

Roots of polynomial equations AS

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What you need to know

By the end of this topic you should be able to:

- 1. Use the links between the roots and the coefficients** of a polynomial equation of degree 2, 3 or 4. Given the equation, write down the sum of the roots, the sum of their products in pairs, and so on, straight from the coefficients. Use these to work out other symmetric expressions in the roots (sums of squares, cubes and higher powers, sums of reciprocals and similar), and go the other way: use facts about the roots to find unknown coefficients.
- 2. Find a new equation by substitution** whose roots are linked in a simple way to the roots of the one you have: for example their reciprocals, their squares, or a linear function such as $2\alpha + 1$. For the simplest cases (reciprocals, squares, a linear function) the question won't tell you the substitution; you must choose it yourself.

Only degrees 2, 3 and 4 are examined. You never need to solve the equation: everything comes from the coefficients.

Understand it

Roots and coefficients

If α and β are the roots of $ax^2 + bx + c = 0$, the quadratic factorises as $a(x - \alpha)(x - \beta)$. Multiply out and compare with the original:

$$a(x - \alpha)(x - \beta) = ax^2 - a(\alpha + \beta)x + a\alpha\beta,$$

so $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

The same idea works for a cubic $ax^3 + bx^2 + cx + d = 0$ with roots α, β, γ . Expanding $a(x - \alpha)(x - \beta)(x - \gamma)$ gives

$$\alpha + \beta + \gamma = -\frac{b}{a}, \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}, \quad \alpha\beta\gamma = -\frac{d}{a}.$$

For example, $(x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6$. The roots 1, 2, 3 have sum 6, pair products $2 + 6 + 3 = 11$ and product 6, matching the coefficients $-6, 11, -6$ with the signs flipped where the pattern says.

A quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$ with roots $\alpha, \beta, \gamma, \delta$ follows the same pattern, one step further:

$$\sum \alpha = -\frac{b}{a}, \quad \sum \alpha\beta = \frac{c}{a}, \quad \sum \alpha\beta\gamma = -\frac{d}{a}, \quad \alpha\beta\gamma\delta = \frac{e}{a}.$$

Here $\sum \alpha\beta$ means all **six** products of two different roots, and $\sum \alpha\beta\gamma$ all **four** products of three. The pattern to remember: divide each coefficient by the leading one, and the signs go $-, +, -, +$.

A missing term means a zero coefficient. In $x^3 + 5x - 2 = 0$ there is no x^2 term, so $\sum \alpha = 0$. Forgetting this is one of the most common slips.

These rules hold whether the roots are real or complex. For example, $x^2 + 2x + 5 = 0$ has roots $-1 \pm 2i$, whose sum is -2 and product is 5 . So a "sum of squares" can come out negative: then the roots cannot all be real, since real numbers have squares that are at least 0 .

Symmetric functions of the roots

An expression like $\alpha^2 + \beta^2 + \gamma^2$ or $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ is **symmetric**: swapping any two roots leaves it unchanged. Every symmetric expression can be written using only $\sum \alpha$, $\sum \alpha\beta$, $\sum \alpha\beta\gamma$ (and $\alpha\beta\gamma\delta$ for a quartic), so you can find its value from the coefficients without knowing the roots.

Sum of squares. Square the sum of the roots: $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$.

Rearranged:

$$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta.$$

This works for any number of roots (for a quartic, $\sum \alpha\beta$ has six terms).

Sum of reciprocals. Put the fractions over a common denominator:

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma}, \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta}.$$

Sums of higher powers: use the equation itself. Write $S_n = \alpha^n + \beta^n + \gamma^n$ (and $+\delta^n$ for a quartic). Each root satisfies the equation. For $x^3 - 2x^2 + 3x - 4 = 0$:

$$\alpha^3 - 2\alpha^2 + 3\alpha - 4 = 0, \quad \beta^3 - 2\beta^2 + 3\beta - 4 = 0, \quad \gamma^3 - 2\gamma^2 + 3\gamma - 4 = 0.$$

Add the three lines: $S_3 - 2S_2 + 3S_1 - 12 = 0$. The constant -4 appears **once per root**, so it becomes $3 \times (-4) = -12$ (in general S_0 is the number of roots). Here $S_1 = 2$ and $S_2 = 2^2 - 2(3) = -2$, so $S_3 = 2(-2) - 3(2) + 12 = 2$.

