Cambridge IGCSE · Mathematics 0580 · Notes

Pythagoras' theorem and trigonometry in 3D

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Read first: **Pythagoras' theorem and right-angled triangles**, **Non-right-angled triangles**, **Estimation and limits of accuracy**

What you need to know

Extended only: this whole topic is Extended content (syllabus E6.6). Core candidates do not study it.

By the end of this topic you should be able to:

1. **Use Pythagoras' theorem in three dimensions** to find lengths inside solids: the diagonal of a cuboid or cube, the height of a pyramid, the length of a rod placed inside a box, and an edge when a diagonal is known.
2. **Use sine, cosine and tangent in three dimensions**, by finding a right-angled triangle inside the solid and working in it exactly as you would on flat paper.
3. **Calculate the angle between a line and a plane**, such as the angle a diagonal makes with the base of a cuboid, or the angle a sloping edge of a pyramid makes with its base.

Angles are in degrees and are given to 1 decimal place; lengths to 3 significant figures unless the question says otherwise.

Understand it

Every 3D problem is a chain of flat triangles

A 3D diagram is hard to measure from, because the drawing squashes the solid and right angles don't look like right angles. The trick is never to work "in 3D" at all. Instead:

1. find a **right-angled triangle** that contains the length or angle you want;
2. **draw it again flat**, beside the diagram, with its right angle marked and the lengths you know;
3. solve it with Pythagoras or SOHCAHTOA.

Often that triangle is missing a side, so you find the missing side first from **another** right-angled triangle, usually one lying flat in the base.

In a cuboid, every edge meets its neighbours at right angles. A vertical edge is at right angles to **every** line in the base that passes through its foot, not only the edges. That is what makes the triangles in this topic right-angled.

Pythagoras' theorem in 3D

Take a cuboid $ABCDEFGH$ with base $ABCD$ and E, F, G, H directly above A, B, C, D. Let $AB = 8$ cm, $BC = 6$ cm and $CG = 5$ cm, as in the diagram below.

Step 1, in the base. Triangle ABC is right-angled at B, so

$$AC^2 = AB^2 + BC^2 = 8^2 + 6^2 = 100, \qquad AC = 10 \text{ cm}.$$

Step 2, standing up. CG is vertical, so it is at right angles to AC. Triangle ACG is right-angled at C, so

$$AG^2 = AC^2 + CG^2 = 100 + 25 = 125, \qquad AG = \sqrt{125} = 11.2 \text{ cm (3 s.f.)}.$$

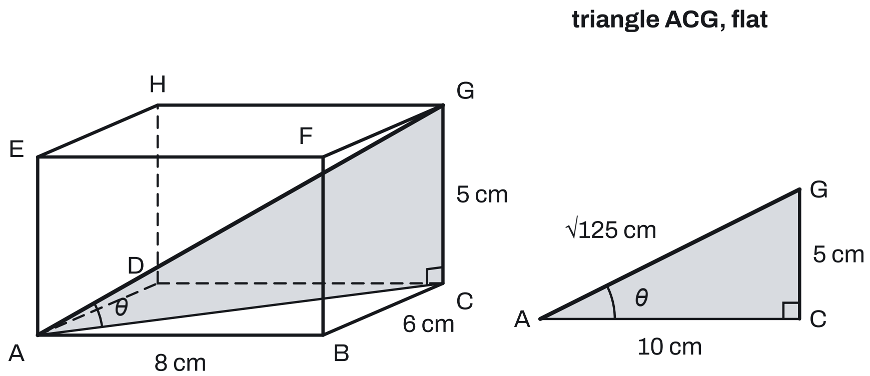
Putting the two steps together, $AG^2 = AB^2 + BC^2 + CG^2$. So for a cuboid with edges a, b and c, the **space diagonal** (the line joining opposite corners through the inside) is

$$d = \sqrt{a^2 + b^2 + c^2}.$$

This is the longest straight line that fits inside the box. In the first step you **don't need** the value of AC itself: you can keep $AC^2 = 100$ and add CG^2 straight away, which avoids rounding.

A **face diagonal** such as AC or AF joins two corners of the same face. A **space diagonal** such as AG joins two corners that don't share a face. Questions that say "a diagonal of the cuboid" mean the space diagonal.

