Cambridge AS & A Level · Mathematics (9709) · Notes

Algebra A2

Read online at pastopic.com/cambridge/mathematics-9709/notes/algebra-p3

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Read first: Algebra, Series

What you need to know

By the end of this topic you should be able to:

1. **Use the modulus.** Sketch $y = |ax + b|$ and solve equations and inequalities with moduli, using $|a| = |b| \iff a^2 = b^2$ and $|x - a| < b \iff a - b < x < a + b$. Only straight lines go inside the modulus; graphs such as $y = |f(x)|$ for curves are not included.
2. **Divide a polynomial** of degree 4 or less by a linear or a quadratic polynomial, and state the quotient and the remainder (which may be zero).
3. **Use the factor theorem and the remainder theorem**, including divisors like $(3x - 2)$, to find factors and remainders, solve polynomial equations and find unknown coefficients.
4. **Split a fraction into partial fractions.** Know which form to use, and find the constants, when the denominator is at most as complicated as $(ax + b)(cx + d)(ex + f)$, $(ax + b)(cx + d)^2$ or $(ax + b)(cx^2 + d)$. The top may have the same degree as the bottom, but never a higher one.
5. **Expand $(1 + x)^n$ when n is any rational number** (negative or a fraction) and $|x| < 1$. Adapt it to expressions like $(3 - 2x)^{-1}$ or $\sqrt{4 + x^2}$, and state the values of x for which the expansion is valid. You never need the general term.

Outcomes 1 to 3 are the Paper 2 algebra again, taught in full in the Paper 2 Algebra note ("Read first"); this note reminds you of the key ideas and shows how Paper 3 asks them. Outcomes 4 and 5 are new, and most of the marks in this topic come from them.

Understand it

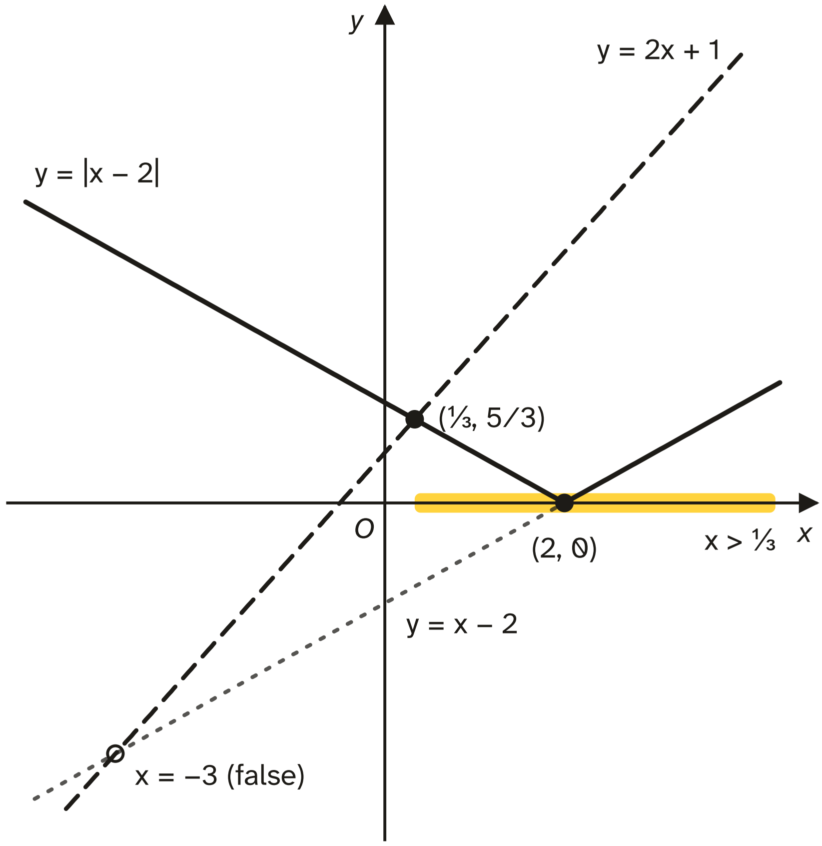
The modulus, again

$|x|$ is the size of x without its sign, so it is never negative. The graph of $y = |ax + b|$ is the line $y = ax + b$ with every part below the x-axis reflected upwards: a V with its vertex on the x-axis where $ax + b = 0$.

To solve an inequality, first find the **critical values** (where the two sides are equal), then choose the regions from a sketch.

- **Both sides are moduli**, as in $|A| < k|B|$: square both sides, $A^2 < k^2B^2$. Squaring is safe because both sides are never negative.

- **One side is not a modulus**, as in $2x + 1 > |x - 2|$: don't square. Solve the two linear equations $2x + 1 = x - 2$ and $2x + 1 = -(x - 2)$, then check each answer with the sketch.



Here $2x + 1 = -(x - 2)$ gives $x = \frac{1}{3}$, a real crossing. But $2x + 1 = x - 2$ gives $x = -3$, where $2x + 1 = -5$ is negative, and a modulus can never equal a negative number: the line meets only the discarded part $y = x - 2$. So the graphs cross once, and the answer is $x > \frac{1}{3}$. Squaring would have given $(3x - 1)(x + 3) = 0$, with the false value -3 mixed in.

