

Vectors

AS

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What you need to know

This topic builds on the vectors in Pure Mathematics 3: position vectors, the scalar (dot) product, the angle between two vectors and the vector equation of a line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. By the end of it you should be able to:

1. **Use the equation of a plane in all three forms** and change from one to another when a problem needs it:
 - Cartesian: $ax + by + cz = d$;
 - scalar product form: $\mathbf{r} \cdot \mathbf{n} = p$;
 - parametric form: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.
2. **Recall the vector product** $\mathbf{a} \times \mathbf{b}$ in two ways: as $|\mathbf{a}||\mathbf{b}| \sin \theta \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector at right angles to both $\mathbf{a}$ and $\mathbf{b}$, and in components.
3. **Solve problems about distances, angles and intersections** with lines and planes, using scalar and vector products. In particular:
 - decide whether a line **lies in** a plane, is **parallel** to it, or **meets** it, and find the point where it meets it;
 - find the **foot of the perpendicular** from a point to a plane;
 - find the **angle between a line and a plane**, and the **angle between two planes**;
 - find an equation of the **line of intersection of two planes**;
 - find the **shortest distance between two skew lines**;
 - find an equation of the **common perpendicular** to two skew lines.

The perpendicular distance from a point to a plane comes up in almost every exam, so this note teaches it too; it sits on the way to the foot of the perpendicular.

Understand it

Position vectors and direction vectors

A **position vector** says where a point is: $\overrightarrow{OA} = \mathbf{a}$ goes from the origin to A . A **direction vector** only says which way something points: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ is the direction from A to B , wherever A is. In the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\mathbf{a}$ is a position vector (a point on the line) and $\mathbf{b}$ is a direction vector. Every method in this topic needs the right kind in the right place.

The scalar product, a reminder

$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}| \cos \theta$, where θ is the angle between the two vectors. So $\mathbf{a} \cdot \mathbf{b} = 0$ exactly when the vectors are at right angles. The scalar product gives a **number**.

The vector product

The vector product gives a **vector**, and that vector is at right angles to both of the vectors you started with:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta \hat{\mathbf{n}},$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\hat{\mathbf{n}}$ is a unit vector perpendicular to both (its direction follows the right-hand rule: first finger along $\mathbf{a}$, second along $\mathbf{b}$, thumb along $\hat{\mathbf{n}}$). In components,

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

A good way to remember it is as a determinant with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ on the top row:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

For example,

$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1(4) - (-2)(-1) \\ (-2)(1) - 3(4) \\ 3(-1) - 1(1) \end{pmatrix} = \begin{pmatrix} 2 \\ -14 \\ -4 \end{pmatrix}.$$

Check it every time: dot the answer with each of the two vectors; both must give 0. Here $6 - 14 + 8 = 0$ and $2 + 14 - 16 = 0$.

Useful facts: $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$ (the order matters, but only for the sign), and $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ when $\mathbf{a}$ and $\mathbf{b}$ are parallel. Any non-zero multiple of a normal is also a normal, so simplify: $(2, -14, -4)$ can be used as $(1, -7, -2)$.

Three ways to write a plane

A plane is fixed by one point on it and its **normal** $\mathbf{n}$, a vector at right angles to every line in the plane.

Scalar product form. If A is on the plane, then for any point R on it, $\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$ lies in the plane, so $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$. That is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = p.$$

Cartesian form. Write $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$: then $\mathbf{r} \cdot \mathbf{n} = p$ becomes $ax + by + cz = p$. So **the coefficients of x, y, z are the normal**, and you can read it straight off: $2x - 5y + z = 4$ has normal $2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$.

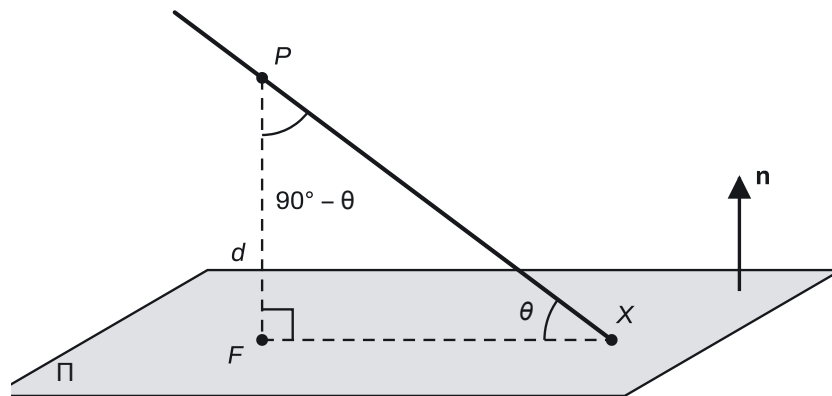
Parametric form. $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$: start at the point A and move any amount along two different directions $\mathbf{b}$ and $\mathbf{c}$ that lie in the plane. Each pair of values of λ and μ gives one point of the plane.