The angle between a line and a plane



A line meets a flat surface (a **plane**) at a point, but "the angle between them" needs a second line to measure against. Use the line's **shadow** in the plane, with the light shining straight down:

1. from the top end of the line, drop a **perpendicular** straight down to the plane (for the base of a cuboid, go down a vertical edge);
2. join the foot of that perpendicular to the point where the line meets the plane;

3. the angle between the line and this shadow is the **angle between the line and the plane**.

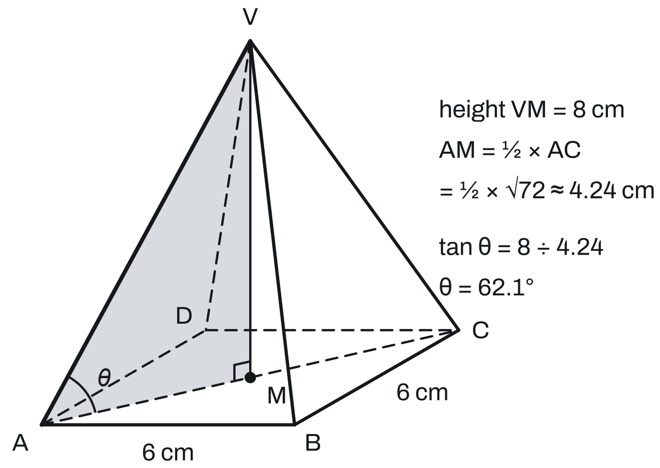
This makes a right-angled triangle, with the right angle at the foot of the perpendicular. For the diagonal AG and the base $ABCD$: G drops to C, the shadow is AC, and the angle is $\theta = \angle GAC$ in triangle ACG:

$$\tan\theta = \frac{CG}{AC} = \frac{5}{10}, \qquad \theta = 26.6^\circ.$$

The **opposite** side is always the vertical height, and the **adjacent** side is always the shadow in the base. If you have found AG instead, $\sin\theta = \frac{5}{\sqrt{125}}$ gives the same angle.

The angle is **not** angle GAB or angle GAD: those measure AG against an edge of the base, not against its shadow AC, so they come out too big. Always ask "where does the top end land if it drops straight down?"

Pyramids and other solids



When the vertex V of a pyramid is **directly above the centre** M of the base, V drops to M. So for a sloping edge such as VA, the shadow is AM, which is **half the diagonal** of the base:

$$AC = \sqrt{6^2+6^2} = \sqrt{72}, \qquad AM = \tfrac{1}{2}\sqrt{72} = 4.24 \text{ cm}, \qquad \tan\theta = \frac{8}{4.24}, \qquad \theta = 62.1^\circ.$$

The same idea works for any solid. Find where the top point lands when it drops vertically, then join up:

- **Vertex above a corner**, such as E directly above D: the shadow of EB is the base diagonal DB.
- **Prism**: the top of a sloping line is on a raised edge. Work out how high that edge is and where directly below it the foot lies, using the cross-section.
- **Cylinder**: a line from a point on the top rim to the far side of the base has a shadow equal to a **diameter**.
- **Cube**: every edge is the same, so the space diagonal is $l\sqrt{3}$ and the base diagonal is $l\sqrt{2}$. The angle between a space diagonal and a face is $\tan^{-1}\frac{1}{\sqrt{2}} = 35.3^\circ$ for **every** cube, whatever its size.

Key facts and formulas

"Given" means it is on the List of formulas printed on page 2 of Papers 2 and 4. "Learn it" means you must remember it. None of the 3D facts are on the list.