Multiply the equation by x first and the same trick links S_4, S_3, S_2, S_1 ; multiply by x^2 for S_5 , and so on. Divide by x (or x^2) and it reaches negative powers such as $S_{-1} = \sum \frac{1}{\alpha}$. This one idea gives every power sum, one step at a time:

$$aS_{n+3} + bS_{n+2} + cS_{n+1} + dS_n = 0 \quad \text{for the cubic } ax^3 + bx^2 + cx + d = 0.$$

Equations with transformed roots

Suppose α, β, γ are the roots of $f(x) = 0$ and you want an equation whose roots are, say, $2\alpha + 1, 2\beta + 1, 2\gamma + 1$. Let $y = 2x + 1$. Then $x = \frac{y-1}{2}$, and x is a root of $f(x) = 0$ exactly when y is one of the new roots. So substituting $x = \frac{y-1}{2}$ into $f(x) = 0$ gives an equation in y whose roots are exactly the new ones.

The recipe is always the same: **write y in terms of x , turn it round to get x in terms of y , substitute into the equation, and tidy up.**

- **Reciprocals**, $y = \frac{1}{x}$: substitute $x = \frac{1}{y}$ and multiply through by the highest power of y . The coefficients simply come out in **reverse order**. For $2x^3 - 5x^2 + x + 3 = 0$ the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is $3y^3 + y^2 - 5y + 2 = 0$.
- **Multiples**, $y = kx$: substitute $x = \frac{y}{k}$. For roots $3\alpha, 3\beta, 3\gamma$ of the same cubic: $2\left(\frac{y}{3}\right)^3 - 5\left(\frac{y}{3}\right)^2 + \frac{y}{3} + 3 = 0$, and multiplying by 27 gives $2y^3 - 15y^2 + 9y + 81 = 0$.
- **Squares**, $y = x^2$: substitute $x = \sqrt{y}$. Collect the terms with $\sqrt{y}$ on one side, then square both sides to get rid of it. For $x^3 + x - 1 = 0$: $y\sqrt{y} + \sqrt{y} = 1$, so $\sqrt{y}(y + 1) = 1$. Squaring: $y(y + 1)^2 = 1$, that is $y^3 + 2y^2 + y - 1 = 0$.
- **Cubes**, $y = x^3$: substitute $x = y^{1/3}$, collect the $y^{1/3}$ terms on one side and cube both sides.

Two less obvious cases turn into these simple ones:

- **Sums of two roots**. For a cubic, $\alpha + \beta = (\alpha + \beta + \gamma) - \gamma = S_1 - \gamma$. So the roots $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ come from $y = S_1 - x$. For $x^3 - 3x^2 + 2 = 0$, $S_1 = 3$, so substitute $x = 3 - y$; this gives $y^3 - 6y^2 + 9y - 2 = 0$.
- **Products of two roots**. For a cubic, $\alpha\beta = \frac{\alpha\beta\gamma}{\gamma}$. For $x^3 - 3x^2 + 2 = 0$ the product of all three roots is -2 , so $\alpha\beta = -\frac{2}{\gamma}$, and the roots $\alpha\beta, \beta\gamma, \gamma\alpha$ come from $y = -\frac{2}{x}$, that is $x = -\frac{2}{y}$. This gives $y^3 - 6y - 4 = 0$.

