

Summation of series

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What you need to know

By the end of this topic you should be able to:

1. **Use the standard results** for $\sum r$, $\sum r^2$ and $\sum r^3$ to find related sums, such as $\sum_{r=1}^n (r+1)(r+2)$ or a sum that starts at $r = n+1$.
2. **Use the method of differences** to find the sum of a finite series: write each term as a difference, for example $f(r) - f(r+1)$, so that almost everything cancels. You may first need **partial fractions** to put the general term in that form.
3. **Decide whether a series is convergent** by looking directly at its sum to n terms, S_n , as $n \rightarrow \infty$, and find the **sum to infinity** when it exists.

Proof by induction is a separate topic, although some questions put the two together.

Understand it

Sigma notation and the standard results

$\sum_{r=1}^n u_r$ means "put $r = 1, 2, 3, \dots, n$ into u_r and add": $u_1 + u_2 + \dots + u_n$. The letter r is only a counter; the answer is in terms of n .

Three sums are given in the formula list (MF19):

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1), \quad \sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1), \quad \sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2.$$

Two rules let you build almost any polynomial sum from them:

- **Split and pull out constants:** $\sum (a u_r + b v_r) = a \sum u_r + b \sum v_r$.
- **A constant added n times is n times the constant:** $\sum_{r=1}^n 5 = 5n$, not 5.

So to find $\sum_{r=1}^n r(r+3)$, expand first: $r^2 + 3r$. Then

$$\sum_{r=1}^n r(r+3) = \frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) = \frac{1}{6}n(n+1)[(2n+1) + 9] = \frac{1}{3}n(n+1)(n+5).$$

Take out the common factor $n(n+1)$ early, as here; the algebra is much shorter than multiplying everything out. Check with $n = 1$: the sum is $1 \times 4 = 4$, and $\frac{1}{3} \times 1 \times 2 \times 6 = 4$.

Sums that don't start at 1. A sum from $r = m$ to $r = n$ is everything up to n minus everything up to $m - 1$:

$$\sum_{r=m}^n u_r = S_n - S_{m-1}, \quad \text{where } S_k = \sum_{r=1}^k u_r.$$

For example, $\sum_{r=n+1}^{2n} r = S_{2n} - S_n = \frac{1}{2}(2n)(2n+1) - \frac{1}{2}n(n+1) = \frac{1}{2}n(3n+1)$. Subtract S_{m-1} , not S_m : the term u_m belongs in the answer.

The method of differences

Suppose each term is a difference of the same function at neighbouring values, $u_r = f(r) - f(r+1)$. Write the terms in a column:

$$\begin{array}{l} r = 1 : f(1) - f(2) \\ r = 2 : f(2) - f(3) \\ r = 3 : f(3) - f(4) \\ \vdots \\ r = n : f(n) - f(n+1) \end{array}$$

Every $-f(2)$ cancels the next $+f(2)$, and so on down the column. Only the first and last pieces survive:

$$\sum_{r=1}^n (f(r) - f(r+1)) = f(1) - f(n+1).$$

This is sometimes called a **telescoping** sum. It works just as well the other way round: $\sum (f(r+1) - f(r)) = f(n+1) - f(1)$.

For example, $\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$, so with $f(r) = \frac{1}{r}$,

$$\sum_{r=1}^n \frac{1}{r(r+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Gaps bigger than one. If $u_r = f(r) - f(r+2)$, each piece cancels with the one **two** rows further down, so **two** pieces survive at each end:

$$\sum_{r=1}^n (f(r) - f(r+2)) = f(1) + f(2) - f(n+1) - f(n+2).$$

With a gap of k , k pieces survive at each end. Writing the column out is the safest way to see exactly which ones.

Three pieces. Partial fractions of a product of three factors often give a pattern like $f(r) - 2f(r+1) + f(r+2)$. The coefficients add up to 0, which is what makes it cancel. Line the terms up so that equal f 's sit in the same column:

$$\begin{array}{rcll}
 r = 1 : & f(1) & -2f(2) & +f(3) \\
 r = 2 : & & f(2) & -2f(3) + f(4) \\
 r = 3 : & & & f(3) -2f(4) + f(5)
 \end{array}$$

Down each column the coefficients 1, -2, 1 add to 0, so the middle columns vanish. What is left is $f(1) - f(2)$ at the top and $-f(n+1) + f(n+2)$ at the bottom:

$$\sum_{r=1}^n (f(r) - 2f(r+1) + f(r+2)) = f(1) - f(2) - f(n+1) + f(n+2).$$

Getting the general term into the right form

The question usually tells you how, in one of these ways.