Converting.

- Parametric to Cartesian: the normal is $\mathbf{n} = \mathbf{b} \times \mathbf{c}$ (perpendicular to both directions), and $d = \mathbf{a} \cdot \mathbf{n}$.
- Three points A, B, C to Cartesian: the directions are $\overrightarrow{AB}$ and $\overrightarrow{AC}$, so $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$, then substitute one point. For a plane through the origin, such as OAB , the directions are simply $\mathbf{a}$ and $\mathbf{b}$, and $d = 0$.
- Cartesian to parametric: find three points on the plane that are not on one line, then use one as $\mathbf{a}$ and the two directions between them. For $x - 2y + 3z = 6$, the points $(6, 0, 0)$, $(0, -3, 0)$ and $(0, 0, 2)$ lie on it, so $\mathbf{r} = 6\mathbf{i} + \lambda(-6\mathbf{i} - 3\mathbf{j}) + \mu(-6\mathbf{i} + 2\mathbf{k})$ is one answer. (There are many correct answers.)

The **distance of the plane** $ax + by + cz = d$ **from the origin** is $\frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$. For $x - 2y + 3z = 6$ it is $\frac{6}{\sqrt{14}} = 1.60$.

A point, a plane and a line



Perpendicular distance from a point to a plane. The shortest way from P to the plane is straight along the normal, to the foot F . For $P(x_1, y_1, z_1)$ and the plane $ax + by + cz = d$,

$$\text{distance} = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

The top is how far P is from satisfying the plane's equation; dividing by $|\mathbf{n}|$ turns that into a length.

Foot of the perpendicular. The line through P along the normal is $\mathbf{r} = \mathbf{p} + t\mathbf{n}$. Where it meets the plane is F .

A line and a plane. A line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ either meets a plane at one point, or never meets it (parallel), or lies in it. The test is $\mathbf{b} \cdot \mathbf{n}$:

- $\mathbf{b} \cdot \mathbf{n} \neq 0$: the line meets the plane at exactly one point. Substitute the line's general point into the plane's equation and solve for λ .
- $\mathbf{b} \cdot \mathbf{n} = 0$: the line is at right angles to the normal, so it runs parallel to the plane. Then test one point: if $\mathbf{a}$ satisfies the plane's equation the line **lies in** the plane; if not, it is parallel and every point of it is the same distance from the plane (use the distance formula with the point A).

Angle between a line and a plane. This is the angle θ between the line and its "shadow" in the plane (the angle PXF in the diagram). The normal is easier to work with: the angle between the line's direction $\mathbf{b}$ and the normal is $90^\circ - \theta$. So

$$\cos(90^\circ - \theta) = \sin \theta = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}||\mathbf{n}|}.$$

Either work out the angle with the normal and take it from 90° , or use $\sin$ directly.

Two planes

Angle between two planes. Two planes meet at the same angle as their normals do, so

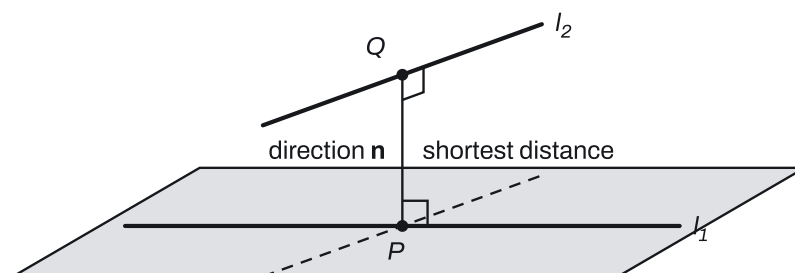
$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}.$$

The modulus gives the **acute** angle. There is **no** 90° step here.

Line of intersection. Two planes that are not parallel meet in a line. That line lies in both planes, so it is at right angles to both normals: its direction is $\mathbf{n}_1 \times \mathbf{n}_2$. For a point on it, choose one coordinate (often $x = 0$, $y = 0$ or $z = 0$) and solve the two plane equations together for the other two. If that gives no solution, the line never has that coordinate equal to 0; choose a different one.