Formula	Status
Space diagonal of a cuboid with edges a, b, c: $d = \sqrt{a^2 + b^2 + c^2}$	Learn it
Pythagoras' theorem: $a^2 + b^2 = c^2$ in a right-angled triangle with hypotenuse c	Learn it
$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \cos\theta = \frac{\text{adj}}{\text{hyp}}, \tan\theta = \frac{\text{opp}}{\text{adj}}$	Learn it
Angle between a line and a plane: in the triangle (line, vertical drop, shadow), with the right angle at the foot of the drop	Learn it
Square of side s: diagonal $s\sqrt{2}$; cube of edge l: space diagonal $l\sqrt{3}$	Learn it
Pyramid with vertex above the centre of the base: the shadow of a sloping edge is half the base diagonal	Learn it
Sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$; cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$ (for a base that is not right-angled)	Given
Volume of a pyramid $= \frac{1}{3} \times \text{base area} \times \text{height}$; volume of a prism $=$ cross-section $\times$ length	Given

How to do it

In the Paper 2 and Paper 4 questions we have tagged for this topic (23 questions, 2021–2025), the types came up this often. 9 of the 23 questions mix two types, so the counts add up to more than 23.

Question type	Questions
Angle between a line and a plane	20
Length of a line inside a solid	7
Finding an edge from a diagonal	4
Bounds in a 3D problem	1

1. Angle between a line and a plane (20 questions)

1. **Identify the angle.** Take the top end of the line, drop it vertically to the plane, and join the foot to the other end of the line. Mark the angle on the diagram and name it with three letters (for example angle GAC). This alone earns a mark.

2. **Find the shadow length** (the adjacent side) with Pythagoras in the base, for example $AC = \sqrt{AB^2 + BC^2}$. Keep the full calculator value, or keep it as a square root.
3. **Write one trig ratio** in the right-angled triangle: $\tan\theta = \dfrac{\text{height}}{\text{shadow}}$. If you already know the sloping length, $\sin\theta = \dfrac{\text{height}}{\text{line}}$ is quicker.
4. **Give the angle to 1 decimal place.**

For example, a cuboid with base 24 cm by 7 cm and height 10 cm: the base diagonal is $\sqrt{576 + 49} = 25$ cm, so the diagonal makes $\tan^{-1}\frac{10}{25} = 21.8^\circ$ with the base.

Variations you will meet:

- **Cube with no lengths given.** Call the edge 1 (or any value): shadow $\sqrt{2}$, height 1, angle $\tan^{-1}\frac{1}{\sqrt{2}} = 35.3^\circ$.
- **"The base" is not the bottom face you expect**, for example "the base $EFGH$" or "the base $BCFE$" of a prism lying on its side. Read which face is named: the height is the edge that sticks out at right angles to **that** face.
- **Pyramid with the vertex above the centre.** Shadow of a sloping edge $=$ half the base diagonal. You may be given the height, or a sloping edge and have to find the height first (Pythagoras with half the diagonal). For a rectangular base, the half-diagonal is $\frac{1}{2}\sqrt{AB^2 + BC^2}$.
- **Pyramid with the vertex above a corner.** The shadow of the edge from the far corner is a full base diagonal.
- **Prism with a trapezium or triangle cross-section.** Find the height of the top end of the line from the cross-section (a vertical side, or a perpendicular height worked out with trigonometry). Then find where it lands: the shadow joins the bottom end of the line to that point, and it is usually the hypotenuse of a right-angled triangle in the base whose sides are the prism's **length** and a horizontal distance across the cross-section.
- **Cylinder.** A line from the top rim to the opposite point of the bottom rim: $\tan\theta = \dfrac{\text{height}}{\text{diameter}}$.
- **Points at different heights**, such as wires between vertical poles. The angle of elevation from one pole top to another uses the **difference** in heights over the horizontal distance between the poles. The midpoint of a wire is at the **average** height of its ends, and it lands at the midpoint of the line between the poles' feet. If the base triangle is not right-angled, find the horizontal distance with the cosine rule first.
- **"Show that" a length is correct**, such as the part of a stick outside a box: find the length inside the box with 3D Pythagoras (not with a rounded angle), subtract, and give more figures than the answer asks for before rounding.

2. Length of a line inside a solid (7 questions)

1. **Cuboid diagonal:** $d = \sqrt{a^2 + b^2 + c^2}$ in one line.
2. **Anything else:** find a right-angled triangle containing the line, with its other two sides either edges or lines you can find with 2D Pythagoras in a face.

For example, a box 10 cm by 6 cm by 3 cm has diagonal $\sqrt{100 + 36 + 9} = \sqrt{145} = 12.0$ cm (3 s.f.).

