Cambridge AS & A Level · Mathematics (9709) · Notes

Complex numbers A2

Read online at pastopic.com/cambridge/mathematics-9709/notes/complex-numbers

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Read first: Quadratics, Coordinate geometry, Trigonometry, Algebra

What you need to know

By the end of this topic you should be able to:

1. **Say what a complex number is** and use its vocabulary: real part $\mathrm{Re}\, z$, imaginary part $\mathrm{Im}\, z$, modulus $|z|$, argument $\arg z$ and conjugate z^*. Two complex numbers are equal only when their real parts are equal **and** their imaginary parts are equal.
2. **Add, subtract, multiply and divide** complex numbers written as $x + \mathrm{i}y$, showing the working for multiplication and division.
3. **Use conjugate pairs**: when a polynomial equation has real coefficients, its non-real roots come in pairs $a \pm b\mathrm{i}$. Use this to solve cubics and quartics when one complex root is given.
4. **Show complex numbers as points on an Argand diagram.**
5. **Multiply and divide in polar form**, $r(\cos\theta + \mathrm{i}\sin\theta)$ or $r\mathrm{e}^{\mathrm{i}\theta}$: moduli multiply and arguments add ($|z_1 z_2| = |z_1||z_2|$, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$), and for division moduli divide and arguments subtract.
6. **Find the two square roots** of a complex number in exact form $x + \mathrm{i}y$, with full working.
7. **Describe in simple terms what each operation does to the diagram**: conjugating, adding, subtracting, multiplying and dividing.
8. **Draw loci and regions**: sketch simple equations and inequalities such as $|z - a| = k$, $|z - a| < k$, $|z - a| = |z - b|$ and $\arg(z - a) = \alpha$, and find things about them such as the greatest or least value of $|z|$ or $\arg z$.

Arguments are usually taken in $-\pi < \theta \leqslant \pi$; $0 \leqslant \theta < 2\pi$ is also accepted unless the question says otherwise. De Moivre's theorem for general powers, roots of unity and loci other than circles, lines and half-lines are **not** in this syllabus.

Understand it

A new number, i

The equation $x^2 = -1$ has no real solution, because no real number squares to a negative. So we invent one: i, with

$$\mathrm{i}^2 = -1.$$

A **complex number** is $z = x + \mathrm{i}y$ with x and y real. x is the **real part**, $\mathrm{Re}\, z$, and y is the **imaginary part**, $\mathrm{Im}\, z$ (note: $\mathrm{Im}\, z$ is the real number y, not $\mathrm{i}y$). For $z = 4 - 7\mathrm{i}$, $\mathrm{Re}\, z = 4$ and $\mathrm{Im}\, z = -7$.

Now every quadratic can be solved. For $z^2 + 6z + 13 = 0$ the discriminant is $36 - 52 = -16$, and $\sqrt{-16} = 4\mathrm{i}$, so $z = \dfrac{-6 \pm 4\mathrm{i}}{2} = -3 \pm 2\mathrm{i}$.

Equal means both parts equal. If $a + \mathrm{i}b = c + \mathrm{i}d$ (all real) then $a = c$ **and** $b = d$. So one complex equation gives you **two** real equations. This single fact drives most of the topic.

Arithmetic in the form $x + \mathrm{i}y$

Treat i like a letter and replace i^2 by -1 whenever it appears.

- Add and subtract part by part: $(3 + \mathrm{i}) + (1 + 2\mathrm{i}) = 4 + 3\mathrm{i}$.
- Multiply out the brackets: $(3 + \mathrm{i})(1 + 2\mathrm{i}) = 3 + 6\mathrm{i} + \mathrm{i} + 2\mathrm{i}^2 = 1 + 7\mathrm{i}$.

The **conjugate** of $z = x + \mathrm{i}y$ is $z^* = x - \mathrm{i}y$. A number times its conjugate is always real and never negative:

$$zz^* = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2.$$

That is how you **divide**: multiply top and bottom by the conjugate of the bottom, which makes the bottom real.

$$\frac{4 + 7\mathrm{i}}{2 + \mathrm{i}} = \frac{(4 + 7\mathrm{i})(2 - \mathrm{i})}{(2 + \mathrm{i})(2 - \mathrm{i})} = \frac{8 - 4\mathrm{i} + 14\mathrm{i} + 7}{5} = \frac{15 + 10\mathrm{i}}{5} = 3 + 2\mathrm{i}.$$

Two more facts about conjugates: $(z + w)^* = z^* + w^*$ and $(zw)^* = z^* w^*$. You may be asked to prove one: write $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$ (or use $r\mathrm{e}^{\mathrm{i}\theta}$ form), work out **each side separately**, and show they are the same.