Skew lines and the common perpendicular

Two lines in space that are not parallel and never meet are **skew**. Between them there is one shortest segment PQ , and it is at right angles to **both** lines. Its direction is $\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2$, the **common perpendicular**.



Shortest distance. Take any point A_1 on the first line and any point A_2 on the second. The vector $\mathbf{a}_2 - \mathbf{a}_1$ goes from one line to the other, but usually not by the shortest route. Its component along $\mathbf{n}$ is the shortest distance:

$$\text{shortest distance} = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}.$$

This works for **any** choice of the two points, because moving along either line doesn't change the component along $\mathbf{n}$. The diagram shows why: the second line is parallel to the plane that contains the first line and has normal $\mathbf{n}$, so the shortest distance is just the distance of any point of l_2 from that plane. If the answer is 0, the lines meet.

The common perpendicular itself. To find the actual points P and Q you need parameters. Write $P = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $Q = \mathbf{a}_2 + \mu \mathbf{b}_2$, so $\overrightarrow{PQ}$ is in terms of λ and μ . Because PQ is perpendicular to both lines, $\overrightarrow{PQ} \cdot \mathbf{b}_1 = 0$ and $\overrightarrow{PQ} \cdot \mathbf{b}_2 = 0$: two equations for two unknowns. Solve them, find P (or Q), and the common perpendicular is $\mathbf{r} = \overrightarrow{OP} + t \mathbf{n}$.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. Only the scalar product is in MF19; learn the rest.

Formula	Status
$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \mathbf{a} \mathbf{b} \cos \theta$	Given
$\mathbf{a} \times \mathbf{b} = \mathbf{a} \mathbf{b} \sin \theta \hat{\mathbf{n}}$	Learn it
$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$	Learn it
$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$; $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ for parallel vectors	Learn it
Line: $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$	Learn it
Plane: $ax + by + cz = d$, $\mathbf{r} \cdot \mathbf{n} = p$, $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$	Learn it
Normal of $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$: $\mathbf{n} = \mathbf{b} \times \mathbf{c}$, and $d = \mathbf{a} \cdot \mathbf{n}$	Learn it
Distance from (x_1, y_1, z_1) to $ax + by + cz = d$: $\frac{ ax_1 + by_1 + cz_1 - d }{\sqrt{a^2 + b^2 + c^2}}$	Learn it
Distance of $ax + by + cz = d$ from O : $\frac{ d }{\sqrt{a^2 + b^2 + c^2}}$	Learn it
Angle between line and plane: $\sin \theta = \frac{ \mathbf{b} \cdot \mathbf{n} }{ \mathbf{b} \mathbf{n} }$	Learn it
Angle between planes: $\cos \theta = \frac{ \mathbf{n}_1 \cdot \mathbf{n}_2 }{ \mathbf{n}_1 \mathbf{n}_2 }$	Learn it
Line of intersection of two planes has direction $\mathbf{n}_1 \times \mathbf{n}_2$	Learn it
Shortest distance between skew lines: $\frac{ (\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2) }{ \mathbf{b}_1 \times \mathbf{b}_2 }$	Learn it
Common perpendicular: $\overrightarrow{PQ} \cdot \mathbf{b}_1 = 0$ and $\overrightarrow{PQ} \cdot \mathbf{b}_2 = 0$	Learn it

Line and plane: $\mathbf{b} \cdot \mathbf{n} \neq 0$ means they meet at one point; $\mathbf{b} \cdot \mathbf{n} = 0$ means parallel, and the line lies in the plane if one of its points does.

How to do it

In the Further Pure 1 questions we have tagged for this topic (32 questions, 2021–2025), the types came up this often. Every one of the 32 questions has several parts of different types, so the counts add up to more than 32.

Question type	Questions
Equation of a plane	31
Perpendicular distance from a point to a plane	15
Shortest distance between skew lines	12
Angle between two planes	11
Angle between a line and a plane	11
Common perpendicular to two skew lines	5
Line of intersection of two planes	5
A line and a plane: parallel, or where they meet	5
Foot of the perpendicular from a point to a plane	4

1. Equation of a plane (31 questions)

1. Get two direction vectors in the plane.

- Three points A, B, C : use $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (subtract position vectors: $\mathbf{b} - \mathbf{a}$).
- Parametric form given: use $\mathbf{b}$ and $\mathbf{c}$ as they are.
- "Contains the line l_1 and is parallel to l_2 ": use the two lines' directions.
- "Contains the line l and the point P ": use the line's direction and $\overrightarrow{AP}$, where A is the point on the line.
- "Contains two lines" that meet: use both directions.