- **Partial fractions.** A fraction with linear factors in the denominator splits into simpler fractions, one per factor. For example,

$$\frac{2}{(r+2)(r+4)} = \frac{1}{r+2} - \frac{1}{r+4}.$$

Find the constants by the cover-up rule or by comparing coefficients. When the factors differ by d , as in $\frac{1}{(dr+a)(dr+a+d)}$, the answer is $\frac{1}{d} \left(\frac{1}{dr+a} - \frac{1}{dr+a+d} \right)$. If the numerator's degree is not lower than the denominator's, divide first: $\frac{r^2+2r+3}{r^2+2r} = 1 + \frac{3}{r(r+2)}$.

- **A given identity**, such as "show that $\frac{r}{r+1} - \frac{r-1}{r} = \frac{1}{r(r+1)}$ ". Put the left side over a common denominator and simplify to the right side; then the right side is your $f(r) - f(r+1)$.
- **Defined terms** such as $w_r = r(r+1)(r+2)$, or powers like $x^{f(r)} - x^{f(r+1)}$, $\ln$ terms or $\tan$ terms. These are already differences, or become them with the laws of logarithms: $\ln \frac{r+1}{r} = \ln(r+1) - \ln r$.

Products telescope too. With $w_r = r(r+1)(r+2)$,

$$w_{r+1} - w_r = (r+1)(r+2)[(r+3) - r] = 3(r+1)(r+2),$$

$$\text{so } \sum_{r=1}^n (r+1)(r+2) = \frac{1}{3}(w_{n+1} - w_1) = \frac{1}{3}((n+1)(n+2)(n+3) - 6).$$

Proving a standard result by differences

The same idea proves the formulas in MF19. Start from an identity whose left side telescopes, for example

$$(r+1)^3 - r^3 = 3r^2 + 3r + 1.$$

Sum both sides from $r = 1$ to n . The left side telescopes to $(n+1)^3 - 1$. The right side is $3 \sum r^2 + 3 \sum r + n$. So

$$3 \sum_{r=1}^n r^2 = (n+1)^3 - 1 - \frac{3}{2}n(n+1) - n,$$

which simplifies to $\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$. Only use results the question allows (here $\sum r$, which is lower down).

Convergence and the sum to infinity

An infinite series $u_1 + u_2 + u_3 + \dots$ is **convergent** if its sum to n terms, S_n , tends to a finite limit as $n \rightarrow \infty$. That limit is the **sum to infinity**, S_∞ . The syllabus asks you to decide this **directly from** S_n , which the method of differences hands you.

$$S_n = \frac{n}{n+1} = 1 - \frac{1}{n+1} \rightarrow 1 \text{ as } n \rightarrow \infty,$$

so $\sum \frac{1}{r(r+1)}$ converges and its sum to infinity is 1. The facts you use most:

- $\frac{1}{n}, \frac{1}{n+3}, \frac{1}{n^2+1}$ and so on all tend to 0.
- $x^n \rightarrow 0$ when $-1 < x < 1$; x^n grows without limit when $x > 1$, and when $x < -1$ it swings between larger and larger positive and negative values.
- $e^{-nx} \rightarrow 0$ when $x > 0$.

A series is **not convergent** when S_n grows without limit (for example $S_n = \ln(n+1)$ or $S_n = n^2$), or when it keeps oscillating (for example S_n contains $\cos 2n$, which keeps swinging between -1 and 1 as n increases, or $(-1)^n$).

When x is a letter, look at the term with n in it and split the cases. For $S_n = x - x^{n+1}$:

- $-1 < x < 1$: $x^{n+1} \rightarrow 0$, so $S_\infty = x$;
- $x = 1$: $S_n = 1 - 1 = 0$ for every n , so $S_\infty = 0$;
- $x = -1$: x^{n+1} alternates between 1 and -1 , so there is no limit;
- $|x| > 1$: x^{n+1} does not settle down, so no limit.

The series converges for $-1 < x \leq 1$. The end points always need checking by hand. If the power is always **even**, as $r(r+1)$ is, then $x = -1$ behaves like $x = 1$.

Limits of a sum divided by a power of n

A different kind of limit question: S_n is a polynomial in n , and you are asked for $\lim_{n \rightarrow \infty} \frac{S_n}{n^k}$, where k is the degree of S_n . Divide every factor by n :

$$\frac{\frac{1}{6}n(n+1)(2n+1)}{n^3} = \frac{1}{6} \times 1 \times \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) \rightarrow \frac{1}{6} \times 2 = \frac{1}{3}.$$

Only the leading term matters: the limit is the coefficient of n^k in S_n .