Conjugate pairs of roots

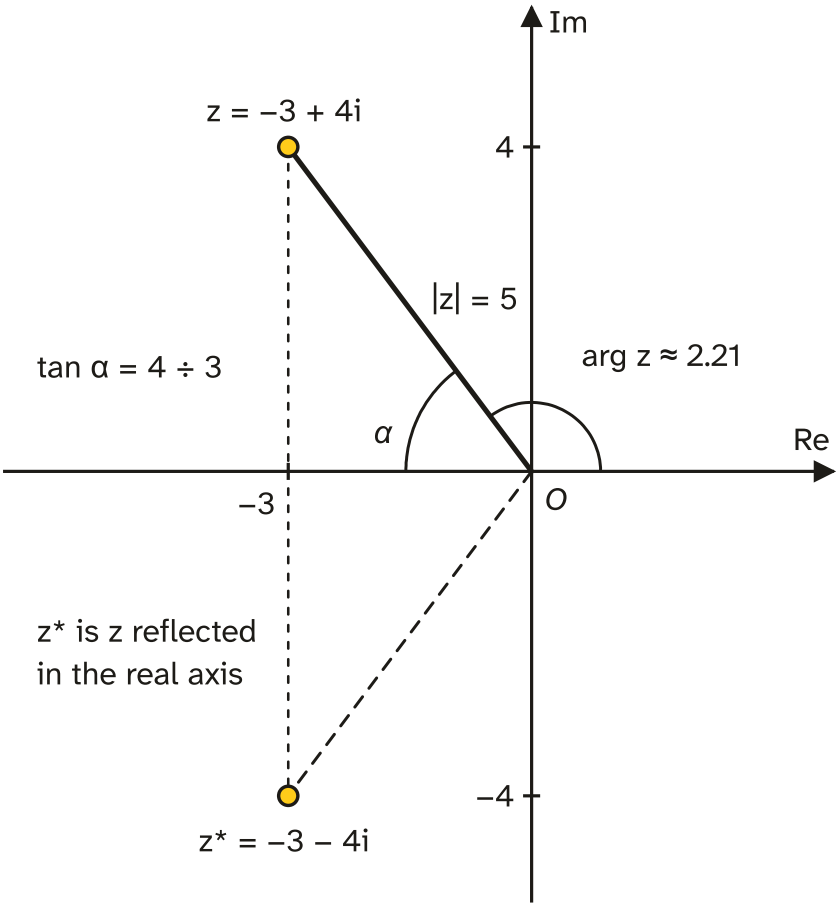
If a polynomial has **real** coefficients and $a + b\mathrm{i}$ is a root, then $a - b\mathrm{i}$ is a root too. (Taking the conjugate of the whole equation leaves the real coefficients unchanged, so z^* satisfies it as well.) The two roots together give a **real quadratic factor**:

$$\big(z - (a + b\mathrm{i})\big)\big(z - (a - b\mathrm{i})\big) = z^2 - 2az + (a^2 + b^2).$$

For roots $2 \pm \mathrm{i}$ this is $z^2 - 4z + 5$. Divide the polynomial by this quadratic to find the remaining factors.

The Argand diagram

Plot $x + \mathrm{i}y$ as the point (x, y): the horizontal axis is the **real axis**, the vertical one the **imaginary axis**. Always use the **same scale on both axes**, or circles turn into ellipses and angles come out wrong.



- The **modulus** $|z|$ is the distance from O to the point: $|x + \mathrm{i}y| = \sqrt{x^2 + y^2}$. For $-3 + 4\mathrm{i}$ it is 5.
- The **argument** $\arg z$ is the angle from the positive real axis to the line Oz, anticlockwise positive, usually in $-\pi < \theta \leqslant \pi$ and always in **radians**. $\tan\theta = \frac{y}{x}$, but your calculator's $\tan^{-1}$ only gives angles in the first or fourth quadrant, so **sketch the point first**. For $-3 + 4\mathrm{i}$ (second quadrant), $\alpha = \tan^{-1}\frac{4}{3} = 0.927$ and $\arg z = \pi - 0.927 = 2.21$.

