lower degree, so $y = 0$. **B1**

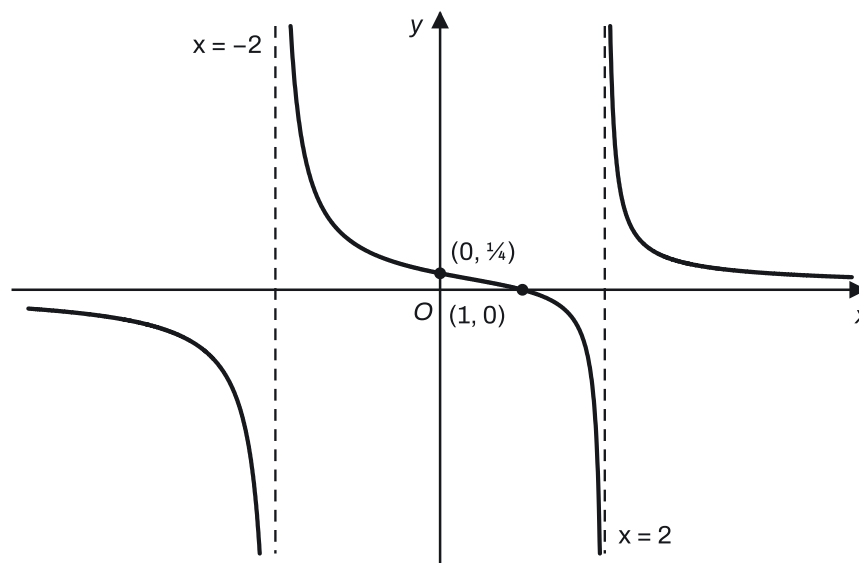
(b)

$$\frac{dy}{dx} = \frac{(x^2 - 4)(1) - (x - 1)(2x)}{(x^2 - 4)^2} = \frac{-x^2 + 2x - 4}{(x^2 - 4)^2}.$$

M1 Setting the numerator to 0: $x^2 - 2x + 4 = 0$. **M1** Its discriminant is $4 - 16 = -12 < 0$, so there are no real roots and no stationary points. **A1**

(In fact $-x^2 + 2x - 4 = -((x - 1)^2 + 3) < 0$, so y decreases on every branch.)

(c) $x = 0$: $y = \frac{-1}{-4} = \frac{1}{4}$, so $(0, \frac{1}{4})$. $y = 0$: $x = 1$, so $(1, 0)$.



B1 axes and asymptotes, **B1** three decreasing branches in the right places, **B1** $(0, \frac{1}{4})$ and $(1, 0)$.

(d) Reflect the parts below the x -axis: the branch for $x < -2$ and the part of the middle branch for $1 < x < 2$. The graph is then above the axis everywhere, with a sharp point at $(1, 0)$. **B1 B1**

(e) Critical values:

$$\begin{aligned} \frac{x-1}{x^2-4} &= \frac{1}{3} \Rightarrow x^2 - 3x - 1 = 0 \Rightarrow x = \frac{3 \pm \sqrt{13}}{2}, \\ \frac{x-1}{x^2-4} &= -\frac{1}{3} \Rightarrow x^2 + 3x - 7 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{37}}{2}. \end{aligned}$$

M1 M1, one for each equation. **A1** all four values.

From the sketch: on the left branch $|y| < \frac{1}{3}$ far to the left; on the middle branch between $y = \frac{1}{3}$ and $y = -\frac{1}{3}$; on the right branch once y has fallen below $\frac{1}{3}$:

$$x < \frac{-3 - \sqrt{37}}{2}, \quad \frac{3 - \sqrt{13}}{2} < x < \frac{-3 + \sqrt{37}}{2}, \quad x > \frac{3 + \sqrt{13}}{2}.$$

A1 (about $x < -4.54$, $-0.30 < x < 1.54$, $x > 3.30$).

Common mistakes

- **Leaving the constant out of the oblique asymptote.** Dividing $2x^2 + 3x - 1$ by $x + 2$ and writing $y = 2x$ is wrong: the asymptote is $y = 2x - 1$. Keep dividing until only a number is left over.
- **Forgetting the y -coordinate.** "Find the coordinates of the stationary points" needs both coordinates of each one.
- **Dropping the denominator in the quotient rule**, or setting the wrong part to zero. Only the numerator $uv' - uv'$ needs to be zero, but it must be expanded carefully; a sign slip there changes every answer after it.
- **Careless approaches to asymptotes.** Examiners want each branch to get steadily closer to its asymptotes without bending away or crossing a vertical one. Label every asymptote with its equation.
- **Believing a curve can never cross an asymptote.** It can cross a horizontal (or oblique) one; drawing the curve to avoid it can give two y -values for one x , which is not a function.
- **A smooth turning point where there should be a sharp point.** On $|f(x)|$ the points on the x -axis are sharp; on $f(|x|)$ the point on the y -axis is usually sharp. Candidates often draw a rounded bottom instead.
- **Forgetting to reflect an asymptote.** When part of the curve that followed an oblique asymptote is reflected, show the reflected line too.
- **Reflecting the wrong half for $f(|x|)$.** It is the part for $x \geq 0$ that is kept and mirrored, not the part for $x < 0$.
- **Solving only one equation for a modulus inequality.** $|f(x)| < k$ needs $f(x) = k$ and $f(x) = -k$. Examiners also see $x = 0$ lost from equations like $3x^2 - 7x = 0$.
- **Working with inequalities throughout.** Multiplying $\frac{\dots}{x-3} < k$ by $x - 3$ is wrong when $x < 3$. Solve equations for the critical values, then use the graph; overlapping or impossible intervals are a sign this went wrong.
- **Proving a range from stationary points alone.** Finding $(1, 1)$ and $(5, 9)$ for the curve in "Understand it" does not by itself show that y never lies between 1 and 9. Use the discriminant, and say whether you need $it < 0$ (no solutions) or ≥ 0 (solutions exist).
- **Ignoring the case where the x^2 coefficient is 0** in the range method, which decides whether the asymptote value itself is included.
- **Drawing only one of two curves** when asked for two on the same diagram, or not labelling which is which.