Why bother? Once you have the new equation, sums of the new roots, their squares, their reciprocals and so on come straight from **its** coefficients. For example, once you have the equation with roots $\alpha^2, \beta^2, \gamma^2$, the value of $\sum \frac{1}{\alpha^2}$ is the sum of the reciprocals of **its** roots: its pair-sum over its product.

Key facts and formulas

"Given" means it is in the MF19 list of formulae. None of this topic is in MF19, so you must learn all of it.

Formula	Status
Quadratic $ax^2 + bx + c = 0$: $\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$	Learn it
Cubic $ax^3 + bx^2 + cx + d = 0$: $\sum \alpha = -\frac{b}{a}, \sum \alpha\beta = \frac{c}{a}, \alpha\beta\gamma = -\frac{d}{a}$	Learn it
Quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$: $\sum \alpha = -\frac{b}{a}, \sum \alpha\beta = \frac{c}{a}, \sum \alpha\beta\gamma = -\frac{d}{a}, \alpha\beta\gamma\delta = \frac{e}{a}$	Learn it

Formula	Status
$\sum \alpha^2 = (\sum \alpha)^2 - 2 \sum \alpha\beta$ (any degree)	Learn it
$\sum \alpha^3 = (\sum \alpha)^3 - 3 \sum \alpha \sum \alpha\beta + 3 \sum \alpha\beta\gamma$ (cubic or quartic)	Learn it
Cubic: $\sum \frac{1}{\alpha} = \frac{\sum \alpha\beta}{\alpha\beta\gamma}$; quartic: $\sum \frac{1}{\alpha} = \frac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta}$	Learn it
Cubic: $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\sum \alpha\beta)^2 - 2\alpha\beta\gamma \sum \alpha$	Learn it
Cubic: $\sum \frac{1}{\alpha^2} = \frac{(\sum \alpha\beta)^2 - 2\alpha\beta\gamma \sum \alpha}{(\alpha\beta\gamma)^2}$	Learn it
Power sums from the equation: $aS_{n+3} + bS_{n+2} + cS_{n+1} + dS_n = 0$ for a cubic (one more term for a quartic); $S_0 =$ number of roots	Learn it
New roots $y = g(\alpha)$: write x in terms of y and substitute into the equation	Learn it
$\sum_{r=1}^n r = \frac{1}{2}n(n+1)$, $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$, $\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$ (for questions that mix in series)	Given

How to do it

In the Paper 1 questions we have tagged for this topic (32 questions, 2021–2025; many appear on two papers of the same session), the types came up this often. 26 of the 32 questions mix two or more types, so the counts add up to more than 32.

Question type	Questions
Sums of powers and other symmetric functions	26
Equation with transformed roots	17
Unknown coefficients from facts about the roots	11
Roots inside a summation $\sum_{r=1}^n$	4
Roots inside a determinant	2

1. Sums of powers and other symmetric functions (26 questions)

- Write down** $\sum \alpha$, $\sum \alpha\beta$, $\sum \alpha\beta\gamma$ (and $\alpha\beta\gamma\delta$) from the coefficients, with the right signs. Put 0 for any missing term.
- Rewrite the expression you want** in terms of these, using the identities in the table.
- Substitute and simplify.** Give exact fractions.

For higher powers, build up with the equation rather than a memorised formula:

1. Find S_1 and S_2 .
2. Multiply the equation by x^k so that the highest power is the one you want, replace each x^m by S_m and the constant term by (number of roots) $\times$ constant.
3. Solve for the new sum. Repeat for the next power.

For $x^4 - x^3 + 2x - 3 = 0$: $x^4 = x^3 - 2x + 3$, so $S_4 = S_3 - 2S_1 + 4 \times 3 = S_3 - 2S_1 + 12$.