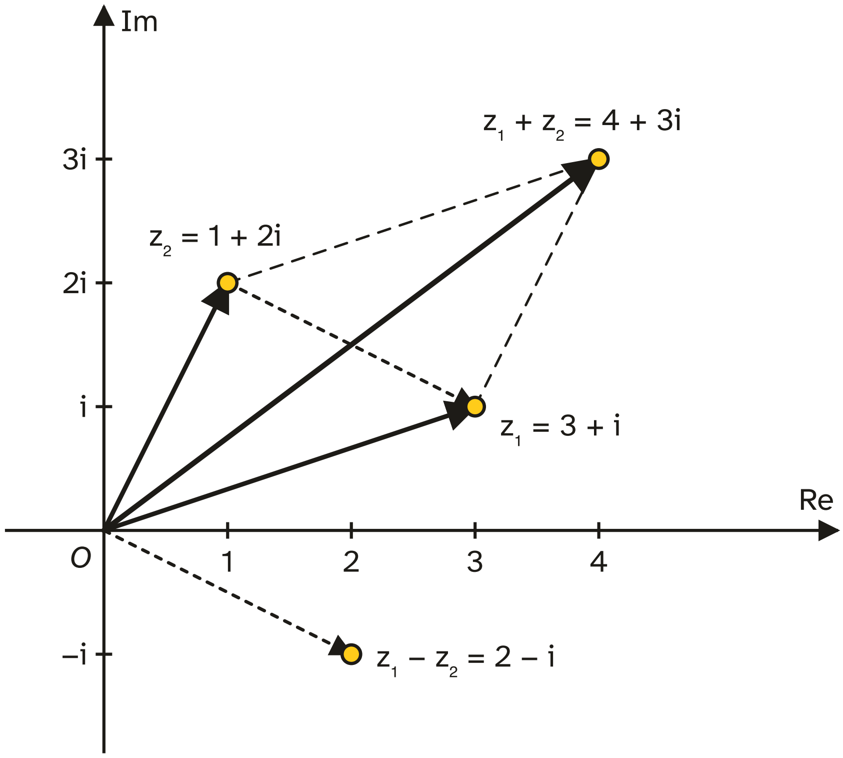
Quadrant of $x + \mathrm{i}y$	**$\arg z$, with $\alpha = \tan^{-1}\left\|\frac{y}{x}\right\|$**
first ($x > 0, y > 0$)	α
second ($x < 0, y > 0$)	$\pi - \alpha$
third ($x < 0, y < 0$)	$-(\pi - \alpha)$
fourth ($x > 0, y < 0$)	$-\alpha$

For example $\arg(-1 - \sqrt{3}\,\mathrm{i}) = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$, and $\arg(4\mathrm{i}) = \frac{\pi}{2}$, $\arg(-5) = \pi$.

Conjugating reflects the point in the real axis: same modulus, argument changes sign.

Adding and subtracting: vectors

A complex number behaves like the vector from O to its point. **Adding** w moves every point by the same vector (a translation): $z_1 + z_2$ is the fourth corner of the parallelogram on Oz_1 and Oz_2.



Subtracting gives the vector between two points: $z_1 - z_2$ is the arrow **from** z_2 **to** z_1. So

$$|z_1 - z_2| = \text{the distance between the points } z_1 \text{ and } z_2.$$

This one idea explains every locus in this topic. Two arrows are **parallel** when one complex number is a real multiple of the other: if $w = kz$ with k real, then w is parallel to z and $|k|$ times as long.

Polar form: $r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta}$

A point is fixed by its distance $r = |z|$ and its angle $\theta = \arg z$. Since $x = r\cos\theta$ and $y = r\sin\theta$,