Dividing polynomials, again

Dividing $p(x)$ by a divisor gives a quotient and a remainder of **lower degree than the divisor**:

$$p(x) \equiv \text{divisor} \times \text{quotient} + \text{remainder}.$$

A linear divisor leaves a number; a quadratic divisor leaves at most $rx + s$. Putting in the value of x that makes a linear divisor zero leaves only the remainder, which gives the two theorems:

- **Remainder theorem:** dividing $p(x)$ by $(ax + b)$ leaves $p\left(-\frac{b}{a}\right)$.
- **Factor theorem:** $(ax + b)$ is a factor exactly when $p\left(-\frac{b}{a}\right) = 0$.

Neither theorem works for a quadratic divisor: then you divide, or compare coefficients.

Partial fractions: the idea

Adding fractions puts them over a common denominator:

$$\frac{3}{2x - 1} + \frac{2}{x + 1} = \frac{3(x + 1) + 2(2x - 1)}{(2x - 1)(x + 1)} = \frac{7x + 1}{(2x - 1)(x + 1)}.$$

Partial fractions does this backwards: it splits one complicated fraction into simple ones. You need it because the simple pieces are easy to expand as series and easy to integrate, while the whole fraction is not.

To go backwards, write the form with unknown constants and multiply through by the whole denominator:

$$\frac{7x+1}{(2x-1)(x+1)} \equiv \frac{A}{2x-1} + \frac{B}{x+1} \quad \Rightarrow \quad 7x+1 \equiv A(x+1) + B(2x-1).$$

This is an **identity**: true for every x. So choose values of x that make brackets vanish:

- $x = -1$: $-6 = -3B$, so $B = 2$;
- $x = \frac{1}{2}$: $\frac{9}{2} = \frac{3}{2}A$, so $A = 3$.

When no value of x kills the remaining unknowns, **compare coefficients** of x^2, x or the constants on the two sides.

Which form to use

The form depends only on the denominator. The top of each piece is always one degree lower than its bottom: a constant over a linear bracket, and $Bx + C$ over a quadratic that doesn't factorise.

Denominator	Form
$(ax+b)(cx+d)(ex+f)$	$\dfrac{A}{ax+b} + \dfrac{B}{cx+d} + \dfrac{C}{ex+f}$
$(ax+b)(cx+d)^2$	$\dfrac{A}{ax+b} + \dfrac{B}{cx+d} + \dfrac{C}{(cx+d)^2}$
$(ax+b)(cx^2+d)$	$\dfrac{A}{ax+b} + \dfrac{Bx+C}{cx^2+d}$
Top and bottom of the same degree	Put a constant in front: $A +$ the form above

Two linear factors work like the first row with one fraction fewer.

A repeated factor needs both $\frac{B}{cx+d}$ and $\frac{C}{(cx+d)^2}$. For $\dfrac{4x}{(x-1)(x+1)^2}$:

$$4x \equiv A(x+1)^2 + B(x-1)(x+1) + C(x-1).$$

$x = 1$ gives $4 = 4A$, so $A = 1$. $x = -1$ gives $-4 = -2C$, so $C = 2$. No value of x isolates B, so compare the x^2 terms: $0 = A + B$, so $B = -1$.

$$\frac{4x}{(x-1)(x+1)^2} = \frac{1}{x-1} - \frac{1}{x+1} + \frac{2}{(x+1)^2}.$$

A quadratic factor such as $x^2 + 3$ has no real roots, so it needs the linear top $Bx + C$. For $\dfrac{3x^2+5}{(x+1)(x^2+3)}$:

$$3x^2 + 5 \equiv A(x^2+3) + (Bx+C)(x+1).$$

$x = -1$ gives $8 = 4A$, so $A = 2$. The x^2 terms give $3 = A + B$, so $B = 1$. The constants give $5 = 3A + C$, so $C = -1$:

$$\frac{3x^2+5}{(x+1)(x^2+3)} = \frac{2}{x+1} + \frac{x-1}{x^2+3}.$$

B or C may come out as 0. That is a correct answer, not a sign of a mistake.

Top and bottom of the same degree (an "improper" fraction): the fraction contains a whole number part, so start with a constant. For $\dfrac{x^2+4x-2}{(x-1)(x+2)}$:

$$x^2+4x-2 \equiv A(x-1)(x+2)+B(x+2)+C(x-1).$$

The x^2 terms give $A=1$; $x=1$ gives $3=3B$, so $B=1$; $x=-2$ gives $-6=-3C$, so $C=2$:

$$\frac{x^2+4x-2}{(x-1)(x+2)}=1+\frac{1}{x-1}+\frac{2}{x+2}.$$

The constant A is simply the ratio of the x^2 coefficients on top and bottom.

The binomial series for any power

For a positive whole number n, you already know that $(1+x)^n$ expands into $n+1$ terms and then stops. The same pattern of coefficients works for **any** rational n:

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

When n is negative or a fraction, the factors $n, n-1, n-2, \ldots$ never reach 0, so the series **never stops**. You write the first few terms only.