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\sum_{r=1}^n r = \frac{1}{2}n(n+1)$	Given
$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$	Given
$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$	Given
$\sum_{r=1}^n c = cn$ for a constant c	Learn it
$\sum (a u_r + b v_r) = a \sum u_r + b \sum v_r$	Learn it
$\sum_{r=m}^n u_r = S_n - S_{m-1}$	Learn it
$\sum_{r=1}^n (f(r) - f(r+1)) = f(1) - f(n+1)$	Learn it
$\sum_{r=1}^n (f(r) - f(r+2)) = f(1) + f(2) - f(n+1) - f(n+2)$	Learn it
$\frac{1}{(dr+a)(dr+a+d)} = \frac{1}{d} \left(\frac{1}{dr+a} - \frac{1}{dr+a+d} \right)$	Learn it
Convergent means S_n tends to a finite limit; $S_\infty = \lim_{n \rightarrow \infty} S_n$	Learn it
$x^n \rightarrow 0$ for $-1 < x < 1$; $\frac{1}{n^k} \rightarrow 0$ for $k > 0$	Learn it
$\ln a - \ln b = \ln \frac{a}{b}$, $\ln(ab) = \ln a + \ln b$, $x^{\log_x a} = a$	Learn it

How to do it

In the Paper 1 questions we have tagged for this topic (37 questions, 2021–2025; many appear on two papers of the same session), the types came up this often. 32 of the 37 questions mix two or more types, so the counts add up to more than 37.

Question type	Questions
Sum to infinity and convergence	26
Standard results for related sums	24
Method of differences with partial fractions	16
Method of differences with a given identity or given terms	11
Sums over other ranges ($r = 2$ to n , $n + 1$ to $2n$, n to n^2)	6
Proving a standard result by differences	5

Question type	Questions
Roots of a polynomial inside a sum	4
Limit of S_n divided by a power of n	3

The most common question has three parts: (a) a polynomial sum from the standard results, (b) a fraction summed by the method of differences, (c) "deduce the sum to infinity".

1. Sum to infinity and convergence (26 questions)

1. **Get S_n first**, usually from the part before.
2. **Let $n \rightarrow \infty$** . Say which terms tend to 0 ("as $n \rightarrow \infty$, $\frac{1}{n+2} \rightarrow 0$ "), and write down the limit.
3. **"Deduce" or "state"** the value: one line is enough, but it must follow from your S_n .

Variations you will meet:

- **A constant from the sum to infinity.** "Given that $S_\infty = \frac{2}{5}$, find c ": find S_∞ in terms of c , set it equal and solve. If you get a quadratic, reject a root that breaks a condition (for example " c is positive").
- **Set of values of x for convergence.** Find S_n , then look at the term that contains n and split into cases as in "Understand it": $|x| < 1$, $x = 1$, $x = -1$, $|x| > 1$ (or $x > 0$ and $x < 0$ for e^{nx}). Give the set, and the sum to infinity for each case where it exists; the end points can have a different sum. For $x^{g(n)}$ with $g(n)$ always even, $x = -1$ gives 1.
- **A power of n with a letter in it**, such as $S_n = 1 - (n+2)^{-x}$: $(n+2)^{-x} \rightarrow 0$ when $x > 0$, it equals 1 when $x = 0$, and it grows without limit when $x < 0$. So the series converges for $x \geq 0$, with sum 1 for $x > 0$ and 0 for $x = 0$. A power such as $a^{\ln m}$ can be rewritten as $m^{\ln a}$ to reach this form.
- **"Explain why the series does not converge."** Show what S_n does: it grows without limit, or it oscillates (for example $\cos 2n$ keeps swinging between -1 and 1). Saying "the terms don't tend to 0" or "it tends to infinity" is not enough when that is not what happens.
- **How close to the limit.** "Find the least n for which $S_\infty - S_n < 0.002$ ": write the difference in terms of n , solve the inequality and round **up** to the next whole number.

2. Standard results for related sums (24 questions)

1. **Expand** the general term into powers of r , for example $(r-1)(r+2)(r+4) = r^3 + 5r^2 + 2r - 8$.
2. **Replace** each $\sum r^k$ by its MF19 formula, and each constant c by cn .
3. **Simplify**: take out common factors such as n or $n(n+1)$ before expanding, then collect like terms. Give the form asked for ("fully factorise", or " $an^3 + bn^2 + cn$ with integers a, b, c ").
4. **Check with $n = 1$** (and $n = 2$ if you have time): the formula must give u_1 (and $u_1 + u_2$).