$$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta}.$$

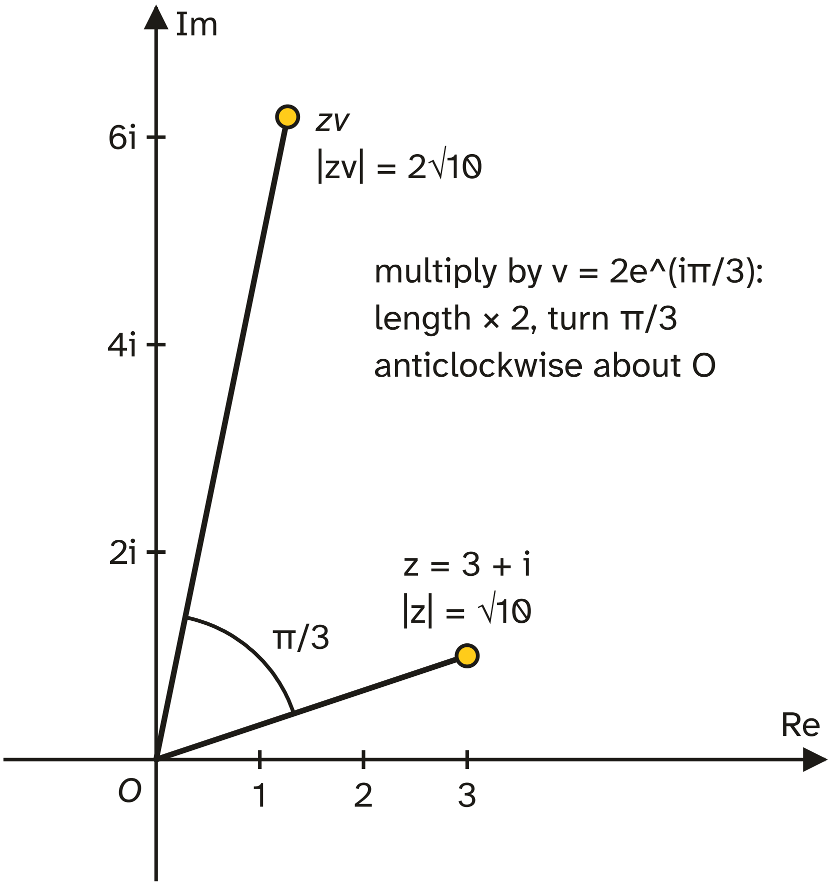
$\mathrm{e}^{\mathrm{i}\theta}$ is short for $\cos\theta + \mathrm{i}\sin\theta$. So $-3+4\mathrm{i} = 5\mathrm{e}^{2.21\mathrm{i}}$, and $2\mathrm{e}^{\mathrm{i}\pi/3} = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = 1 + \sqrt{3}\,\mathrm{i}$.

Polar form makes multiplying easy, because powers of e add:

$$r_1\mathrm{e}^{\mathrm{i}\theta_1} \times r_2\mathrm{e}^{\mathrm{i}\theta_2} = r_1 r_2\,\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}, \qquad \frac{r_1\mathrm{e}^{\mathrm{i}\theta_1}}{r_2\mathrm{e}^{\mathrm{i}\theta_2}} = \frac{r_1}{r_2}\,\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)}.$$

Moduli multiply, arguments add; moduli divide, arguments subtract. If the new argument falls outside $-\pi < \theta \leqslant \pi$, add or subtract 2π. Repeating the rule gives powers: $z^2 = r^2\mathrm{e}^{2\mathrm{i}\theta}$, $z^3 = r^3\mathrm{e}^{3\mathrm{i}\theta}$.

So **multiplying by** $r\mathrm{e}^{\mathrm{i}\alpha}$ is an **enlargement** with scale factor r and a **rotation** through α anticlockwise, both about O. Dividing by it is an enlargement with scale factor $\frac{1}{r}$ and a rotation through α **clockwise**. Multiplying by $\mathrm{i} = \mathrm{e}^{\mathrm{i}\pi/2}$ is just a quarter turn.



Square roots

Every non-zero complex number has **two** square roots, and they are negatives of each other. To find the square roots of $8+6\mathrm{i}$, write $(x+\mathrm{i}y)^2=8+6\mathrm{i}$:

$$x^2-y^2+2xy\,\mathrm{i}=8+6\mathrm{i}\ \Rightarrow\ x^2-y^2=8 \text{ and } 2xy=6.$$

From the second, $y=\frac{3}{x}$. Then $x^2-\frac{9}{x^2}=8$, so $x^4-8x^2-9=0$, $(x^2-9)(x^2+1)=0$. As x is real, $x^2=9$, $x=\pm3$, $y=\pm1$. The roots are $\pm(3+\mathrm{i})$. Check: $(3+\mathrm{i})^2=9+6\mathrm{i}-1=8+6\mathrm{i}$.