Variations you will meet:

- **"Express S_{n+3} in terms of S_{n+2} and S_n ":** multiply the equation by x^n and sum over the roots. Make S_{n+3} the subject, as asked.
- **Negative powers.** $S_{-1} = \sum \frac{1}{\alpha}$ comes from the reciprocal identity. S_{-2} is $(S_{-1})^2 - 2 \sum \frac{1}{\alpha\beta}$, where $\sum \frac{1}{\alpha\beta} = \frac{\sum \alpha}{\alpha\beta\gamma}$ for a cubic. Or divide the equation by x^2 and use the power-sum rule.
- **Powers of the new roots**, such as $\sum (2\alpha + 1)^2$: if the question has just given you the new equation, use its coefficients with the same identities. Otherwise expand: $\sum (2\alpha + 1)^2 = 4S_2 + 4S_1 + 3$ for a cubic (the 1 is added once per root).
- **Mixed products**, such as $\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \dots$ (all six for a cubic): factor as $\alpha^2(\beta + \gamma) + \dots = \alpha^2(S_1 - \alpha) + \dots = S_1S_2 - S_3$. The same idea works for a quartic. For a cubic this also equals $\sum \alpha \sum \alpha\beta - 3\alpha\beta\gamma$; for $x^3 - 2x^2 + 3x - 4 = 0$ that is $2 \times 3 - 3 \times 4 = -6$.
- **Products of squares**, $\sum \alpha^2\beta^2$: use $(\sum \alpha\beta)^2 - 2\alpha\beta\gamma \sum \alpha$ for a cubic, or find the equation with roots $\alpha^2, \dots$ and read off its pair-sum coefficient.
- **Power sums given, not the equation.** "It is given that $\sum \alpha = 2, \sum \alpha^2 = 3, \sum \alpha^3 = 4$ ": find $\sum \alpha\beta$ from the sum-of-squares identity, then carry on as usual.

2. Equation with transformed roots (17 questions)

1. **Name the substitution:** $y =$ (new root in terms of x). Write it down; this line earns a mark.
2. **Invert it:** $x =$ (in terms of y).
3. **Substitute** into the original equation.
4. **Clear** fractions (multiply by a power of y or a number) and radicals (collect the $\sqrt{y}$ or $y^{1/3}$ terms on one side, then square or cube both sides).
5. **Expand and simplify** to a polynomial equation with integer coefficients, written " $= 0$ ". If the answer is given ("show that"), your last line must match it exactly.
6. **Read off** what the later parts ask for from the new equation's coefficients.

Variations you will meet:

- **Linear**, such as $y = 2x + 1$ or $y = 3x + 1$: substitute $x = \frac{y-1}{2}$, multiply through to clear the fractions, expand with the binomial theorem.
- **Squares or reciprocals of squares** ($y = x^2, y = \frac{1}{x^2}$): substitute $x = y^{1/2}$ or $x = y^{-1/2}$, get the half-powers on one side, square.

- **Cubes or reciprocals of cubes** ($y = x^3, y = \frac{1}{x^3}$): substitute $x = y^{1/3}$ or $x = y^{-1/3}$, get the cube roots on one side, cube.
- **Fourth powers** of the roots of a quartic with only an x^3 (or x) term besides x^4 and the constant, such as $x^4 + 2x^3 - 1 = 0$: rearrange to $2x^3 = 1 - x^4$, put $x^4 = y$, then raise both sides to the fourth power: $16x^{12} = (1 - y)^4$, that is $16y^3 = (1 - y)^4$.
- **Use the equation first.** If $\alpha^3 + 2\alpha + 1 = 0$, then $\alpha^3 - 1 = -2\alpha - 2$. So the roots $\alpha^3 - 1, \dots$ come from the **linear** substitution $y = -2x - 2$, which avoids cube roots altogether.
- **Sums or products of two roots:** $y = S_1 - x$ or $y = \frac{\alpha\beta\gamma}{x}$, as shown in "Understand it".
- **"State the value of $\sum \alpha^2$ "** after finding the "squares" equation: it is just minus the new y^2 coefficient over the leading one; no more working needed.