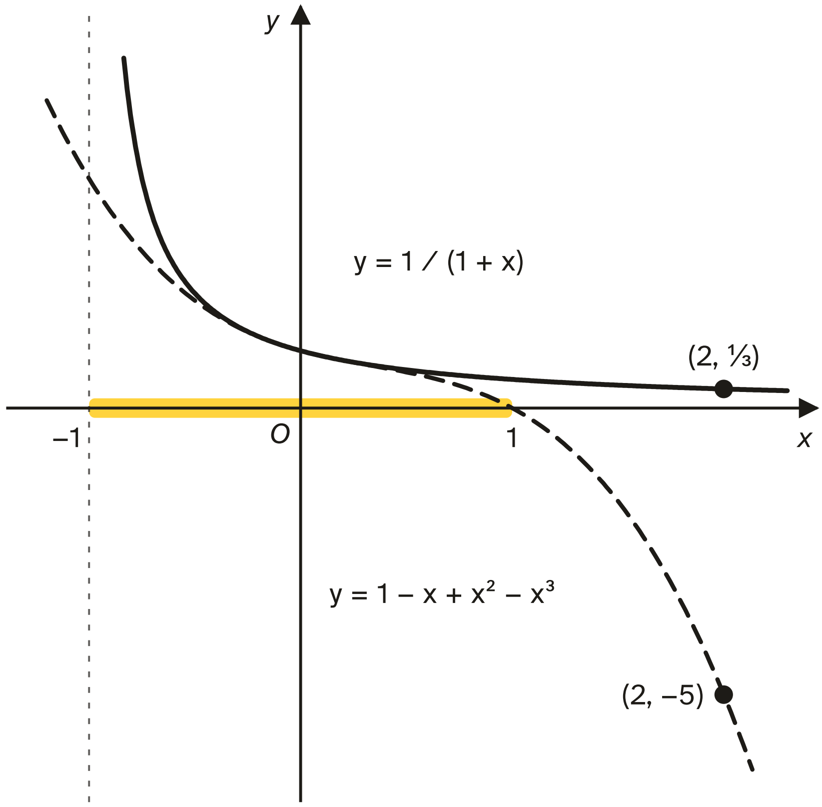
Replace x by the whole term, in brackets. For $(1-3x)^{-2}$, use $n=-2$ and put $(-3x)$ wherever the formula has x:

$$1+(-2)(-3x)+\frac{(-2)(-3)}{2!}(-3x)^2+\frac{(-2)(-3)(-4)}{3!}(-3x)^3=1+6x+27x^2+108x^3+\cdots$$

The bracket matters: $(-3x)^2=9x^2$, not $-3x^2$.

When is the expansion valid?

An endless series only makes sense if its terms get smaller and settle on a total. $(1+x)^{-1}=1-x+x^2-x^3+\cdots$ is a geometric series with common ratio $-x$, which has a sum only when $|x|<1$.



At $x = 0.1$ the four terms give $1 - 0.1 + 0.01 - 0.001 = 0.909$, very close to $\frac{1}{1.1} = 0.90909\ldots$ At $x = 2$ they give $1 - 2 + 4 - 8 = -5$, nowhere near $\frac{1}{3}$, and each extra term makes it worse.

The rule is: **the expansion of** $(1 + \text{term})^n$ **is valid when** $|\text{term}| < 1$**.** So $(1 - 3x)^{-2}$ is valid for $|3x| < 1$, that is $|x| < \frac{1}{3}$. (For a positive whole number n the series stops, so it is true for every x.)

When the bracket doesn't start with 1

The formula needs a 1 in front. For $(a + bx)^n$, take a out **to the power** n:

$$(a + bx)^n = a^n \left(1 + \frac{b}{a}x\right)^n.$$

For $(2 + x)^{-3}$: $2^{-3}\left(1 + \frac{x}{2}\right)^{-3} = \frac{1}{8}\left(1 - \frac{3}{2}x + \frac{3}{2}x^2 - \cdots\right) = \frac{1}{8} - \frac{3}{16}x + \frac{3}{16}x^2 - \cdots$, because $\frac{(-3)(-4)}{2}\left(\frac{x}{2}\right)^2 = \frac{3}{2}x^2$. It is valid for $\left|\frac{x}{2}\right| < 1$, that is $|x| < 2$.

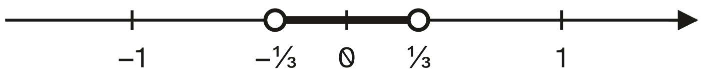
Products and quotients. Expand each power separately, then multiply, keeping only the terms up to the power you need. A quotient becomes a product with a negative power:

$$(2 - x)(1 - 3x)^{-2} = (2 - x)(1 + 6x + 27x^2 + \cdots) = 2 + 12x + 54x^2 - x - 6x^2 + \cdots = 2 + 11x + 48x^2 + \cdots$$

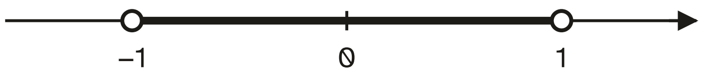
Partial fractions, then a series

A fraction like $\dfrac{7x^2 - 4x + 5}{(1 + 3x)(1 - x)^2}$ has no single binomial expansion. Split it into partial fractions first (Example 4), then expand each piece and add. Each piece has its own condition, and the whole expansion is valid only where **every** piece is valid, so take the strictest condition.