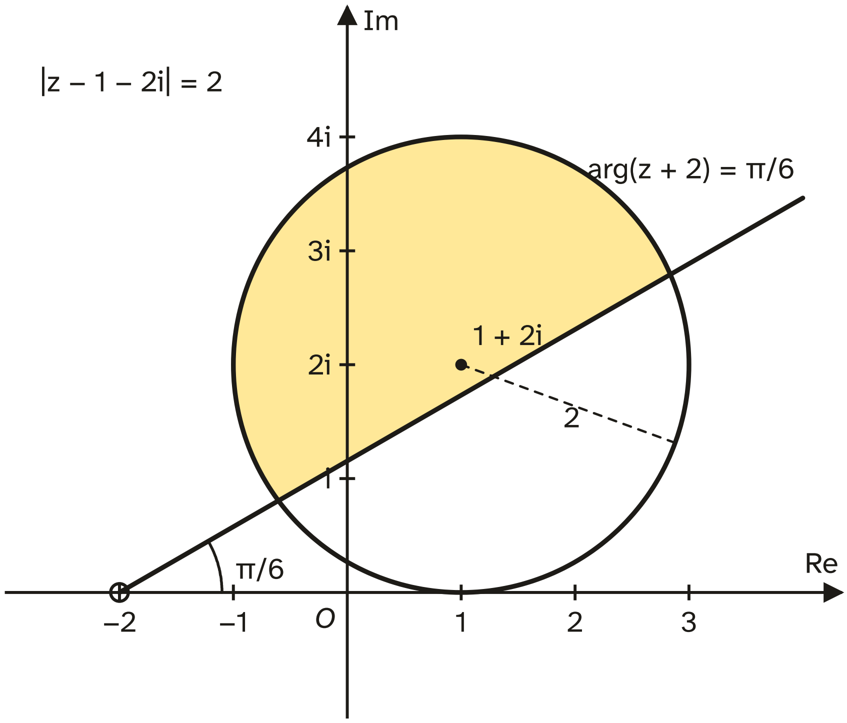
In polar form it is quicker: the square roots of $r\mathrm{e}^{\mathrm{i}\theta}$ are $\sqrt{r}\,\mathrm{e}^{\mathrm{i}\theta/2}$ and the same with π added to (or taken from) the argument.

Loci: where can z be?

A **locus** is the set of points that obey a condition. Read each condition as a statement about **distance** or **direction**.

- $|z-a|=k$: the distance from z to the point a is k. A **circle**, centre a, radius k. Watch the signs: $|z-1-2\mathrm{i}|=|z-(1+2\mathrm{i})|$ has centre $1+2\mathrm{i}$, and $|z+2|=|z-(-2)|$ has centre -2.
- $|z-a|<k$ is the **inside** of that circle; $|z-a|>k$ the outside. Use a dashed circle for $<$ and a solid one for $\leqslant$ if you like, but the region is what matters.

- $|z-a|=|z-b|$: equally far from a and b. The **perpendicular bisector** of the line joining a and b. $|z-a|\leqslant|z-b|$ is the side containing a.
- $\arg(z-a)=\alpha$: the arrow from a to z points in direction α. A **half-line** starting at a (the point a itself is not included) at angle α to the positive real direction.
- $\operatorname{Re} z=c$ is a vertical line, $\operatorname{Im} z=c$ a horizontal one.



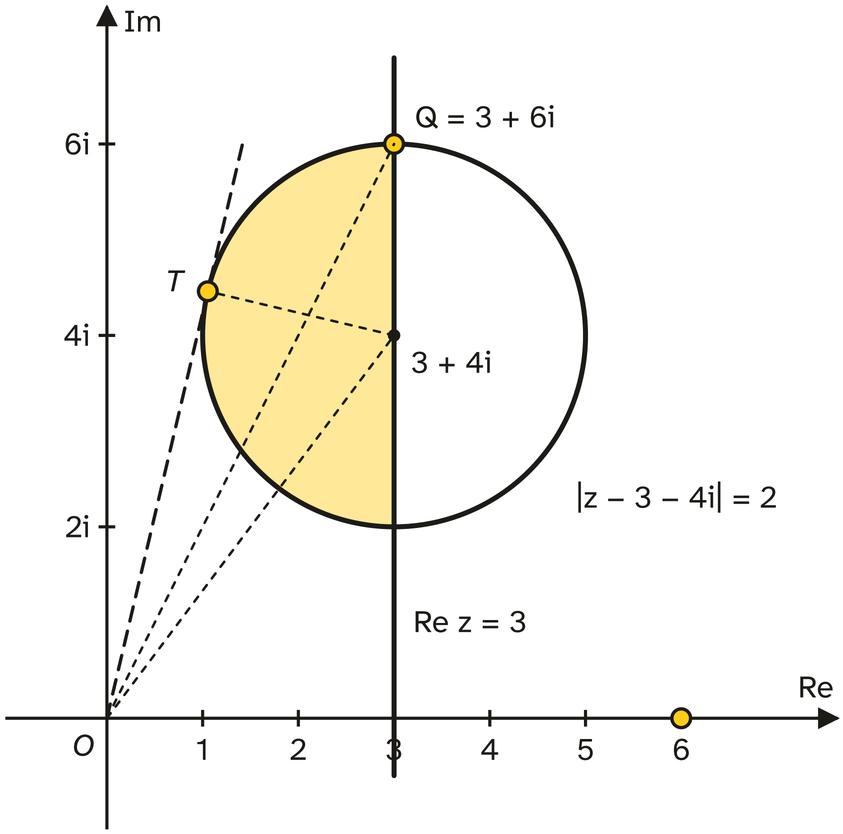
A region given by **two** inequalities is where both shadings overlap.