3. Unknown coefficients from facts about the roots (11 questions)

1. **Write the relations** between the roots and the (unknown) coefficients, one equation per coefficient. Include any that are 0 because a term is missing.
2. **Bring in the extra facts:** given power sums, a special relation between the roots, or a condition from another part.
3. **Solve** the equations together. Check any condition given (such as $d \neq 0$ or "positive constants") and reject what breaks it.

Variations you will meet:

- **Power sums given** ($\sum \alpha, \sum \alpha^2, \sum \alpha^3$ or $\sum \frac{1}{\alpha}$): $\sum \alpha$ gives b at once, the sum-of-squares identity gives $\sum \alpha\beta$ and so c , and the power-sum rule gives the last coefficient. If $\sum \alpha^3$ is not given but $\sum \alpha^4$ is, write S_3 in terms of the unknown d from one power-sum equation, then use the next one to find d .
- **A relation between roots**, such as one root being the reciprocal of another ($\gamma = \frac{1}{\alpha}$), two roots being equal ($\alpha = \beta$), or roots in a pattern. Write the roots using as few letters as possible, then use each relation. For $\gamma = \frac{1}{\alpha}$ in a monic cubic, $\alpha\beta\gamma = \beta$, so $\beta = -d$ at once.
- **Roots in arithmetic progression** $\beta - k, \beta, \beta + k$: the sum is 3β , so β comes from $\sum \alpha$. **Geometric progression** $\frac{\beta}{r}, \beta, \beta r$: the product is β^3 , so β comes from $\alpha\beta\gamma$. Then β is a root: substitute it into the equation.
- **A repeated root** α satisfies both $f(\alpha) = 0$ and $f'(\alpha) = 0$. For example, $x^3 - 3x + k = 0$ has a repeated root when $3\alpha^2 - 3 = 0$, so $\alpha = \pm 1$ and $k = 2$ or -2 . The relations between roots and coefficients (with roots α, α, γ) do the same job.
- **"Show that $4b^3 + 27d^2 = 0$ "**: find each root in terms of the coefficients, then substitute into the remaining relation and tidy up to the given result.

4. Roots inside a summation (4 questions)

These combine this topic with summation of series: find $\sum_{r=1}^n ((\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2)$ and similar.

1. **Expand one bracket:** $(\alpha + r)^2 = \alpha^2 + 2\alpha r + r^2$; $(\alpha + r)^3 = \alpha^3 + 3\alpha^2 r + 3\alpha r^2 + r^3$.

- Add over the roots first**, keeping r as it is: for a cubic $\sum(\alpha + r)^2 = S_2 + 2rS_1 + 3r^2$ (the r^2 appears once per root, so $3r^2$, or $4r^2$ for a quartic). If the bracket holds a power of the root, such as $(\alpha^2 + r)^2 = \alpha^4 + 2\alpha^2r + r^2$, the sum over the four roots of a quartic is $S_4 + 2rS_2 + 4r^2$, so find S_2 and S_4 first.
- Put in the numbers** for $S_1, S_2, \dots$ so that you have a simple polynomial in r .
- Sum over r** with the MF19 results $\sum r, \sum r^2, \sum r^3$ (and $\sum 1 = n$), then factorise to the form asked for. Take out common factors such as n or $n(n+1)$ early to keep the algebra short.

5. Roots inside a determinant (2 questions)

These combine this topic with matrices: a matrix has the roots as entries, and you show it is singular or use the fact that it is.

- Find the determinant** in terms of the roots. For the symmetric matrix with 1s down the diagonal and α, β, γ off it,

$$\begin{vmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{vmatrix} = 1 - \alpha^2 - \beta^2 - \gamma^2 + 2\alpha\beta\gamma.$$

- Rewrite it with symmetric functions**, here $1 - \sum \alpha^2 + 2\alpha\beta\gamma$, and substitute values from the coefficients.
- Singular** means the determinant is 0. For "show that it is singular", show every substitution, since the answer 0 is given.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
-

Example 1

The cubic equation $2x^3 - 3x^2 + 4x - 5 = 0$ has roots α, β, γ .