2. **Normal $\mathbf{n}$** = the vector product of the two directions. Check it by dotting with both. Divide out any common factor.

3. **Constant $d = \mathbf{n} \cdot \mathbf{a}$** for any point A on the plane. Check with a second point if you have one.

4. Write the answer in the form asked for: $ax + by + cz = d$, or $\mathbf{r} \cdot \mathbf{n} = p$, or $\mathbf{r} = \dots$

Variations you will meet:

- "Write down" the parametric form.** Just write $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$ with a point and two directions from the question (for example the direction of a line and the common perpendicular you found). It is worth one or two marks and needs no working, but it must start with $\mathbf{r} =$.
- A plane through the origin**, such as OAB : the normal is $\mathbf{a} \times \mathbf{b}$ and $d = 0$. Read the letters carefully; OAB is not ABC .
- A point with a letter in it**: work the same way, keeping the letter. If the normal comes out as, say, $(3 - s)\mathbf{j}$ and you are told $s \neq 3$, you may divide by $3 - s$ and use $\mathbf{j}$.
- A plane containing a given line of intersection and a point**: use the line's direction and the direction from a point on the line to the given point.

2. Perpendicular distance from a point to a plane (15 questions)

1. Write the plane as $ax + by + cz = d$ (find it first if needed).

2. Substitute the point: distance = $\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$. Keep the $-d$.
3. Give an exact answer, or 3 significant figures.

Variations you will meet:

- **From the origin:** the top is just $|d|$.
- **The distance between a parallel line and the plane:** first show the line is parallel ($\mathbf{b} \cdot \mathbf{n} = 0$), then use the distance of any point of the line.
- **A given distance, an unknown in the point:** set the formula equal to the distance and solve. Because of the modulus there may be two answers (the point could be on either side of the plane); reject any that break a condition in the question, such as " a is positive".
- **Another way:** find the foot F (type 9) and work out $|PF|$. It takes longer but gets the same answer.

3. Shortest distance between skew lines (12 questions)

1. Write each line as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. When lines are given by points, such as "the lines AB and CD ", the directions are $\overrightarrow{AB}$ and $\overrightarrow{CD}$ and the points are A and C .
2. $\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2$, simplified.
3. A vector between the lines: $\mathbf{a}_2 - \mathbf{a}_1$ (any point of one minus any point of the other, for example $\overrightarrow{AC}$).
4. Shortest distance = $\frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{n}|}{|\mathbf{n}|}$.

Variations you will meet:

- **An unknown constant and a given distance.** Everything goes through with the letter in it; then set the distance equal to the given value, square both sides and solve. For example, the lines $\mathbf{r} = s(2\mathbf{i} + \mathbf{k})$ and $\mathbf{r} = a\mathbf{i} + \mathbf{j} + t(\mathbf{j} + \mathbf{k})$ have $\mathbf{n} = (2\mathbf{i} + \mathbf{k}) \times (\mathbf{j} + \mathbf{k}) = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, with $|\mathbf{n}| = 3$, and $(a\mathbf{i} + \mathbf{j}) \cdot \mathbf{n} = -a - 2$. If the shortest distance is 2, then $\frac{|a + 2|}{3} = 2$, so $(a + 2)^2 = 36$ and $a = 4$ or $a = -8$. If $\mathbf{n}$ has the letter in it too, $|\mathbf{n}|$ is a square root of an expression; squaring gives a quadratic (a "show that" usually gives you the quadratic to reach).
- **"Find the length PQ ",** where PQ is perpendicular to both lines: this is the shortest distance, so use this method; you don't need P and Q themselves.

4. Angle between two planes (11 questions)

1. Find both normals (from the Cartesian equations, or by vector products).
2. $\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$.
3. Give the acute angle in degrees to 1 decimal place (or radians to 3 significant figures). Don't take it from 90° .

Variations: one plane is described by points (ABD , OAB) or by a line and a point, so find its normal first with a vector product. If your angle is obtuse because you left out the modulus, take it from 180° .

5. Angle between a line and a plane (11 questions)

1. Direction of the line $\mathbf{b}$ (for a line through two points, the vector between them; for OA , just $\mathbf{a}$) and the plane's normal $\mathbf{n}$.
2. $\cos \phi = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}||\mathbf{n}|}$ gives the angle ϕ between the line and the **normal**.
3. The angle with the plane is $\theta = 90^\circ - \phi$. (Or go straight there with $\sin \theta = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}||\mathbf{n}|}$.)