Greatest and least values on a locus

For points in a region, the greatest or least $|z|$ is the furthest or nearest point to O; the greatest or least $\arg z$ is the point where the line from O is turned furthest round. For a circle with centre c and radius r, and O outside it, with $d=|c|$:

- least $|z|=d-r$ and greatest $|z|=d+r$, on the line through O and the centre;
- greatest and least $\arg z$ are on the two **tangents from** O. Each tangent makes an angle $\sin^{-1}\frac{r}{d}$ with the line to the centre (the radius meets the tangent at a right angle), so they are $\arg c\pm\sin^{-1}\frac{r}{d}$.

If a line cuts off part of the circle, the extreme point may be a **corner** where the line meets the circle instead. Draw the diagram and check that the tangent point is actually in the region.



Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it. Almost nothing in this topic is given.

Formula	Status
$\mathrm{i}^2 = -1$; $z = x + \mathrm{i}y$, $\mathrm{Re}\, z = x$, $\mathrm{Im}\, z = y$	Learn it
$a + \mathrm{i}b = c + \mathrm{i}d \iff a = c$ and $b = d$	Learn it
$z^* = x - \mathrm{i}y$; $zz^* = x^2 + y^2 = \|z\|^2$	Learn it
Divide: multiply top and bottom by the conjugate of the bottom	Learn it
$(z + w)^* = z^* + w^*$, $(zw)^* = z^*w^*$	Learn it
$\|z\| = \sqrt{x^2 + y^2}$; $\tan(\arg z) = \dfrac{y}{x}$, quadrant from a sketch	Learn it
$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta}$, $x = r\cos\theta$, $y = r\sin\theta$	Learn it
$\|z_1 z_2\| = \|z_1\|\|z_2\|$, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ $(\pm 2\pi)$	Learn it
$\left\|\dfrac{z_1}{z_2}\right\| = \dfrac{\|z_1\|}{\|z_2\|}$, $\arg\dfrac{z_1}{z_2} = \arg z_1 - \arg z_2$ $(\pm 2\pi)$	Learn it
Real coefficients: roots $a \pm b\mathrm{i}$ in pairs, factor $z^2 - 2az + a^2 + b^2$	Learn it

Formula	Status
Quadratic formula $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (also with complex a, b, c)	Given
$\lvert z_1 - z_2 \rvert$ = distance between the points z_1 and z_2	Learn it
$\lvert z - a\rvert = k$: circle; $\lvert z - a\rvert = \lvert z - b\rvert$: perpendicular bisector; $\arg(z - a) = \alpha$: half-line from a	Learn it
Circle centre c, radius r, $d = \lvert c\rvert$: $\lvert z\rvert$ from $d - r$ to $d + r$; extreme $\arg z = \arg c \pm \sin^{-1} \frac{r}{d}$	Learn it

Exact values you will need often: $e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $e^{i\pi/4} = \frac{1}{\sqrt{2}}(1 + i)$, $e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $e^{i\pi/2} = i$, $e^{i\pi} = -1$.

How to do it

In the Paper 3 questions we have tagged for this topic (61 questions, 2021–2025), the types came up this often. 23 of the 61 questions mix two or more types, so the counts add up to more than 61.