- Find the value of $\alpha^2 + \beta^2 + \gamma^2$ and explain what it tells you about the roots. **[3]**
- Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **[2]**
- Find the value of $\alpha^3 + \beta^3 + \gamma^3$. **[3]**

(a) $\sum \alpha = \frac{3}{2}$, $\sum \alpha\beta = \frac{4}{2} = 2$, $\alpha\beta\gamma = \frac{5}{2}$. **B1** for all three.

$$\sum \alpha^2 = \left(\frac{3}{2}\right)^2 - 2(2) = \frac{9}{4} - 4 = -\frac{7}{4}.$$

M1 for the sum-of-squares identity, **A1**. The squares of real numbers can't add to a negative number, so the roots are not all real.

(b) $\sum \frac{1}{\alpha} = \frac{\sum \alpha\beta}{\alpha\beta\gamma} = \frac{2}{5/2} = \frac{4}{5}$. **M1 A1**

(c) Each root satisfies $2\alpha^3 - 3\alpha^2 + 4\alpha - 5 = 0$. Adding for all three roots: **M1**

$$2S_3 - 3S_2 + 4S_1 - 15 = 0.$$

A1 (the constant is $3 \times (-5)$). So $2S_3 = 3\left(-\frac{7}{4}\right) - 4\left(\frac{3}{2}\right) + 15 = \frac{15}{4}$, giving $S_3 = \frac{15}{8}$. **A1**

Example 2

The cubic equation $x^3 - 4x^2 + x + 3 = 0$ has roots α, β, γ .

(a) Find a cubic equation whose roots are $\alpha^2, \beta^2, \gamma^2$. **[4]**

(b) Find the value of $\alpha^4 + \beta^4 + \gamma^4$. **[2]**

(c) Find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$. **[2]**

(a) Let $y = x^2$, so $x = \sqrt{y}$. **B1** for the substitution.

$$y\sqrt{y} - 4y + \sqrt{y} + 3 = 0 \Rightarrow \sqrt{y}(y + 1) = 4y - 3.$$

Squaring both sides: $y(y + 1)^2 = (4y - 3)^2$. **M1** for an equation with no square roots.

$$y^3 + 2y^2 + y = 16y^2 - 24y + 9 \Rightarrow y^3 - 14y^2 + 25y - 9 = 0.$$

M1 for expanding, **A1**.

Check: the new roots add to 14, and $\sum \alpha^2 = 4^2 - 2(1) = 14$. **✓**

(b) The roots of the new equation are $\alpha^2, \beta^2, \gamma^2$, so use the sum-of-squares identity on it: **M1**

$$\sum \alpha^4 = 14^2 - 2(25) = 146.$$

A1

(c) This is the sum of the reciprocals of the new roots: $\frac{25}{9}$ (pair sum over product, both from the new equation).

M1 A1

Example 3

The cubic equation $x^3 - 7x^2 + kx - 8 = 0$, where k is a constant, has three real roots in geometric progression.

(a) Show that one of the roots is 2. **[2]**

(b) Find the value of k and the other two roots. **[4]**

(a) Write the roots as $\frac{\beta}{r}, \beta, \beta r$. **B1** for a form that uses the progression.

Their product is $\beta^3 = -\frac{-8}{1} = 8$, so $\beta = 2$ (the only real cube root). **B1**

(b) 2 is a root, so $8 - 28 + 2k - 8 = 0$, giving $k = 14$. **M1 A1**

Then $x^3 - 7x^2 + 14x - 8 = (x - 2)(x^2 - 5x + 4) = (x - 2)(x - 1)(x - 4)$. **M1** for factorising out $(x - 2)$.