Variation: an unknown in the direction, angle given. Set $\sin \theta$ (or $\cos \phi$) equal to the value that angle gives, square and solve; keep only the values that fit the conditions (for example " b is positive").

6. Common perpendicular to two skew lines (5 questions)

1. **General points.** $P = \mathbf{a}_1 + \lambda \mathbf{b}_1$ on the first line, $Q = \mathbf{a}_2 + \mu \mathbf{b}_2$ on the second, and $\overrightarrow{PQ} = Q - P$ in terms of λ and μ . Say what P and Q are.
2. **Two equations.** $\overrightarrow{PQ} \cdot \mathbf{b}_1 = 0$ and $\overrightarrow{PQ} \cdot \mathbf{b}_2 = 0$. Simplify each to a linear equation in λ and μ .
3. **Solve** them together, then substitute to get $\overrightarrow{OP}$ (and $\overrightarrow{OQ}$ as a check: $\overrightarrow{PQ}$ must be a multiple of $\mathbf{b}_1 \times \mathbf{b}_2$).
4. **Answer:** $\mathbf{r} = \overrightarrow{OP} + t(\mathbf{b}_1 \times \mathbf{b}_2)$. Write the $\mathbf{r} =$.

Variations you will meet:

- **Lines given by points**, such as OC and AB : OC is $\mathbf{r} = \lambda \mathbf{c}$ and AB is $\mathbf{r} = \mathbf{a} + \mu(\mathbf{b} - \mathbf{a})$.
- **Another way to set up the equations:** find $\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2$ first and write $\overrightarrow{PQ} = k\mathbf{n}$. That gives three equations in λ , μ and k . You need the extra unknown k ; $\overrightarrow{PQ}$ is a multiple of $\mathbf{n}$, not $\mathbf{n}$ itself.
- **"Find the points P and Q "** or **"the length PQ "**: the same working gives both.

7. Line of intersection of two planes (5 questions)

1. Direction: $\mathbf{n}_1 \times \mathbf{n}_2$, simplified. Check it by dotting with both normals.
2. A point: put one coordinate equal to 0 and solve the two plane equations for the other two. Check the point in both equations.
3. Answer: $\mathbf{r} = (\text{point}) + t(\text{direction})$, starting with $\mathbf{r} =$.

Variation: one plane is given in parametric or $\mathbf{r} \cdot \mathbf{n}$ form. Turn it into Cartesian form first (type 1); the equation you need for the point is the Cartesian one.

8. A line and a plane: parallel, or where they meet (5 questions)

Where they meet.

1. Write the line's general point as one vector, for example $(2 + t, -3, 5 + t)$. For a line through O and D , it is $t\mathbf{d}$.
2. Substitute into the plane's Cartesian equation and solve for t .

3. Substitute t back. Give **coordinates** if the question says "coordinates" and a **position vector** if it says "position vector".

"Show that the line is parallel to the plane": show $\mathbf{b} \cdot \mathbf{n} = 0$ with the working, and say what it means ("so the line is perpendicular to the normal, so parallel to the plane"). If the question says the line is not in the plane, you are done; if not, show one point of the line does not satisfy the plane's equation. **"Lies in the plane"** needs $\mathbf{b} \cdot \mathbf{n} = 0$ **and** one point of the line on the plane.

9. Foot of the perpendicular from a point to a plane (4 questions)

1. The perpendicular from P runs along the **normal**: its general point is $\mathbf{p} + t\mathbf{n}$. (Not along any line given in the question.)
2. Substitute into the plane's equation and solve for t .
3. Substitute t back to get $\overrightarrow{OF}$. Check that F satisfies the plane's equation.

Variations: "the point Q is such that $\overrightarrow{OQ} = k\overrightarrow{OP}$ ": find Q first, then do the same. The distance $|PF| = |t| |\mathbf{n}|$ comes for free.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
 - **A1** accuracy mark: for a correct answer; it needs the M mark before it.
 - **B1** mark for a correct result on its own, with no method needed.
-

Example 1

The points A, B, C have position vectors $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $3\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{j} + 4\mathbf{k}$ respectively.