Question type	Questions
Loci and regions on an Argand diagram	24
Greatest or least modulus or argument	15
Polar and exponential form, geometric effects	15
Equations solved by equating real and imaginary parts	9
Arithmetic and division in the form $x + iy$	9
Square roots of a complex number	6
Quadratic equations with complex coefficients	5
Polynomials with real coefficients: conjugate roots	5

1. Loci and regions on an Argand diagram (24 questions)

1. **Rewrite each condition** in the standard forms: $\lvert z - (\text{centre})\rvert \leqslant k$, $\lvert z - a\rvert \leqslant \lvert z - b\rvert$, $\arg(z - a) \geqslant \alpha$, $\operatorname{Re} z \leqslant c$ or $\operatorname{Im} z \geqslant c$. Change the signs inside carefully: $\lvert z + 3i\rvert = \lvert z - (-3i)\rvert$.
2. **Draw axes with equal scales** and mark numbers (or dashes) on both, so the examiner can check positions.
3. **Draw each boundary.** Circle: mark the centre and make it pass through the right points (centre $\pm k$ across and up). Perpendicular bisector: mark both points (or their midpoint) and draw the line at right angles to the join. Half-line: start at a, draw only one direction, at the right angle.
4. **Shade the region** that satisfies every condition. Test a point if unsure: for $\lvert z - a\rvert \leqslant \lvert z - b\rvert$, the point a itself is always in the region.

Variations you will meet:

- **Perpendicular bisector in the plane.** For $|z+1-\mathrm{i}| \leqslant |z-3-\mathrm{i}|$ the points are $-1+\mathrm{i}$, at $(-1,1)$, and $3+\mathrm{i}$, at $(3,1)$; the midpoint is $(1,1)$ and the join is horizontal, so the bisector is the vertical line $\operatorname{Re} z = 1$ and the region is the side containing $(-1,1)$: $\operatorname{Re} z \leqslant 1$. When the join is sloping, its gradient m gives the bisector gradient $-\frac{1}{m}$ through the midpoint.
- **A pair of half-lines**, such as $-\frac{\pi}{3} \leqslant \arg(z+2-\mathrm{i}) \leqslant \frac{\pi}{6}$: both start at $-2+\mathrm{i}$ and the region is the wedge between them.
- **Read the inequalities off a given diagram**: name the circle's centre and radius and write $|z-(\text{centre})| \leqslant r$; name the line, for example $\operatorname{Re} z \geqslant -1$. Use $\leqslant$ or $\geqslant$ if the boundary is included.
- **Loci of related numbers**, such as z^2 when $|z|=3$ and $-\frac{\pi}{6} \leqslant \arg z \leqslant \frac{\pi}{3}$: squaring squares the modulus and doubles the argument, so z^2 lies on the arc $|w|=9$, $-\frac{\pi}{3} \leqslant \arg w \leqslant \frac{2\pi}{3}$.
- **An equation in x and y that is a circle**: substitute $z = x+\mathrm{i}y$, then complete the square, for example $x^2+4x+y^2-2y=4 \Rightarrow (x+2)^2+(y-1)^2=9$, a circle with centre $-2+\mathrm{i}$ and radius 3.

2. Greatest or least modulus or argument (15 questions)

1. **Draw the region first**, with the centre and radius marked.
2. **Decide where the extreme point is**: on the line through O and the centre (for $|z|$), on a tangent from O (for $\arg z$), or at a corner where two boundaries meet.
3. **Calculate.** $|z|$: distance d from O to the centre, then $d \pm r$. $\arg z$ at a tangent: $\arg(\text{centre}) \pm \sin^{-1}\frac{r}{d}$, using a right-angled triangle with hypotenuse d and opposite side r. At a corner: find the point exactly (solve the circle with the line, for example put $x=4$ into $(x-2)^2+(y-1)^2=9$, giving $y=1\pm\sqrt{5}$), then use its modulus or argument.
4. **Give radians** unless degrees are asked for, to 3 significant figures (or the accuracy stated), with the right sign: points below the real axis have negative arguments.

For example, on the circle $|z-5-12\mathrm{i}|=4$: $d=13$, so $|z|$ goes from 9 to 17, and the greatest argument is $\tan^{-1}\frac{12}{5}+\sin^{-1}\frac{4}{13}=1.176+0.313=1.49$ radians.

Variations you will meet:

- **Least distance between two loci**, such as a point on a circle and a point on a line: the shortest path runs along the perpendicular from the centre to the line, so it is (distance from centre to line) $-r$. The perpendicular distance from (p,q) to the line $ax+by+c=0$ is $\frac{|ap+bq+c|}{\sqrt{a^2+b^2}}$.
- **Greatest or least $\operatorname{Im} z$ or $\operatorname{Re} z$**: the top, bottom or side of a circle, or a corner.
- **Half-line regions**: the extreme argument may be at the starting point of the half-line or along it.