Variations you will meet:

- **A sum of sums**: if $S_n = 3n^2 - n$, then $\sum_{n=1}^N S_n = 3 \sum n^2 - \sum n$, written with N instead of n : $\frac{1}{2}N(N+1)(2N+1) - \frac{1}{2}N(N+1) = N^2(N+1)$.

- **A pattern to spot**, such as $1 + 2 + 4 + 5 + 7 + 8 + \dots$ up to $3n$ (the whole numbers that are not multiples of 3): take all the numbers up to $3n$ and subtract the multiples of 3, giving $\sum_{r=1}^{3n} r - 3 \sum_{r=1}^n r = \frac{3}{2}n(3n+1) - \frac{3}{2}n(n+1) = 3n^2$. In general, split the series into pieces that are each a sum over r .
- **Find a constant**: "given $\sum_{r=1}^n r(r+a) = \frac{1}{3}n(n+1)(n+5)$, find a ". Work out the left side in terms of a and n , then compare or substitute a value of n : with $n = 1$, $1 + a = 4$, so $a = 3$ (as in "Understand it").
- **A new result from a given one**: if a question gives you a formula for a sum that mixes a higher power (such as r^4 or r^5) with powers in MF19, subtract the MF19 parts and divide by the coefficient to get the sum of the higher power on its own.
- **Logarithms**: with $u_r = 2^r e^{r^2}$, $\ln u_r = r \ln 2 + r^2$, so $\sum \ln u_r = \frac{1}{2}n(n+1) \ln 2 + \frac{1}{6}n(n+1)(2n+1)$. Use the log laws first, then the standard results.

3. Method of differences with partial fractions (16 questions)

1. **Split into partial fractions**, dividing first if the fraction is improper. Check by recombining or with a value of r .
2. **Write it with one function f** , for example $\frac{3}{r+2} - \frac{4}{r+3} + \frac{1}{r+4} = 3f(r) - 4f(r+1) + f(r+2)$ with $f(r) = \frac{1}{r+2}$. Pull a constant like $\frac{1}{k}$ outside the sum, and remember to put it back.
3. **Write out the rows**: at least the first two or three values of r and the last two, lined up so the cancelling shows.
4. **Collect what survives** at the top and bottom, and simplify. It must be in terms of n (not r).

Variations you will meet:

- **A letter in the denominator**, such as $\frac{1}{(r+c)(r+c+1)} = \frac{1}{r+c} - \frac{1}{r+c+1}$, or $\frac{1}{(ar+b)(ar+a+b)}$, whose factors differ by a , so it is $\frac{1}{a}$ times a difference. Treat the letter as a number.
- **Starting at $r = 2$** (because $r = 1$ makes a factor zero): the first row is $r = 2$, so the survivors are $f(2)$ -type terms, not $f(1)$.
- **Improper fraction**, such as $\frac{2r^2+8r+7}{(r+1)(r+3)}$: divide to get $2 + \frac{1}{(r+1)(r+3)}$; then the 2 sums to $2n$.

4. Method of differences with a given identity or given terms (11 questions)

1. **"Show that"**: put the left side over a common denominator, expand and simplify until you reach the printed right side exactly.
2. **"Hence"**: sum both sides; the left side telescopes. Write out at least three rows including the last.
3. Simplify, then use it for the sum to infinity if asked.

Variations you will meet:

- **Powers**: $\sum (x^{f(r)} - x^{f(r+1)}) = x^{f(1)} - x^{f(n+1)}$, then a convergence question in x . An exponent that is a logarithm simplifies with $x^{\log_x a} = a$, for example $x^{\log_x(r+5)} = r+5$.
- **Products**: $w_r = (2r-1)(2r+1)(2r+3)$ gives $w_{r+1} - w_r = (2r+1)(2r+3)[(2r+5) - (2r-1)] = 6(2r+1)(2r+3)$: take out the factors the two products share. So $\sum_{r=1}^n (2r+1)(2r+3) = \frac{1}{6}(w_{n+1} - w_1) = \frac{1}{6}((2n+1)(2n+3)(2n+5) - 15)$.