Variations you will meet:

- **"Show that the rod does (or does not) fit."** Work out the space diagonal, give it to more figures than needed (for example 9.95), and write a comparison sentence: "$9.95 < 10$, so the rod does not fit." The sentence earns the last mark.
- **A rod between two points that are not corners**, for example from the midpoint of one edge to a point that splits a top edge in a ratio. Find how far apart the two ends are in each of the three directions (along the length, across the width and up the height), then $PQ = \sqrt{x^2 + y^2 + z^2}$. Use the ratio to find distances along the edge: $BQ : QF = 2 : 3$ on an edge of 15 cm puts Q 6 cm from B and 9 cm from F.
- **Part of a line outside the box**: total length minus the diagonal inside the box.
- **A line up to the top of a vertical pole**: the pole is at right angles to the ground, so the line is the hypotenuse of (distance along the ground, pole height).

3. Finding an edge from a diagonal (4 questions)

1. **Call the unknown edge** x and write every other edge in terms of x.
2. **Write the 3D Pythagoras equation** with the diagonal: $\text{edge}_1^2 + \text{edge}_2^2 + \text{edge}_3^2 = d^2$.
3. **Solve for** x^2, then square-root.

Common set-ups:

- **Two edges and the diagonal known:** third edge $= \sqrt{d^2 - a^2 - b^2}$.
- **Square base of side** x**, height** h**:** $x^2 + x^2 + h^2 = d^2$, so $2x^2 = d^2 - h^2$.
- **One edge twice another** ($AB = 2BC = 2x$): $(2x)^2 + x^2 + h^2 = d^2$, so $5x^2 = d^2 - h^2$. Put the $2x$ in a bracket before squaring.
- **Cube of edge** l**:** $3l^2 = d^2$, so $l = \dfrac{d}{\sqrt{3}}$.

Once you have the edges, the angle with the base usually follows as in type 1, and $\sin\theta = \dfrac{\text{height}}{\text{diagonal}}$ needs no further lengths.

4. Bounds in a 3D problem (1 question)

When the lengths are given "correct to the nearest centimetre", each could be up to half a unit either way. To make a trig ratio as **small** as possible, use the **smallest** top and the **largest** bottom.

1. Write the bounds of each length: 30 cm to the nearest cm lies from 29.5 to 30.5.
2. Choose the ratio for the angle ($\sin$ if you know height and line, $\tan$ if you know height and shadow).
3. **Lower bound of the angle:** lower bound of the height ÷ upper bound of the other length. **Upper bound:** the other way round.
4. Use inverse trig and give the angle to 1 decimal place.

For example, a height of 9 cm and a diagonal of 15 cm, both to the nearest cm: the upper bound of the angle is $\sin^{-1}\frac{9.5}{14.5} = 40.9^\circ$.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

A cuboid $ABCDEFGH$ has base $ABCD$, with E, F, G and H vertically above A, B, C and D. $AB = 9$ cm, $BC = 4$ cm and $CG = 7$ cm.

(a) Calculate the length of the diagonal AG. **[3]**

(b) Calculate the angle between AG and the base $ABCD$. **[3]**

(a) $AC^2 = 9^2 + 4^2 = 97$. **M1** for Pythagoras in the base (or any pair of edges).

$AG^2 = AC^2 + CG^2 = 97 + 49 = 146$. **M1**

$AG = \sqrt{146} = 12.1$ cm (3 s.f.). **A1**

(b) G is vertically above C, so the angle is GAC, in triangle ACG with the right angle at C. **B1** for identifying the angle.

$$\sin GAC = \frac{7}{\sqrt{146}} \qquad \left(\text{or } \tan GAC = \frac{7}{\sqrt{97}}\right)$$

M1 for a correct trig statement in the right triangle.

Angle $= 35.4^\circ$. **A1**

Example 2

$VABCD$ is a pyramid with a square base $ABCD$ of side 8 cm. The vertex V is vertically above M, the centre of the base. Each sloping edge is 13 cm long.