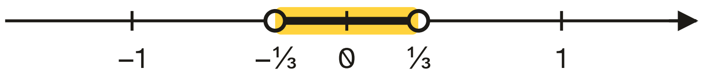
Valid for $(1 + 3x)^{-1}$: $|x| < \frac{1}{3}$



Valid for $(1 - x)^{-1}$ and $(1 - x)^{-2}$: $|x| < 1$



Valid for the whole expansion: $|x| < \frac{1}{3}$



Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \cdots$, where n is rational and $\lvert x\rvert < 1$	Given
$(a+b)^n = a^n + \dbinom{n}{1}a^{n-1}b + \dbinom{n}{2}a^{n-2}b^2 + \cdots + b^n$, where n is a positive integer	Given
$(a+bx)^n = a^n\left(1 + \dfrac{b}{a}x\right)^n$, valid for $\lvert x\rvert < \left\lvert\dfrac{a}{b}\right\rvert$	Learn it
The expansion of $(1+\text{term})^n$ is valid when $\lvert\text{term}\rvert < 1$; for a sum or product of expansions, use the strictest condition	Learn it
Partial fraction forms (the table in "Understand it"), with a constant in front when top and bottom have the same degree	Learn it
$p(x) \equiv \text{divisor} \times \text{quotient} + \text{remainder}$, with the remainder of lower degree than the divisor	Learn it
Remainder theorem: dividing $p(x)$ by $(ax+b)$ leaves $p\left(-\frac{b}{a}\right)$	Learn it
Factor theorem: $(ax+b)$ is a factor $\iff p\left(-\frac{b}{a}\right) = 0$	Learn it
$\lvert a\rvert = \lvert b\rvert \iff a^2 = b^2$; $\lvert x-a\rvert < b \iff a-b < x < a+b$	Learn it
Quadratic formula $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$	Given

Useful first terms: $(1+x)^{-1} = 1 - x + x^2 - \cdots$, $(1+x)^{-2} = 1 - 2x + 3x^2 - \cdots$ and $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$. Work them out from the formula if you forget.

How to do it

In the Paper 3 questions we have tagged for this topic (76 questions, 2021–2025), the types came up this often. 13 of the 76 questions mix two types, most often partial fractions followed by a series, so the counts add up to more than 76. Many partial fraction questions go on to integration or a differential equation; those parts belong to the Integration and Differential equations topics.

Question type	Questions
Partial fractions	28
Binomial expansion for any rational power	20
Modulus inequalities	16
Factor and remainder theorems	13
Dividing polynomials: quotient and remainder	12

1. Partial fractions (28 questions)

1. **Check the degrees.** If the top has the same degree as the bottom, the form starts with a constant. If the bottom is given multiplied out, such as $3x^2 + 5x - 2$, factorise it first: $(3x - 1)(x + 2)$.
2. **Write the form** from the table, with a separate letter for every constant. Then multiply through by the whole denominator to get an identity. Put brackets round $(Bx + C)$ before you multiply it by anything.
3. **Find the constants.** Substitute the values of x that make each linear bracket zero; then compare coefficients (the x^2 terms, or the constants) for any that are left.
4. **Write the final answer** as partial fractions with your values in. The constants alone are not the answer.

Variations you will meet:

- **Three linear factors** in the denominator: three fractions, and three substitutions find all three constants.
- **A repeated factor** $(cx + d)^2$: include both $\frac{B}{cx+d}$ and $\frac{C}{(cx+d)^2}$. A factor x^2 counts as the repeated factor $x \cdot x$, giving $\frac{A}{x} + \frac{B}{x^2}$.
- **A quadratic factor** like $x^2 + 3$ or $5 + x^2$: use $\frac{Bx+C}{x^2+3}$, never a constant on its own over it.
- **A letter in the fraction**, such as $\dfrac{5a}{(a + x)(3a - 2x)}$ with $a > 0$. Work exactly as usual with a carried along: substitute $x = -a$ and $x = \frac{3a}{2}$. Your constants may contain a.
- **A different variable**, such as $\frac{4}{1-16y^2}$ in a differential equation: factorise the difference of two squares as $(1 - 4y)(1 + 4y)$ and split as usual.
- **"Hence"**: the next part uses your partial fractions to expand as a series (type 2) or to integrate. Keep the form tidy: $\frac{-3}{2+x}$, not a fraction inside a fraction.

2. Binomial expansion for any rational power (20 questions)

1. **Get a 1 in front.** Rewrite roots and fractions as powers, then take out the constant: $\sqrt{9 + x} = 3\left(1 + \frac{x}{9}\right)^{1/2}$ and $\frac{1}{5-x} = \frac{1}{5}\left(1 - \frac{x}{5}\right)^{-1}$.
2. **Use the formula** with the whole term in brackets in place of x, up to the power asked for.
3. **Simplify each coefficient** to a single number or fraction, and write the terms in ascending powers of x.