How to get the marks

- **Asymptotes** earn marks only as **equations**: $x = 3$, $y = x + 2$, not "3" or "the line $x + 2$ ". Exact forms such as $x = -2 + \sqrt{3}$ are expected.
- **Sketch marks** are usually one for axes and labelled asymptotes, and one or two for the branches (correct shape, position and approach to the asymptotes). The last mark is often for coordinates of the intersections, which may be written on the diagram. The sketch must be consistent with what you found earlier.
- **"Show that there are no stationary points"** needs a reason (negative discriminant, or a derivative that is always positive) and a conclusion in words.
- **"Show that ... for all real x "** is a given answer: set up the quadratic in x , state the discriminant condition, factorise, and explain any strict sign. Every step must be visible.
- **"Hence"** means use the previous part, for example read the stationary points from the boundary values of the range, or use your sketch to solve the inequality.
- **"Sketch the curve with equation $y = |f(x)|$ "** is marked from your earlier sketch (follow-through), plus a mark for the right shape at the axis: sharp points and reflected asymptotes.
- **Inequalities**: typically one method mark for each equation giving critical values, an accuracy mark for the values, and a final mark for the correct set. "In exact form" means surds, not decimals; decimals lose the accuracy marks.
- **State the final set clearly**, with the right strict signs, and don't include the x -values of vertical asymptotes.

Quick check

1. Find the equations of the asymptotes of $y = \frac{2x^2 + 3x - 1}{x + 2}$.
2. Show that $-1 \leq \frac{4x}{x^2 + 4} \leq 1$ for all real x .
3. Find the set of values of x for which $\left| \frac{x - 1}{x + 2} \right| < 2$.
4. The curve $y = f(x)$ has a maximum at $(2, 4)$ and a minimum at $(6, -3)$. State the coordinates of the corresponding stationary points of $y = \frac{1}{f(x)}$, saying which is a maximum, and the points of $y^2 = f(x)$ with $x = 2$.
5. For $f(x) = \frac{x^2 - x - 2}{x - 3}$, the function in "Understand it", find the set of values of k for which $f(x) = k$ has no real solutions.

Answers

- Vertical: $x = -2$. $2x^2 + 3x - 1 = (x + 2)(2x - 1) + 1$, so $y = 2x - 1 + \frac{1}{x+2}$ and the oblique asymptote is $y = 2x - 1$.
- $yx^2 - 4x + 4y = 0$. If $y \neq 0$, real x needs $16 - 16y^2 \geq 0$, so $-1 \leq y \leq 1$; $y = 0$ is taken at $x = 0$. So $-1 \leq y \leq 1$ for all real x .
- $y = \frac{x-1}{x+2}$ has asymptotes $x = -2$ and $y = 1$, and $\frac{dy}{dx} = \frac{3}{(x+2)^2} > 0$, so both branches rise. $\frac{x-1}{x+2} = 2$ gives $x = -5$; $\frac{x-1}{x+2} = -2$ gives $x = -1$. For $x < -2$, $y > 1$ and $y < 2$ when $x < -5$; for $x > -2$, $y < 1$ and $y > -2$ when $x > -1$. So $x < -5$ or $x > -1$.
- $(2, \frac{1}{4})$ is a **minimum** and $(6, -\frac{1}{3})$ is a **maximum** of $y = \frac{1}{f(x)}$. For $y^2 = f(x)$ at $x = 2$: $y^2 = 4$, so $(2, 2)$ and $(2, -2)$. (There are no points at $x = 6$, as $f(6) < 0$.)
- From the discriminant $(y - 1)(y - 9) \geq 0$ in "Understand it", real solutions exist only for $k \leq 1$ or $k \geq 9$. So there are no solutions for $1 < k < 9$.

Past questions to try

- 9231/13/O/N/24 Q6**: asymptotes, stationary points, and $|f(x)| = k$ with four solutions.
- 9231/13/M/J/22 Q1**: sketching $f(|x|)$ with a sharp point on the y -axis, then an inequality.
- 9231/12/O/N/23 Q7**: the reciprocal graph $\frac{1}{f(x)}$ and its stationary points.
- 9231/13/M/J/23 Q7**: $y = f(x)$ and $y^2 = f(x)$ on one diagram.