(a) Calculate the height VM of the pyramid. **[3]**

(b) Calculate the angle between the edge VA and the base. **[2]**

(c) N is the midpoint of BC. Calculate the angle between VN and the base. **[2]**

(a) $AC = \sqrt{8^2 + 8^2} = \sqrt{128}$, so $AM = \frac{1}{2}\sqrt{128} = 5.657\ldots$ cm. **M1** for half the diagonal.

$VM^2 = 13^2 - AM^2 = 169 - 32 = 137$. **M1**

$VM = \sqrt{137} = 11.7$ cm (3 s.f.). **A1**

(b) The shadow of VA is AM, so in triangle VMA:

$$\cos VAM = \frac{AM}{VA} = \frac{5.657\ldots}{13}$$

M1. Angle $= 64.2^\circ$. **A1**

(c) M is the centre of the square, so MN is half a side: $MN = 4$ cm, at right angles to VM.

$$\tan VNM = \frac{VM}{MN} = \frac{\sqrt{137}}{4}$$

M1 using the unrounded height. Angle $= 71.1^\circ$. **A1**

(The face VBC is steeper than the edge VA, because N is closer to the centre than A is.)

Example 3

A cuboid $ABCDEFGH$ has a square base $ABCD$ of side x cm. The height CG is 12 cm and the diagonal AG is 20 cm.

(a) Calculate the value of x. **[3]**

(b) The lengths 12 cm and 20 cm are each correct to the nearest centimetre. Calculate the lower bound of the angle between AG and the base. **[3]**

(a) $x^2 + x^2 + 12^2 = 20^2$. **M1** for the 3D Pythagoras equation.

$2x^2 = 400 - 144 = 256$, so $x^2 = 128$. **M1**

$x = \sqrt{128} = 11.3$ (3 s.f.). **A1**

(b) In triangle ACG, $\sin GAC = \dfrac{CG}{AG}$. **B1** for 11.5 and 20.5 seen.

For the smallest angle, use the smallest height and the largest diagonal:

$$\sin GAC = \frac{11.5}{20.5}$$

M1. Lower bound $= 34.1^\circ$. **A1**

Example 4

A box is a cuboid $ABCDEFGH$ with base $ABCD$ and E, F, G, H vertically above A, B, C, D. $AB = 12$ cm, $BC = 8$ cm and $CG = 9$ cm. P is on the base edge AB with $AP : PB = 1 : 3$. Q is on the vertical edge CG with $CQ : QG = 2 : 1$.

(a) Calculate the length of the straight line PQ. **[4]**

(b) Calculate the angle between PQ and the base $ABCD$. **[2]**

(a) $AP = \frac{1}{4} \times 12 = 3$ cm, so $PB = 9$ cm; $CQ = \frac{2}{3} \times 9 = 6$ cm. **B1** for both.

Q is directly above C, so first find PC in the base. Triangle PBC is right-angled at B:

$$PC^2 = PB^2 + BC^2 = 9^2 + 8^2 = 145$$

M1. Triangle PCQ is right-angled at C (because CG is vertical):

$$PQ = \sqrt{PC^2 + CQ^2} = \sqrt{145 + 36} = \sqrt{181} = 13.5 \text{ cm } (3 \text{ s.f.})$$

M1 A1

(b) Q drops to C, so the shadow of PQ is PC and the height is $CQ = 6$ cm: $\sin\theta = \dfrac{6}{\sqrt{181}}$ (or $\tan\theta = \dfrac{6}{\sqrt{145}}$). **M1**

$\theta = 26.5^\circ$. **A1**

Common mistakes

- **Finding a face diagonal instead of the space diagonal.** "A diagonal of the cuboid" joins corners that don't share a face. $\sqrt{a^2 + b^2}$ is only half the job.
- **Using the wrong angle.** Examiners often see angle PCB or PBA (in a face) instead of PCA (to the shadow), or the angle with the vertical instead of the horizontal. Drop the top end straight down, then mark the angle before calculating.
- **Rounding in the middle.** Writing $\sqrt{29^2 + 21^2}$ as 35.8 and carrying on pushes the final answer out of the accepted range, and a rounded angle used to find a length loses a "show that". Keep full values in the calculator, or at least 4 significant figures, until the last line.
- **Forgetting to square a diagonal** when you use it in the second stage, or writing $x^2 + x^2 = d^2$ for a cube's space diagonal when it should be $x^2 + x^2 + x^2 = d^2$.
- **Assuming an angle is** 45° (or 90°) because it looks that way on the drawing. Only use right angles that the solid guarantees: edges of a cuboid, a vertical height, a pole on horizontal ground.
- **Adding or mixing angles**, such as adding angle CAB to the correct angle, or using the sine rule in a triangle that is right-angled. One right-angled triangle and one trig ratio is enough.
- **Wrong heights for a midpoint.** The midpoint of a wire between heights 3 m and 4 m is at 3.5 m. Don't assume the line to it has the same length as the wire, or that a line to a midpoint bisects an angle.
- **Too few figures.** An edge of 4.9 instead of 4.91, or an angle of 35 instead of 35.3, loses the accuracy mark.