- **Logarithms:** $\ln \frac{r+3}{r+1} = \ln(r+3) - \ln(r+1)$, a gap of two, so $\sum_{r=1}^n \ln \frac{r+3}{r+1} = \ln(n+2) + \ln(n+3) - \ln 2 - \ln 3 = \ln \frac{(n+2)(n+3)}{6}$. A log of a longer fraction splits into two or more telescoping sums the same way.
- **Trigonometry:** the compound-angle formulae give identities such as $\cos(2r-1) - \cos(2r+1) = 2 \sin 1 \sin 2r$, so $\sum_{r=1}^n \sin 2r = \frac{\cos 1 - \cos(2n+1)}{2 \sin 1}$. This sum oscillates, so $\sum \sin 2r$ does not converge.
- **Exponentials:** a factorised term such as $e^{-rx}(1 - e^{-x})$ is really $e^{-rx} - e^{-(r+1)x}$, so the sum is $e^{-x} - e^{-(n+1)x}$, which tends to e^{-x} when $x > 0$. Expand first to see the difference. It is also a geometric series, and "or otherwise" allows that.
- **Quadratic factors:** when the identity has factors like $r^2 + 1$ and $(r+1)^2 + 1$, notice that the second is the first with $r+1$ in place of r .

5. Sums over other ranges (6 questions)

1. **Find S_N as a formula** (from the standard results or the method of differences).
2. **Use $\sum_{r=m}^N = S_N - S_{m-1}$** , putting in the right expressions for N and $m-1$. For $\sum_{r=n+1}^{2n}$ that is $S_{2n} - S_n$; for $\sum_{r=n}^{n^2}$ it is $S_{n^2} - S_{n-1}$.
3. **Or use differences directly:** the telescoped answer is "first surviving piece at the bottom limit minus last surviving piece at the top limit", for example $\sum_{r=m}^N (f(r) - f(r+1)) = f(m) - f(N+1)$.
4. Simplify over a common denominator if the question wants one fraction.

A sum from $r = n$ to ∞ is $S_\infty - S_{n-1}$.

6. Proving a standard result by differences (5 questions)

1. **Expand the given difference**, for example $(r+1)^3 - r^3 = 3r^2 + 3r + 1$ (as in "Understand it").
2. **Sum the left side** by the method of differences; write enough rows to show the cancelling. Here it gives $(n+1)^3 - 1$.
3. **Sum the right side** term by term, remembering that a constant sums to n times itself: $3 \sum r^2 + 3 \sum r + n$.
4. **Set them equal and rearrange** to the printed formula, showing every step (it is a "show that").

A difference of fourth or fifth powers brings in $\sum r^3$ or $\sum r^4$, and the same steps prove those formulas.

Differences of odd numbers, such as $(2r+1)^k - (2r-1)^k$, work the same way: the left side telescopes to $(2n+1)^k - 1$.

7. Roots of a polynomial inside a sum (4 questions)

1. **Find the sums of powers of the roots** first, as in the roots topic. From the coefficients, $\sum \alpha^2 = (\sum \alpha)^2 - 2 \sum \alpha\beta$. For a higher power, put each root into the equation and add: for $x^4 + bx^3 + cx^2 + dx + e = 0$ this gives $S_4 + bS_3 + cS_2 + dS_1 + 4e = 0$, where $S_k = \sum \alpha^k$. For example, $x^4 + 5x^2 + 2x + 9 = 0$ has $S_1 = 0$ and $S_2 = 0 - 2 \times 5 = -10$, so $S_4 = -(5S_2 + 2S_1 + 36) = 14$.

2. **Expand one bracket**, for example $(\alpha r - 1)^2 = \alpha^2 r^2 - 2\alpha r + 1$, and add over the roots. Each power of r gets a coefficient made of root sums: here $S_2 r^2 - 2S_1 r + 4$ for four roots. A term without a root in it appears once **per root**, so the 1 becomes 4 for a quartic.
3. **Only then** sum over r with the standard results.

Putting in the numbers for the root sums before summing over r keeps the algebra short.

8. Limit of S_n divided by a power of n (3 questions)

1. Write S_n as a product of factors.
2. Divide by n^k by dividing each factor by n (one n per factor of degree one).
3. Let $n \rightarrow \infty$: every $\frac{1}{n}$ term goes to 0. The limit is the leading coefficient.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
-

Example 1

(a) Use standard results from the list of formulae to show that

$$\sum_{r=1}^n (4r - 1)(4r + 3) = \frac{1}{3}n(16n^2 + 36n + 11).$$

[3]

(b) Use the method of differences to find $\sum_{r=1}^n \frac{1}{(4r - 1)(4r + 3)}$ in terms of n . **[4]**

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(4r - 1)(4r + 3)}$. **[1]**

(a) $(4r - 1)(4r + 3) = 16r^2 + 8r - 3$. **B1**

$$\sum_{r=1}^n (16r^2 + 8r - 3) = 16 \times \frac{1}{6}n(n+1)(2n+1) + 8 \times \frac{1}{2}n(n+1) - 3n.$$

M1 for using both formulae and $-3n$.

$$= \frac{1}{3}n[8(2n^2 + 3n + 1) + 12(n + 1) - 9] = \frac{1}{3}n(16n^2 + 36n + 11).$$

A1, the given answer with the step shown. Check with $n = 1$: $3 \times 7 = 21$ and $\frac{1}{3}(16 + 36 + 11) = 21$.