- (a) Find the equation of the plane ABC , giving your answer in the form $ax + by + cz = d$. **[5]**
- (b) The point D has position vector $6\mathbf{i} + \mathbf{j} + 7\mathbf{k}$. Find the perpendicular distance from D to the plane ABC . **[2]**
- (c) Find the position vector of the foot of the perpendicular from D to the plane ABC . **[3]**
- (a) $\overrightarrow{AB} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\overrightarrow{AC} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. **B1**

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{vmatrix} = \begin{pmatrix} 2(2) - (-1)(2) \\ (-1)(-1) - 2(2) \\ 2(2) - 2(-1) \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

M1 A1 for the vector product and the normal. Check: $(2, -1, 2) \cdot (2, 2, -1) = 0$ and $(2, -1, 2) \cdot (-1, 2, 2) = 0$.

Substitute A : $2(1) - (-1) + 2(2) = 7$, so the plane is $2x - y + 2z = 7$. **M1 A1** (B gives $6 - 1 + 2 = 7$ too.)

(b) $|\mathbf{n}| = \sqrt{4 + 1 + 4} = 3$ and $2(6) - 1 + 2(7) = 25$, so

$$\text{distance} = \frac{|25 - 7|}{3} = 6.$$

M1 A1

(c) The perpendicular from D is $\mathbf{r} = (6 + 2t)\mathbf{i} + (1 - t)\mathbf{j} + (7 + 2t)\mathbf{k}$. **M1**

$$2(6 + 2t) - (1 - t) + 2(7 + 2t) = 7 \Rightarrow 25 + 9t = 7 \Rightarrow t = -2.$$

M1

The foot is $2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$. **A1** Check: $4 - 3 + 6 = 7$, and $|t||\mathbf{n}| = 2 \times 3 = 6$ agrees with (b). The first diagram in "Understand it" shows this set-up, with $P = D$ and $d = 6$.

Example 2

The plane Π_1 has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j}) + \mu(2\mathbf{j} - \mathbf{k})$.

(a) Find a Cartesian equation of Π_1 . [4]

(b) The line l has equation $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + t(3\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$. Show that l is parallel to Π_1 , and find the distance between l and Π_1 . [4]

(c) The plane Π_2 has equation $x + y - z = 2$. Find a vector equation of the line of intersection of Π_1 and Π_2 . [4]

(d) Find the acute angle between Π_1 and Π_2 . [3]

(a)

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ 0 & 2 & -1 \end{vmatrix} = \begin{pmatrix} (-2)(-1) - 0(2) \\ 0(0) - 1(-1) \\ 1(2) - (-2)(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

M1 A1

The point $(1, 0, 3)$ gives $2 + 0 + 6 = 8$, so Π_1 is $2x + y + 2z = 8$. **M1 A1**

(b) $(3\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 6 - 2 - 4 = 0$, so l is perpendicular to the normal of Π_1 , and so l is parallel to Π_1 . **M1 A1**

The point $(3, 4, 2)$ on l gives $6 + 4 + 4 = 14 \neq 8$, so l is not in the plane. Every point of l is the same distance from Π_1 :

$$\text{distance} = \frac{|14 - 8|}{\sqrt{4 + 1 + 4}} = \frac{6}{3} = 2.$$

M1 A1

(c) Direction:

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1(-1) - 2(1) \\ 2(1) - 2(-1) \\ 2(1) - 1(1) \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}.$$

M1 A1

A point: put $y = 0$. Then $2x + 2z = 8$ and $x - z = 2$, so $x + z = 4$ and $x - z = 2$, giving $x = 3, z = 1$. **B1**
 Check: $6 + 0 + 2 = 8$ and $3 + 0 - 1 = 2$.

$$\mathbf{r} = 3\mathbf{i} + \mathbf{k} + s(-3\mathbf{i} + 4\mathbf{j} + \mathbf{k}).$$

A1(d) $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 + 1 - 2 = 1, |\mathbf{n}_1| = 3, |\mathbf{n}_2| = \sqrt{3}$. **M1**

$$\cos \theta = \frac{1}{3\sqrt{3}} \Rightarrow \theta = 78.9^\circ.$$

A1 A1**Example 3**

The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{j} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} - 3\mathbf{k} + \mu(\mathbf{j} + 2\mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2 . **[5]**

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both lines.