How to get the marks

- **Identify the angle.** Marking the correct angle on the diagram, or naming it (angle GAC), is usually worth a mark on its own, even if the rest goes wrong.
- **Show the Pythagoras.** Write $\sqrt{9^2 + 4^2 + 7^2}$ or $AC^2 = 9^2 + 4^2$ before the value. Method marks are given for the correct expression.
- **Write the trig statement with numbers in**, for example $\tan\theta = \dfrac{7}{\sqrt{97}}$. This is the main method mark.
- **Accuracy:** angles to 1 decimal place, lengths to 3 significant figures, unless the question says otherwise. A value found by a different correct route (sine instead of tangent) gets full marks.
- **"Show that"** means the answer is printed. Show every step, give your value to more figures than the printed one (for example 4.76 for "4.8 to 1 decimal place"), and don't start from the printed value.
- **"Show that it does not fit"** needs the comparison stated in words at the end.
- **Method marks follow through.** If part (a) was wrong, still use your value in part (b) in a correct method.

Quick check

1. A box is a cuboid 12 cm long, 5 cm wide and 3 cm high. Show that a straight pencil of length 13.5 cm cannot fit inside the box.
2. The space diagonal of a cube is 15 cm. Calculate the length of an edge of the cube.
3. A wedge has a horizontal rectangular base $JKLM$ with $JK = 15$ cm and $KL = 8$ cm. Its face $KLNR$ is a vertical rectangle, with R and N directly above K and L and $KR = 6$ cm; the sloping face is $JMNR$. Calculate the angle between the line JN and the base $JKLM$.
4. A cylinder has radius 5 cm and height 12 cm. A straight line joins a point on the top rim to the point directly opposite it on the bottom rim. Calculate the angle this line makes with the base.
5. A pyramid has a horizontal rectangular base $ABCD$ with $AB = 9$ cm and $BC = 12$ cm. Its vertex E is vertically above D, with $DE = 8$ cm. Calculate the angle between EB and the base.

Answers

1. Diagonal $= \sqrt{12^2 + 5^2 + 3^2} = \sqrt{178} = 13.34$ cm. $13.34 < 13.5$, so the pencil is longer than the longest line in the box and cannot fit.
2. $3l^2 = 15^2 = 225$, so $l^2 = 75$ and $l = \sqrt{75} = 8.66$ cm (3 s.f.).
3. N is above L, so the shadow of JN is $JL = \sqrt{15^2 + 8^2} = 17$ cm, and the height is $LN = KR = 6$ cm. $\tan\theta = \dfrac{6}{17}$, so $\theta = 19.4^\circ$.
4. The shadow is a diameter, 10 cm. $\tan\theta = \dfrac{12}{10}$, so $\theta = 50.2^\circ$.
5. E drops to D, so the shadow of EB is $DB = \sqrt{9^2 + 12^2} = 15$ cm. $\tan\theta = \dfrac{8}{15}$, so $\theta = 28.1^\circ$.

Past questions to try

- **0580/41/M/J/25 Q20**: the angle a pyramid's edge makes with its base, using half the diagonal.
- **0580/41/O/N/24 Q9**: a rod from the midpoint of an edge to a point dividing a top edge in a ratio.
- **0580/43/M/J/22 Q10**: an edge from the diagonal when one edge is twice another.
- **0580/41/O/N/22 Q8**: wires between poles of different heights, with a non-right-angled base.