The other roots are 1 and 4 (the progression 1, 2, 4, with $r = 2$). **A1**

Example 4

The quartic equation $x^4 - 2x^3 + 3x - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find the values of $\sum \alpha^2$ and $\sum \alpha^3$. **[3]**

(b) Find the value of $\sum \alpha^4$. **[2]**

(c) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^n \left((\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2 + (\delta + r)^2 \right) = \frac{4}{3} n (n^2 + 3n + 5).$$

[5]

(a) $\sum \alpha = 2$, $\sum \alpha\beta = 0$ (no x^2 term), $\sum \alpha\beta\gamma = -3$. **B1**

$$\sum \alpha^2 = 2^2 - 2(0) = 4. \text{ **B1**}$$

$$\sum \alpha^3 = 2^3 - 3(2)(0) + 3(-3) = -1. \text{ **B1**}$$

(b) $x^4 = 2x^3 - 3x + 1$, so adding for all four roots, $S_4 = 2S_3 - 3S_1 + 4$. **M1**

$$S_4 = 2(-1) - 3(2) + 4 = -4. \text{ **A1**}$$

(c) $(\alpha + r)^2 = \alpha^2 + 2\alpha r + r^2$. **B1** for expanding.

Adding over the four roots: $\sum (\alpha + r)^2 = S_2 + 2rS_1 + 4r^2 = 4 + 4r + 4r^2$. **M1** for collecting terms, **A1**.

$$\sum_{r=1}^n (4 + 4r + 4r^2) = 4n + 4 \cdot \frac{1}{2}n(n+1) + 4 \cdot \frac{1}{6}n(n+1)(2n+1).$$

M1 for using the MF19 results.

$$= \frac{2}{3}n \left(6 + 3(n+1) + (n+1)(2n+1) \right) = \frac{2}{3}n (2n^2 + 6n + 10) = \frac{4}{3}n (n^2 + 3n + 5).$$

A1, the given answer with the factorising shown. Check with $n = 1$: $4 + 4 + 4 = 12$ and $\frac{4}{3}(1 + 3 + 5) = 12$.

✓

Common mistakes

- **Missing a zero coefficient.** In $x^3 + 5x - 2 = 0$, $\sum \alpha = 0$, and in $x^4 + 2x^3 - 1 = 0$, $\sum \alpha\beta = 0$ and $\sum \alpha\beta\gamma = 0$. Examiners report candidates leaving out the zero term and getting stuck, both in the sum of squares and in "show that" questions about the roots.
- **Sign slips in the relations.** The signs alternate $-, +, -, +$. $\alpha\beta\gamma = -\frac{d}{a}$ for a cubic, but $\alpha\beta\gamma\delta = +\frac{e}{a}$ for a quartic.
- **Forgetting to divide by the leading coefficient.** For $2x^3 - 3x^2 + \dots$, $\sum \alpha = \frac{3}{2}$, not 3.
- **Adding the constant only once.** When you sum the equation over the roots, the constant term is added once for **each** root: $3d$ for a cubic, $4e$ for a quartic. The same goes for the r^2 term in summation questions.
- **Misremembering a formula for the sum of cubes.** Examiners note that candidates who summed the equation over the roots were more often right than those who used a half-remembered formula. If unsure, use the equation.
- **Using the wrong equation.** After finding a new equation, read the next part carefully: does it want sums of the old roots or the new ones? Using the old coefficients for new roots (or the other way round) is a common error.
- **Answering only half the question.** "Find a quartic equation whose roots are ... and state the value of ..." needs both. Examiners see candidates give only one.
- **Squaring before collecting the root terms.** For $y = x^2$, get all the $\sqrt{y}$ terms on one side **first**; squaring a sum with $\sqrt{y}$ on both sides leaves radicals behind.
- **Leaving the new equation unfinished.** Expand fully, collect terms, clear fractions and write " $= 0$ ". A "show that" answer must match the printed equation exactly, with the expansion shown.
- **Not using conditions given.** If the question says $d \neq 0$ or the constants are positive, use it to reject solutions, and say that you have.