(b) The factors differ by 4, so

$$\frac{1}{(4r-1)(4r+3)} = \frac{1}{4} \left(\frac{1}{4r-1} - \frac{1}{4r+3} \right).$$

M1 A1 for the partial fractions.

$$\sum_{r=1}^n = \frac{1}{4} \left[\left(\frac{1}{3} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{11} \right) + \cdots + \left(\frac{1}{4n-1} - \frac{1}{4n+3} \right) \right].$$

M1 for three complete terms, including the last, showing the cancelling.

$$= \frac{1}{4} \left(\frac{1}{3} - \frac{1}{4n+3} \right) = \frac{n}{3(4n+3)}.$$

A1

(c) As $n \rightarrow \infty$, $\frac{1}{4n+3} \rightarrow 0$, so the sum to infinity is $\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$. **B1**

Example 2

(a) Show that $\frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1} = \frac{2r}{r^4 + r^2 + 1}$. **[2]**

(b) Hence use the method of differences to find $\sum_{r=1}^n \frac{r}{r^4 + r^2 + 1}$ in terms of n . **[3]**

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{r}{r^4 + r^2 + 1}$. **[1]**

(a) Over a common denominator:

$$\frac{(r^2 + r + 1) - (r^2 - r + 1)}{(r^2 - r + 1)(r^2 + r + 1)} = \frac{2r}{(r^2 + 1)^2 - r^2} = \frac{2r}{r^4 + r^2 + 1}.$$

M1 for the common denominator, **A1** for expanding the product to $r^4 + r^2 + 1$.

(b) Let $f(r) = \frac{1}{r^2 - r + 1}$. Then $f(r+1) = \frac{1}{(r+1)^2 - (r+1) + 1} = \frac{1}{r^2 + r + 1}$, so each term is $\frac{1}{2}(f(r) - f(r+1))$. **M1** for seeing this.

$$\sum_{r=1}^n \frac{r}{r^4 + r^2 + 1} = \frac{1}{2} \left[\left(1 - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{7} \right) + \cdots + \left(\frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1} \right) \right]$$

M1 for three complete terms, including the last.

$$= \frac{1}{2} \left(1 - \frac{1}{n^2 + n + 1} \right) = \frac{n(n+1)}{2(n^2 + n + 1)}.$$

A1

(c) $\frac{1}{n^2+n+1} \rightarrow 0$, so the sum to infinity is $\frac{1}{2}$. **B1**

Example 3

(a) Express $\frac{r+3}{r(r+1)(r+2)}$ in partial fractions. **[2]**

(b) Hence show that $\sum_{r=1}^n \frac{r+3}{r(r+1)(r+2)} = \frac{5}{4} - \frac{2n+5}{2(n+1)(n+2)}$. **[4]**

(c) Find $\sum_{r=n+1}^{2n} \frac{r+3}{r(r+1)(r+2)}$, giving your answer as a single fraction. **[3]**

(a) $\frac{r+3}{r(r+1)(r+2)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$, so $r+3 = A(r+1)(r+2) + Br(r+2) + Cr(r+1)$.

$r = 0: 3 = 2A, A = \frac{3}{2}$. $r = -1: 2 = -B, B = -2$. $r = -2: 1 = 2C, C = \frac{1}{2}$. **M1 A1**

$$\frac{r+3}{r(r+1)(r+2)} = \frac{1}{2} \left(\frac{3}{r} - \frac{4}{r+1} + \frac{1}{r+2} \right).$$

(b) With $f(r) = \frac{1}{r}$, the term is $\frac{1}{2}(3f(r) - 4f(r+1) + f(r+2))$. Write the rows lined up:

$$\begin{array}{rcl} r = 1 : & 3f(1) - 4f(2) + f(3) \\ r = 2 : & 3f(2) - 4f(3) + f(4) \\ r = 3 : & 3f(3) - 4f(4) + f(5) \\ & \vdots \\ r = n-1 : & 3f(n-1) - 4f(n) + f(n+1) \\ r = n : & 3f(n) - 4f(n+1) + f(n+2) \end{array}$$

M1 for enough rows, including the first two and last two. **A1** for seeing that each f from $f(3)$ to $f(n)$ has total coefficient $3 - 4 + 1 = 0$.