4. **Validity**, if asked: $|\text{term}| < 1$, then rearrange to $|x| < \ldots$

Variations you will meet:

- **A product** such as $(4-x)\sqrt{1+2x}$ or $(1+5x)(1-x)^{-3/2}$: expand the power, multiply by the bracket, and collect only the terms you need.
- **Only one coefficient** ("find the coefficient of x^3"): find just the pieces that multiply to give x^3, for example the constant times the x^3 term plus the x times the x^2 term.
- **A square root of a quotient**, such as $\sqrt{\dfrac{1-x}{1+3x}}$: write it as $(1-x)^{1/2}(1+3x)^{-1/2}$, expand both, and **multiply** them.
- **An x^2 inside the bracket**, such as $(3+x^2)^{-1} = \frac{1}{3}\left(1+\frac{x^2}{3}\right)^{-1}$: the "term" is $\frac{x^2}{3}$, so the series goes up in steps of x^2 and the validity condition is $\frac{x^2}{3} < 1$, that is $|x| < \sqrt{3}$. Give this in exact form.
- **Unknown constants from coefficients**, such as "the coefficients of x and x^2 in $(a+bx)\sqrt{1+8x}$ are 7 and -20". Expand $(1+8x)^{1/2} = 1+4x-8x^2+\cdots$, multiply by $(a+bx)$, and set the coefficients equal: $4a+b=7$ and $-8a+4b=-20$, so $a=2$ and $b=-1$.
- **Partial fractions first** (11 of the 20 questions): split, expand each piece (Example 4), then add. The validity is the strictest of the pieces' conditions. With a letter a, a condition such as $|x| < \frac{a}{2}$ is fine.

3. Modulus inequalities (16 questions)

With both sides moduli, such as $|x+1| > 2|x-1|$ (7 questions):

1. Square both sides, squaring the number in front too: $(x+1)^2 > 4(x-1)^2$.
2. Collect into a three-term quadratic: $0 > 3x^2 - 10x + 3$.
3. Solve for the critical values: $(3x-1)(x-3) = 0$ gives $\frac{1}{3}$ and 3.
4. Choose the region: a positive quadratic is negative between its roots, so $\frac{1}{3} < x < 3$.

With a sketch first, such as "sketch $y = |2x - 3a|$, then solve an inequality with a line on the other side" (9 questions):

1. **Sketch the V**: vertex on the x-axis where the inside is zero, y-intercept the size of the constant, straight arms on both sides of the vertex. With a letter, mark the points in terms of a.
2. **Decide which arm the line crosses** from the sketch, and solve just that linear equation (the left arm is $y = -(ax+b)$). Where the line is steeper than the arms there is only one crossing.
3. **Read off the region** where the inequality is true.

Variations you will meet:

- **A letter** $a > 0$ in the inequality: square as usual and factorise, or use the quadratic formula for x. Critical values like $-\frac{2a}{5}$ are perfectly possible even though a is positive.
- **Two regions**: write them as $x < p$ **or** $x > q$, never as one double inequality.

4. Factor and remainder theorems (13 questions)

1. **Turn each condition into an equation.** A factor $(ax+b)$ gives $p\left(-\frac{b}{a}\right) = 0$; a remainder R gives $p\left(-\frac{b}{a}\right) = R$.
2. **Work out every power carefully**, then clear fractions.

3. **Solve the simultaneous equations.**
4. **Use the values** in any later part (factorise, solve, and so on).

Variations you will meet:

- **A remainder in terms of another**, such as "the remainder on dividing by $(x-3)$ is twice the remainder on dividing by $(x+1)$": $p(3) = 2p(-1)$.
- **A letter in every term**, such as $2x^3 + pax^2 + qa^2x + 4a^3$. Substitute $x = a$ (say); every term is then a multiple of a^3, so divide through by a^3 (allowed, as $a \neq 0$) to get an equation in p and q.
- **The derivative $p'(x)$ is involved.** "$(x-1)$ is a factor of $p(x)$ and dividing $p'(x)$ by $(x-1)$ leaves 10" means $p(1) = 0$ and $p'(1) = 10$ (differentiate $p(x)$ term by term). If $(x+1)$ is a factor of both $p(x)$ and $p'(x)$, then $(x+1)^2$ is a factor of $p(x)$. The reverse is also asked: if $f(x) = (x-a)^2 g(x)$, the product rule gives $f'(x) = (x-a)\left[2g(x) + (x-a)g'(x)\right]$, so $(x-a)$ is a factor of $f'(x)$.
- **Factorise completely, then solve an inequality** such as $p(x) < 0$. Sketch the cubic through its roots, or look at signs: a squared factor $(x+2)^2$ is never negative, and a quadratic factor with a negative discriminant has one sign for every x.
- **A substitution after factorising**: an equation such as $4\cos^3\theta + \cdots = 0$ is $f(\cos\theta) = 0$, so set $\cos\theta$ equal to each root of $f(x) = 0$ and reject any root outside $-1 \leqslant \cos\theta \leqslant 1$.
- **Remainder on dividing by a quadratic.** The theorems don't apply. Write $p(x) \equiv (x^2 + x + 2)(Ax^2 + Bx + C) + 5x - 1$ and compare coefficients (or divide).
- **With complex numbers**: show a linear factor with the factor theorem, divide to get the quadratic factor, then solve it; the complex part belongs to the Complex numbers topic.