(b) Find a vector equation for the line PQ . **[7]**(c) The plane Π contains l_1 and is parallel to l_2 . Find the equation of Π in the form $ax + by + cz = d$. **[2]**(d) Find the acute angle between Π and the plane $3x + y - z = 4$. **[3]**(a) A vector from l_1 to l_2 : $(3\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) - \mathbf{j} = 3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$. **B1**

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \begin{pmatrix} 1(2) - 0(1) \\ 0(0) - 1(2) \\ 1(1) - 1(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

M1 A1 for the common perpendicular.

$$\text{shortest distance} = \frac{|(3, -3, -3) \cdot (2, -2, 1)|}{\sqrt{4 + 4 + 1}} = \frac{|6 + 6 - 3|}{3} = \frac{9}{3} = 3.$$

M1 A1

$$(b) \overrightarrow{OP} = \begin{pmatrix} \lambda \\ 1 + \lambda \\ 0 \end{pmatrix} \text{ and } \overrightarrow{OQ} = \begin{pmatrix} 3 \\ -2 + \mu \\ -3 + 2\mu \end{pmatrix}, \text{ so } \overrightarrow{PQ} = \begin{pmatrix} 3 - \lambda \\ -3 + \mu - \lambda \\ -3 + 2\mu \end{pmatrix}. \text{ **M1 A1** }$$

PQ is perpendicular to both lines: **M1**

$$\overrightarrow{PQ} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 3 - \lambda - 3 + \mu - \lambda = 0 \Rightarrow \mu = 2\lambda.$$

A1

$$\overrightarrow{PQ} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = -3 + \mu - \lambda - 6 + 4\mu = 0 \Rightarrow 5\mu - \lambda = 9.$$

A1

So $10\lambda - \lambda = 9$: $\lambda = 1$ and $\mu = 2$. Then $\overrightarrow{OP} = \mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{OQ} = 3\mathbf{i} + \mathbf{k}$. **M1** Check: $\overrightarrow{PQ} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, the common perpendicular from (a), with length 3.

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + t(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}).$$

A1 The second diagram in "Understand it" shows these two lines, with $PQ = 3$.

(c) Π contains the direction of l_1 and is parallel to l_2 , so its normal is $(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + 2\mathbf{k}) = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, from (a). The point $(0, 1, 0)$ on l_1 gives $d = -2$: $2x - 2y + z = -2$. **M1 A1**

$$(d) \frac{|(2, -2, 1) \cdot (3, 1, -1)|}{3\sqrt{11}} = \frac{|6 - 2 - 1|}{3\sqrt{11}} = \frac{1}{\sqrt{11}}. \text{ **M1 A1**}$$

$$\theta = \cos^{-1} \frac{1}{\sqrt{11}} = 72.5^\circ. \text{ **A1**}$$

Common mistakes

- **Forgetting the 90° step for a line and a plane.** The scalar product of the line's direction and the normal gives the angle with the **normal**. Examiners in 2021 and 2022 saw many answers stop there. Take it from 90° , or use $\sin$.
- **Taking an angle between two planes from 90° .** The angle between the normals already *is* the angle between the planes. Only the line-and-plane angle needs the extra step.
- **Arithmetic and sign slips in the vector product.** This is the error examiners mention most, and one slip spoils every part that follows. Dot your answer with both vectors: two zeros means it is right.
- **Mixing up position vectors and direction vectors**, for example crossing two position vectors to get a normal to ABC . Directions between points come from subtracting: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.
- **Leaving out the $-d$ or the division by $|\mathbf{n}|$** in the distance from a point to a plane. The formula is $\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$; both pieces are needed.
- **Using the wrong direction for the foot of the perpendicular.** The perpendicular from a point to a plane runs along the plane's **normal**, not along a line mentioned in the question.
- **Writing an expression instead of an equation.** A line or plane must start with $\mathbf{r} =$. Examiners reported that few answers for a line of intersection or a common perpendicular did this, and it costs the final mark.

- **Not knowing the parametric form.** "Give your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$ " means a point and two directions, not a Cartesian equation.
- **Getting stuck on a point common to two planes.** Setting one coordinate to 0 turns it into two simple equations in two unknowns.
- **Answering for the wrong object:** the plane ABC when OAB was asked for, or a position vector when the question asked for coordinates. Reread the question before the last line.
- **Setting $\overrightarrow{PQ}$ equal to the common perpendicular** instead of to a multiple $k\mathbf{n}$ of it. If you use that method, bring in the extra unknown.
- **Losing a root when there is a modulus.** Squaring both sides handles the modulus; then check each root against the conditions in the question (such as a positive constant) before rejecting one.