How to get the marks

- **Write the relations down.** Mark schemes often give a **B1** just for stating $\sum \alpha$, $\sum \alpha\beta$ or the product correctly, even when it is only used later ("SOI": seen or implied). Write each one with its value.
- **Name the substitution.** In transformed-root questions the first mark is for the correct substitution, such as $y = 2x + 1 \Rightarrow x = \frac{1}{2}(y - 1)$. Then a method mark is for an equation with no radicals or fractions left.
- **Show the identity with numbers in**, for example $\sum \alpha^2 = (-2)^2 - 2(3)$. Method marks need the method visible; a bare answer can lose them.
- **"Show that"** means the answer is printed. Every step must be visible, including the substitution of values. A determinant that you claim is 0, or an equation you claim to have reached, needs the working that gets there.
- **"State"** means no working is needed: for example, $\sum \alpha^2$ read off the "squares" equation you have just found.
- **"Hence"** means use the part before. After finding a new equation, the next parts nearly always want you to read values from it.

- **"Hence or otherwise"** means another valid method gets full marks too.
- **Exact answers.** Leave fractions as fractions, such as $\frac{15}{8}$ or $-\frac{7}{4}$; don't round.
- **Method marks follow through.** A wrong value in an early part used correctly later still earns the later method marks (the mark scheme's "FT"), so keep going.

Quick check

1. The cubic equation $x^3 + 3x^2 - 2x + 5 = 0$ has roots α, β, γ . Find $\sum \alpha^2$ and $\sum \frac{1}{\alpha}$.
2. The cubic equation $x^3 - 6x^2 + 5x + 1 = 0$ has roots α, β, γ . Find a cubic equation whose roots are $\alpha - 2, \beta - 2, \gamma - 2$.
3. The cubic equation $x^3 + px^2 + qx + s = 0$ has roots α, β, γ , where $\sum \alpha = 4, \sum \alpha^2 = 10$ and $\sum \alpha^3 = 34$. Find p, q and s .
4. The quartic equation $x^4 - x^3 + 2x - 3 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find $\sum \alpha^2, \sum \alpha^3$ and $\sum \alpha^4$.

Answers

1. $\sum \alpha = -3$, $\sum \alpha\beta = -2$, $\alpha\beta\gamma = -5$. $\sum \alpha^2 = (-3)^2 - 2(-2) = 13$. $\sum \frac{1}{\alpha} = \frac{-2}{-5} = \frac{2}{5}$.
2. $y = x - 2$, so $x = y + 2$: $(y + 2)^3 - 6(y + 2)^2 + 5(y + 2) + 1 = 0$. Expanding: $y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 5y + 10 + 1 = 0$, that is $y^3 - 7y - 5 = 0$.
3. $p = -4$. $\sum \alpha\beta = \frac{4^2 - 10}{2} = 3$, so $q = 3$. $34 = 4^3 - 3(4)(3) + 3\alpha\beta\gamma = 28 + 3\alpha\beta\gamma$, so $\alpha\beta\gamma = 2$ and $s = -2$.
4. $\sum \alpha = 1$, $\sum \alpha\beta = 0$, $\sum \alpha\beta\gamma = -2$. $\sum \alpha^2 = 1 - 0 = 1$. $\sum \alpha^3 = 1 - 0 + 3(-2) = -5$. $x^4 = x^3 - 2x + 3$, so $S_4 = S_3 - 2S_1 + 12 = -5 - 2 + 12 = 5$.

Past questions to try

- [9231/12/O/N/23 Q4](#): a linear substitution, then power sums including negative ones.
- [9231/13/M/J/23 Q3](#): the equation with squared roots, then reciprocals and fourth powers.
- [9231/13/O/N/22 Q1](#): unknown coefficients from two equal roots.
- [9231/12/M/J/24 Q1](#): products of pairs of roots and a new equation from them.