5. Dividing polynomials: quotient and remainder (12 questions)

1. **Write the polynomial in descending powers with 0 for missing ones**: $2x^4 + x^3 + 0x^2 - 3x + 1$, and the divisor too: $x^2 + 0x - 2$.
2. **Divide leading term by leading term, multiply back, subtract**, and repeat until what is left has a lower degree than the divisor.
3. **State the quotient and the remainder.**

Step	Divide by x^2	Multiply (x^2-2) by it	Subtract to leave
1	$2x^2$	$2x^4 - 4x^2$	$x^3 + 4x^2 - 3x + 1$
2	x	$x^3 - 2x$	$4x^2 - x + 1$
3	4	$4x^2 - 8$	$-x + 9$

The quotient is $2x^2 + x + 4$ and the remainder is $9 - x$.

Variations you will meet:

- **A quadratic divisor** (most of these questions): the remainder is usually linear.
- **Same degree top and bottom**, such as $6x^2 \div (2x^2+1)$: the quotient is just a number. $6x^2 = 3(2x^2+1) - 3$, so the quotient is 3 and the remainder is -3.
- **"Hence integrate" or "hence solve the differential equation"**: rewrite $\dfrac{p(x)}{\text{divisor}} = \text{quotient} + \dfrac{\text{remainder}}{\text{divisor}}$. The remainder goes **over the divisor**.

- **Comparing coefficients** instead of long division is always allowed: write $p(x) \equiv \text{divisor} \times (Ax^2 + Bx + C) + Dx + E$ and match terms.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

(a) Expand $\sqrt[3]{8+6x}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. **[4]**

(b) State the set of values of x for which the expansion is valid. **[1]**

(a) Take out $8^{1/3} = 2$:

$$(8+6x)^{1/3} = 2\left(1+\tfrac{3}{4}x\right)^{1/3}.$$

B1 for the factor 2.

$$\left(1+\tfrac{3}{4}x\right)^{1/3} = 1 + \tfrac{1}{3}\left(\tfrac{3}{4}x\right) + \frac{\tfrac{1}{3}\left(-\tfrac{2}{3}\right)}{2}\left(\tfrac{3}{4}x\right)^2 = 1 + \tfrac{1}{4}x - \tfrac{1}{9} \times \tfrac{9}{16}x^2 = 1 + \tfrac{1}{4}x - \tfrac{1}{16}x^2.$$

B1 for $1 + \frac{1}{4}x$; **M1** for the x^2 term with the correct structure.

$$(8+6x)^{1/3} = 2 + \tfrac{1}{2}x - \tfrac{1}{8}x^2.$$

A1

(b) Valid when $\left|\frac{3}{4}x\right| < 1$, that is $|x| < \frac{4}{3}$. **B1**

Example 2

Solve the inequality $|2x-a| < 3|x+a|$, where a is a positive constant. **[4]**

Both sides are moduli, so square, squaring the 3 as well:

$$(2x-a)^2 < 9(x+a)^2 \;\Rightarrow\; 4x^2 - 4ax + a^2 < 9x^2 + 18ax + 9a^2.$$

B1 for the inequality without moduli.

$$0 < 5x^2 + 22ax + 8a^2 = (5x+2a)(x+4a).$$

M1 for solving the three-term quadratic. The critical values are $x = -4a$ and $x = -\frac{2a}{5}$. **A1**

A positive quadratic is positive outside its roots, and $-4a < -\frac{2a}{5}$ because $a > 0$:

$$x < -4a \quad \text{or} \quad x > -\tfrac{2a}{5}.$$

A1. Check with $x = 0$: $|-a| < 3|a|$ is true, and $0 > -\frac{2a}{5}$, so 0 is in the answer, as it should be.

Example 3

The polynomial $p(x) = 3x^3 + ax^2 + bx - 8$, where a and b are constants, has $(3x - 2)$ as a factor. When $p(x)$ is divided by $(x - 1)$ the remainder is 9.

(a) Find the values of a and b. **[5]**

(b) Factorise $p(x)$ completely. **[3]**

(c) Hence solve the inequality $p(x) < 0$. **[2]**

(a) Factor theorem with $x = \frac{2}{3}$:

$$3\left(\tfrac{8}{27}\right) + a\left(\tfrac{4}{9}\right) + b\left(\tfrac{2}{3}\right) - 8 = 0.$$

M1 for substituting $x = \frac{2}{3}$ and equating to 0. Multiply by 9: $8 + 4a + 6b - 72 = 0$, so $2a + 3b = 32$. **A1**

Remainder theorem with $x = 1$: $3 + a + b - 8 = 9$, so $a + b = 14$. **M1 A1**

Then $2a + 3b - 2(a + b) = 32 - 28$ gives $b = 4$, and $a = 10$. **A1**

(b) $p(x) = 3x^3 + 10x^2 + 4x - 8$. Divide by $(3x - 2)$:

Step	Divide by $3x$	Multiply $(3x - 2)$ by it	Subtract to leave
1	x^2	$3x^3 - 2x^2$	$12x^2 + 4x - 8$
2	$4x$	$12x^2 - 8x$	$12x - 8$
3	4	$12x - 8$	0

M1 for the division, **A1** for the quotient $x^2 + 4x + 4$.

$$p(x) = (3x - 2)(x^2 + 4x + 4) = (3x - 2)(x + 2)^2.$$

A1

(c) $(x + 2)^2$ is positive except at $x = -2$, where $p(x) = 0$. So $p(x) < 0$ exactly when $3x - 2 < 0$ and $x \neq -2$. **M1**

$$x < \tfrac{2}{3}, \quad x \neq -2.$$

A1

Example 4

Let $f(x) = \dfrac{7x^2 - 4x + 5}{(1 + 3x)(1 - x)^2}$.