How to get the marks

- **Show the vectors you are using.** Write $\overrightarrow{AB} = \dots$ and the vector product set out as a determinant, or with the working of each component. The method marks can then be given even after a numerical slip.
- **Normals and planes** usually earn **M1 A1** for the vector product and **M1 A1** for substituting a point. Any multiple of the normal is fine, and so is any equivalent equation; simplifying makes later parts easier.
- **"Find the equation ... in the form $ax + by + cz = d$ "** means integers, in exactly that form.
- **"Find a vector equation" or "an equation for the line"** means $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, with the $\mathbf{r} =$.
- **"Write down"** means no working is needed; one or two marks, for a correct point and correct directions.
- **"Show that"** (for example "show that l is parallel to Π ", or a given quadratic for an unknown constant) needs every step: write the scalar product with its numbers and a sentence of conclusion.
- **Angles** are marked on the final value: give the acute angle, in degrees to 1 decimal place (radians to 3 significant figures are accepted). Once you have it, don't do anything else to it; an extra " $90^\circ -$ " at the end loses the mark.
- **Distances:** an exact form such as $\frac{6}{\sqrt{14}}$ or a 3 significant figure decimal is accepted.
- **Common perpendicular** questions are long (7 to 10 marks) and marked in steps: $\overrightarrow{PQ}$ in terms of two parameters, each of the two equations, solving, the point, and the equation. Say what each new point is ("let P be $\lambda\mathbf{c}$ on OC "), so the examiner can follow you.
- **Method marks follow through.** A wrong normal used correctly in the next part still earns the method marks, so keep going.

Quick check

1. Find $(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$, and a unit vector perpendicular to both vectors.
2. The line $\mathbf{r} = 3\mathbf{i} - \mathbf{k} + t(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ meets the plane $2x + y + z = 13$ at the point X . Find the coordinates of X and the acute angle between the line and the plane.
3. Find the acute angle between the planes $x + 2y - 2z = 1$ and $4x - 4y + 7z = 3$.
4. Find a vector equation of the line of intersection of the planes $x + y + z = 6$ and $2x - y + z = 3$.

5. The line $\mathbf{r} = \mathbf{i} + k\mathbf{j} + 2\mathbf{k} + t(3\mathbf{i} + \mathbf{j} - \mathbf{k})$ lies in the plane $x - 2y + z = -1$. Find the value of the constant k .

Answers

1. $\begin{pmatrix} (-1)(-2) - 3(4) \\ 3(1) - 2(-2) \\ 2(4) - (-1)(1) \end{pmatrix} = \begin{pmatrix} -10 \\ 7 \\ 9 \end{pmatrix}$. Check: $-20 - 7 + 27 = 0$ and $-10 + 28 - 18 = 0$. Its length is $\sqrt{100 + 49 + 81} = \sqrt{230}$, so a unit vector is $\frac{1}{\sqrt{230}}(-10\mathbf{i} + 7\mathbf{j} + 9\mathbf{k})$ (or its negative).
2. $2(3 + t) + (-t) + (-1 + 3t) = 13$, so $5 + 4t = 13$ and $t = 2$: X is $(5, -2, 5)$. For the angle, $(1, -1, 3) \cdot (2, 1, 1) = 4$, so the angle with the normal has $\cos \phi = \frac{4}{\sqrt{11}\sqrt{6}}$, $\phi = 60.5^\circ$. The angle between the line and the plane is $90^\circ - 60.5^\circ = 29.5^\circ$.
3. $(1, 2, -2) \cdot (4, -4, 7) = 4 - 8 - 14 = -18$; the lengths are 3 and 9. $\cos \theta = \frac{18}{27} = \frac{2}{3}$, so $\theta = 48.2^\circ$.
4. Direction $(1, 1, 1) \times (2, -1, 1) = (2, 1, -3)$. With $z = 0$: $x + y = 6$ and $2x - y = 3$, so $x = 3$, $y = 3$. $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} + t(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$.
5. $(3, 1, -1) \cdot (1, -2, 1) = 3 - 2 - 1 = 0$, so the line is parallel to the plane for every k . It lies in the plane when its point $(1, k, 2)$ does: $1 - 2k + 2 = -1$, so $k = 2$.

Past questions to try

- **9231/12/O/N/23 Q5**: a plane from parametric form, a line parallel to it and its distance, then a line of intersection.
- **9231/12/M/J/22 Q2**: the plane through three points, its distance from the origin, and where a line meets it.
- **9231/13/O/N/22 Q4**: the plane containing two lines, the angle with a line, and the foot of a perpendicular.
- **9231/12/M/J/21 Q6**: the shortest distance between skew lines with an unknown constant, then angles with a plane.