The survivors are $3f(1), 3f(2) - 4f(2) = -f(2), -4f(n+1) + f(n+1) = -3f(n+1)$ and $f(n+2)$:

$$S_n = \frac{1}{2} \left(3 - \frac{1}{2} - \frac{3}{n+1} + \frac{1}{n+2} \right) = \frac{5}{4} - \frac{3(n+2) - (n+1)}{2(n+1)(n+2)} = \frac{5}{4} - \frac{2n+5}{2(n+1)(n+2)}.$$

M1 A1, the given answer. Check with $n = 1: \frac{4}{6} = \frac{2}{3}$ and $\frac{5}{4} - \frac{7}{12} = \frac{8}{12} = \frac{2}{3}$.

(c) $\sum_{r=n+1}^{2n} = S_{2n} - S_n$. **M1** for subtracting S_n (not S_{n+1}).

$$S_{2n} = \frac{5}{4} - \frac{4n+5}{2(2n+1)(2n+2)} = \frac{5}{4} - \frac{4n+5}{4(n+1)(2n+1)}, \text{ so}$$

$$S_{2n} - S_n = \frac{2n+5}{2(n+1)(n+2)} - \frac{4n+5}{4(n+1)(2n+1)} = \frac{2(2n+5)(2n+1) - (4n+5)(n+2)}{4(n+1)(n+2)(2n+1)}.$$

M1 for a common denominator. The numerator is $8n^2 + 24n + 10 - (4n^2 + 13n + 10) = 4n^2 + 11n$, so

$$\sum_{r=n+1}^{2n} \frac{r+3}{r(r+1)(r+2)} = \frac{n(4n+11)}{4(n+1)(n+2)(2n+1)}.$$

A1**Example 4**

Let $u_r = x^{r(r+1)} - x^{(r+1)(r+2)}$, where x is a real constant.

(a) Find $\sum_{r=1}^n u_r$ in terms of n and x . **[2]**

(b) Find the set of values of x for which the infinite series $u_1 + u_2 + u_3 + \dots$ is convergent, and give the sum to infinity when it exists. **[3]**

(a) With $f(r) = x^{r(r+1)}$, $u_r = f(r) - f(r+1)$, because $(r+1)(r+2)$ is $r(r+1)$ with $r+1$ in place of r .

$$\sum_{r=1}^n u_r = (x^2 - x^6) + (x^6 - x^{12}) + \dots + (x^{n(n+1)} - x^{(n+1)(n+2)}) = x^2 - x^{(n+1)(n+2)}.$$

M1 for three terms including the last, **A1**.

(b) The power $(n+1)(n+2)$ grows without limit, and it is always even (one of two neighbouring whole numbers is even).

- If $-1 < x < 1$: $x^{(n+1)(n+2)} \rightarrow 0$, so the sum to infinity is x^2 . **M1 A1**
- If $x = 1$ or $x = -1$: $x^{(n+1)(n+2)} = 1$ and $x^2 = 1$, so every $S_n = 0$ and the sum to infinity is 0.
- If $|x| > 1$: $x^{(n+1)(n+2)}$ grows without limit, so the series diverges.

The series converges for $-1 \leq x \leq 1$, with sum x^2 when $-1 < x < 1$ and 0 when $x = \pm 1$. **A1**

Common mistakes

- **Summing a constant to itself.** $\sum_{r=1}^n 3 = 3n$, not 3. Examiners saw this when candidates summed both sides of an identity and left the constant on the right as it was, instead of multiplying it by n .
- **Too few terms written out.** A single row, or only the first and last, doesn't show the cancelling and loses the method mark. Write at least three complete rows including the first and last (with a three-piece pattern, the first two and last two), lined up in columns.
- **Cancelling terms that don't match.** Only identical pieces cancel. If your partial fractions are wrong, the cancelling won't work: check them before you start.
- **Leaving the answer in r .** The sum to n terms has only n in it. Candidates also sometimes wrote $f(r+1)$ where the last survivor is $f(n+1)$.
- **Losing a factor taken outside.** If you take out $\frac{1}{k}$ or $\frac{1}{2}$ to make the writing easier, put it back in the final answer.