(a) Express $f(x)$ in partial fractions. **[5]**

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. **[5]**

(c) State the set of values of x for which the expansion in (b) is valid. **[1]**

(a)

$$\frac{7x^2-4x+5}{(1+3x)(1-x)^2} \equiv \frac{A}{1+3x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}.$$

B1 for the correct form. Multiply through:

$$7x^2-4x+5 \equiv A(1-x)^2 + B(1+3x)(1-x) + C(1+3x).$$

M1 for a correct method for one constant.

- $x=1$: $8=4C$, so $C=2$. **A1**
- $x=-\frac{1}{3}$: $\frac{7}{9}+\frac{4}{3}+5=\frac{64}{9}$ and $A\left(\frac{4}{3}\right)^2=\frac{16}{9}A$, so $A=4$. **A1**
- x^2 terms: $7=A-3B$, so $B=-1$. **A1**

$$f(x)=\frac{4}{1+3x}-\frac{1}{1-x}+\frac{2}{(1-x)^2}.$$

(b) Expand each piece:

$$4(1+3x)^{-1}=4\left(1-3x+9x^2\right)=4-12x+36x^2$$

M1 for the first two terms of an expansion, **A1**.

$$-(1-x)^{-1}=-\left(1+x+x^2\right)=-1-x-x^2$$

A1

$$2(1-x)^{-2}=2\left(1+2x+3x^2\right)=2+4x+6x^2$$

A1. Adding:

$$f(x)=5-9x+41x^2+\cdots$$

A1. A quick check: $f(0)=\frac{5}{1}=5$, which matches the constant term.

(c) $(1+3x)^{-1}$ needs $|3x|<1$ and the other two need $|x|<1$. Both hold when $|x|<\frac{1}{3}$. **B1** (The third diagram in "Understand it" shows this.)

Common mistakes

- **The wrong partial fraction form.** Examiners regularly see forms like $\frac{A}{1+4x}+\frac{B}{(2-x)^2}$ (missing a $\frac{C}{2-x}$ term), $\frac{Dx+E}{(3+x)^2}$ for a repeated factor, or a constant over a quadratic factor like x^2+5. The form earns the first mark, and a wrong form caps what the rest of your work can earn.
- **Forgetting the constant when top and bottom have the same degree.** Starting $\frac{3x^2+x-1}{(x+2)(3x-1)}$ with just $\frac{A}{x+2}+\frac{B}{3x-1}$ cannot work, whatever you do after.
- **Dropping brackets**: writing $Bx+C(x+4)$ instead of $(Bx+C)(x+4)$, or later $-3x+2$ where you meant $-(3x+2)$.
- **Doubting a zero.** If a constant comes out as $B=0$, that is often right; check it by substituting another value of x rather than starting again.
- **Taking out the wrong power of the constant.** $(4-x^2)^{-2}=4^{-2}\left(1-\frac{x^2}{4}\right)^{-2}$, and $(5+x)^{-2}$ needs $5^{-2}=\frac{1}{25}$, not 5^{-1}. The constant goes to the same power as the bracket, and a negative power puts it in the denominator.

- **Using x instead of the whole term.** In $\sqrt{1+5x}$ the term is $5x$, and $(5x)^2 = 25x^2$. Squaring only part of the term is common.
- **Adding expansions that should be multiplied**, as with the square root of a quotient.
- **Spoiling a correct expansion**: multiplying through to clear fractions, or writing the terms in descending order. Leave $\frac{1}{3} - \frac{x}{27} + \cdots$ as it is.
- **Not squaring the number in front** of a modulus: $|A| < 3|B|$ squares to $A^2 < 9B^2$, not $A^2 < 3B^2$.
- **Squaring when one side is a line.** It brings in a false critical value (like $x = -3$ in the first diagram), which then ends up in the answer. When a sketch is asked for first, it is there to show you which single crossing you need.
- **Sketching the V badly**: curves instead of straight lines, the vertex on the y-axis instead of the x-axis, or stopping the graph at the y-axis.
- **Writing an impossible region**, such as "$x < -5$ and $x > -\frac{1}{2}$" or $\frac{5}{2} > x > 4$. Two separate regions need "or".
- **Treating a critical value like $-\frac{2a}{5}$ (Example 2) as impossible** because a is positive. x can be negative even when a is not.
- **Long division when the factor theorem is quicker.** Dividing a polynomial with unknown letters rarely ends with a correct remainder. Substitute instead.
- **Substituting the wrong value**: for $(x+2)$, use $x = -2$, not $x = 2$ or $x = 3$.
- **Sign slips at the end of a long division**, especially in the last subtraction, which gives the remainder.
- **Thinking the polynomial equals divisor $\times$ remainder.** It is divisor $\times$ quotient $+$ remainder.
- **Factorising with a calculator.** Roots from an equation solver, with no division shown, earn no marks for the factorising.

How to get the marks

- **Partial fractions**: the form earns a mark, the method earns a mark, and each correct constant earns a mark. Finish by writing the fractions out with your constants.
- **"Expand ... in ascending powers of x ... simplifying the coefficients"**: marks are usually given term by term. Each coefficient must be a single simplified number or fraction, and extra terms beyond the one asked for are ignored.
- **"Hence obtain the expansion"**: use your partial fractions, expanding each one. The method mark needs a correct binomial expansion of at least one piece.
- **"State the set of values of x for which the expansion is valid"**: a strict inequality such as $|x| < \frac{1}{2}$ (or $-\frac{1}{2} < x < \frac{1}{2}$). "State" means no working is needed, but the answer must be exact when the question says so, such as $|x| < \sqrt{3}$.
- **Modulus inequalities**: the marks are for the non-modulus inequality or equations, for solving them, for the critical values, and for the final region. Two regions must be joined by "or" (or $\cup$), never "and". No marks are given for an answer with no working.
- **"Sketch"**: a V made of two straight lines, symmetrical about the vertex, with the vertex and the y-intercept marked, and both arms drawn.
- **Factor and remainder theorems**: write each substitution with the numbers in and "$= 0$" or "$=$ the remainder". Each equation earns its own marks, so show both before solving.

- **"Find the quotient and the remainder"**: state both clearly and correctly. Writing "remainder $= \frac{7}{x-2}$" is wrong: the remainder is 7.
- **"Factorise completely"**: show the division (or comparing coefficients), then factorise the quadratic if you can.
- **"Show that" and "hence"**: every step must be visible, and "hence" means use the part before.
- **"Exact"** means fractions, surds and letters, no decimals. Answers in this topic are almost always exact anyway.
- **Method marks follow through.** A wrong constant in partial fractions, used correctly in the expansion, still earns the method marks in the next part.

Quick check

1. Express $\dfrac{x^2 - x + 6}{(x+2)(x^2+2)}$ in partial fractions.
2. Express $\dfrac{2x^2 + 2x - 3}{(x+1)(x-2)}$ in partial fractions.
3. Expand $(9+2x)^{-1/2}$ in ascending powers of x, up to and including the term in x^2, and state the set of values of x for which the expansion is valid.
4. Solve the inequality $|x-3| > 2|x|$.
5. The polynomial $p(x) = 2x^3 + ax^2 - 7x + b$ has $(x-2)$ as a factor, and leaves remainder $-\frac{5}{2}$ when divided by $(2x+1)$. Find a and b.

Answers

1. $x^2 - x + 6 \equiv A(x^2+2) + (Bx+C)(x+2)$. $x=-2$: $12 = 6A$, so $A=2$. x^2 terms: $1 = A+B$, so $B=-1$. Constants: $6 = 2A+2C$, so $C=1$. Answer: $\dfrac{2}{x+2} + \dfrac{1-x}{x^2+2}$.
2. Same degree, so $2x^2+2x-3 \equiv A(x+1)(x-2) + B(x-2) + C(x+1)$. x^2 terms: $A=2$. $x=-1$: $-3 = -3B$, so $B=1$. $x=2$: $9=3C$, so $C=3$. Answer: $2 + \dfrac{1}{x+1} + \dfrac{3}{x-2}$.
3. $(9+2x)^{-1/2} = \frac{1}{3}\left(1+\frac{2x}{9}\right)^{-1/2} = \frac{1}{3}\left(1 - \frac{x}{9} + \frac{3}{8}\cdot\frac{4x^2}{81}\right) = \frac{1}{3} - \frac{x}{27} + \frac{x^2}{162}$. Valid for $\left|\frac{2x}{9}\right| < 1$, that is $|x| < \frac{9}{2}$.
4. Square: $(x-3)^2 > 4x^2$, so $x^2 - 6x + 9 > 4x^2$, that is $0 > 3x^2 + 6x - 9 = 3(x+3)(x-1)$. The quadratic is negative between its roots: $-3 < x < 1$.
5. $p(2) = 16 + 4a - 14 + b = 0$, so $4a + b = -2$. $p\left(-\frac{1}{2}\right) = -\frac{1}{4} + \frac{a}{4} + \frac{7}{2} + b = -\frac{5}{2}$, so $a + 4b = -23$. Solving: $a=1$, $b=-6$.

Past questions to try

- **9709/32/O/N/23 Q1**: sketch a V, then an inequality with a line on one side.
- **9709/31/M/J/22 Q2**: a binomial expansion with x^2 inside the bracket, and its validity.
- **9709/33/O/N/23 Q3**: two remainder conditions, one with a fraction.
- **9709/33/M/J/23 Q10**: partial fractions with a repeated factor, then the expansion.