- **Wrong numerators in partial fractions.** With a quadratic factor such as $r^2 + 5$, the numerator is $Ar + B$; assuming $B = 0$ gives a wrong answer. Often the question gives an identity that makes partial fractions unnecessary: use it.
- $S_n - S_m$ **instead of** $S_n - S_{m-1}$. A sum from $r = n + 1$ needs S_n subtracted, and a sum from $r = n$ needs S_{n-1} .
- **Guessing about convergence.** When S_n contains $\cos 2n$ or $(-1)^n$, it oscillates; it doesn't "tend to infinity". Say what actually happens to S_n .
- **Forgetting end points and conditions.** Check $x = 1$ and $x = -1$ separately, and throw out a value of a constant that breaks a condition such as " $a > 0$ ".
- **Log laws.** $\ln(ab) = \ln a + \ln b$, but $\ln(a + b)$ does not split. Candidates who split logs wrongly lost the rest of the question.
- **Missing zero coefficients.** For a polynomial with no x^3 term, $\sum \alpha = 0$; forgetting that spoils every root sum that follows.

How to get the marks

- **"Use standard results from the list of formulae"** means write each formula in with the right coefficient, then simplify. The substitution line earns the method mark even if the algebra slips later.
- **"Show that"** means the answer is given: show the expansion, the common factor and the simplification, and end with the printed form exactly.
- **"Use the method of differences"** needs the cancelling to be visible: enough complete rows, the first and last included. Just writing the answer, even a correct one, may get no marks, or a single special-case mark.
- **"Hence"** means use the previous part: the identity you just proved, or the S_n you just found.
- **"Deduce" or "state"** the sum to infinity is usually one mark, awarded on your own S_n if it has the right form. Saying which terms tend to 0 makes it safe.
- **"Find the set of values"** needs an interval (with the end points right) and the sum for each case.
- **Give answers in the form asked** ("in terms of n ", "fully factorised", "as a single fraction"). An equivalent correct form is usually accepted unless the form is specified.
- **Follow-through marks.** A wrong S_n used correctly still earns the sum-to-infinity mark, so always finish the question.

Quick check

1. Use standard results to show that $\sum_{r=1}^n r^2(r-1) = \frac{1}{12}n(n+1)(n-1)(3n+2)$.
2. Find $\sum_{r=1}^n \frac{1}{(r+1)(r+3)}$ in terms of n , and deduce its sum to infinity.
3. Find $\sum_{r=n+1}^{2n} r^2$ in terms of n , fully factorised.

4. Show that the series $\sum_{r=1}^{\infty} \ln \left(1 + \frac{1}{r} \right)$ is not convergent.
5. Show that $r \times r! = (r + 1)! - r!$, and hence find $\sum_{r=1}^n r \times r!$.

Answers

1. $\sum(r^3 - r^2) = \frac{1}{4}n^2(n+1)^2 - \frac{1}{6}n(n+1)(2n+1) = \frac{1}{12}n(n+1)[3n(n+1) - 2(2n+1)] = \frac{1}{12}n(n+1)(3n^2 - n - 2)$, and $3n^2 - n - 2 = (n-1)(3n+2)$. Check $n=2$: $0+4=4$ and $\frac{1}{12} \times 2 \times 3 \times 1 \times 8 = 4$.
2. $\frac{1}{(r+1)(r+3)} = \frac{1}{2} \left(\frac{1}{r+1} - \frac{1}{r+3} \right)$, a gap of two, so two pieces survive at each end: $\frac{1}{2} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right) = \frac{5}{12} - \frac{2n+5}{2(n+2)(n+3)}$. As $n \rightarrow \infty$ the fractions in n tend to 0, so the sum to infinity is $\frac{5}{12}$.
3. $S_{2n} - S_n = \frac{1}{6}(2n)(2n+1)(4n+1) - \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}n(2n+1)[2(4n+1) - (n+1)] = \frac{1}{6}n(2n+1)(7n+1)$.
4. $\ln\left(1 + \frac{1}{r}\right) = \ln(r+1) - \ln r$, so $S_n = \ln(n+1) - \ln 1 = \ln(n+1)$. This grows without limit as $n \rightarrow \infty$, so the series is not convergent.
5. $(r+1)! - r! = r![(r+1) - 1] = r \times r!$. The sum telescopes: $(2! - 1!) + (3! - 2!) + \dots + ((n+1)! - n!) = (n+1)! - 1$.

Past questions to try

- **9231/12/O/N/25 Q1**: standard results, then partial fractions with a sum that starts at $r=2$.
- **9231/12/O/N/24 Q5**: differences of powers of x , convergence in x , and a sum of sums.
- **9231/13/M/J/24 Q4**: proving $\sum r^4$ by differences, then a limit.
- **9231/13/M/J/23 Q2**: an improper fraction, the method of differences, and the sum from $n+1$ to $2n$.