3. Polar and exponential form, geometric effects (15 questions)

1. **Modulus** $r=\sqrt{x^2+y^2}$, simplified exactly: $\sqrt{8}=2\sqrt{2}$.
2. **Argument**: sketch the point, find $\alpha=\tan^{-1}\left|\frac{y}{x}\right|$, and adjust for the quadrant (table in "Understand it"). Give exact values such as $\frac{3\pi}{4}$ when the question says "exact".
3. **Write** $r\mathrm{e}^{\mathrm{i}\theta}$ or $r(\cos\theta+\mathrm{i}\sin\theta)$ as asked.

4. **Multiply or divide**: multiply or divide the moduli, add or subtract the arguments, then bring the argument into $-\pi < \theta \leqslant \pi$.

Variations you will meet:

- **Geometric effect of multiplying or dividing by $k\mathrm{e}^{\mathrm{i}\alpha}$**: say both parts: "enlargement, scale factor k, centre O" and "rotation through α anticlockwise about O" (clockwise for division). Don't call a scale factor less than 1 a "reduction" or "shrink": it is still an enlargement.
- **Powers**: $z^3 = r^3\mathrm{e}^{3\mathrm{i}\theta}$. z^n is **real** when $n\theta$ is a multiple of π; real and positive when it is a multiple of 2π. For example $\left(\mathrm{e}^{3\pi\mathrm{i}/8}\right)^n$ is real first when $n = 8$ (it is -1 then) and real and positive first when $n = 16$.
- **"zw is real and $|w|$ is known; find w"**: $\arg z + \arg w$ must be 0 or π, so there are two answers.
- **Exact trig values**: find the same number in both forms, then compare. If $\frac{u}{v} = \mathrm{e}^{\mathrm{i}\theta} \times r$ and also $= x + \mathrm{i}y$, then $\cos\theta = \frac{x}{r}$ and $\tan\theta = \frac{y}{x}$.
- **Shapes on the diagram**: $|z_1 - z_2|$ gives side lengths; the difference of two arguments gives the angle at O. "OAB is equilateral" needs equal moduli and an angle of $\frac{\pi}{3}$ between them; a right angle at O means the arguments differ by $\frac{\pi}{2}$. "Parallel" means one difference is a real multiple of the other, and state the ratio of lengths too.
- **Proofs with conjugates**, such as $(z_1z_2)^* = z_1^*z_2^*$: with $z = r\mathrm{e}^{\mathrm{i}\theta}$, $z^* = r\mathrm{e}^{-\mathrm{i}\theta}$. Work out each side separately and conclude.

4. Equations solved by equating real and imaginary parts (9 questions)

1. **Substitute** $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$. If there is a fraction, multiply through by its denominator first.
2. **Expand**, using $\mathrm{i}^2 = -1$, and collect into (real part) $+\mathrm{i}$(imaginary part) $=$ right-hand side.
3. **Equate** real parts and imaginary parts: two equations. If the right side is real, its imaginary part is 0.
4. **Solve** them together, often a linear equation with a quadratic. When a factor such as y appears on both sides, **factorise, don't cancel**, or you lose the solution $y = 0$.
5. Write each answer as $x + \mathrm{i}y$ and apply any condition in the question (for example $\mathrm{Im}\, z > 0$).

For example, $2z - \mathrm{i}z^* = 3 + 3\mathrm{i}$: $2x + 2\mathrm{i}y - \mathrm{i}x - y = 3 + 3\mathrm{i}$, so $2x - y = 3$ and $2y - x = 3$, giving $x = y = 3$ and $z = 3 + 3\mathrm{i}$.

Variations you will meet:

- **"w is real" or "w is purely imaginary"** for a fraction w: put $z = x + \mathrm{i}y$, multiply top and bottom by the conjugate of the bottom, and set the imaginary part (or real part) of the **top** to 0. For example, $\frac{z-1}{z-\mathrm{i}}$ is real when $x + y = 1$. Geometrically, $\frac{z-a}{z-b}$ is real when z, a and b lie on one straight line. A second condition, such as a given $|z|$, then gives a quadratic.
- **Unknown real constants** in an identity, such as $\frac{p+3\mathrm{i}}{2-\mathrm{i}} = q(1+\mathrm{i})$: multiply out and equate real and imaginary parts to get two equations for the constants.

5. Arithmetic and division in the form $x + \mathrm{i}y$ (9 questions)

1. **Multiply** out brackets, replacing i^2 by -1.

2. **Divide** by multiplying top and bottom by the conjugate of the bottom; show the product on the bottom, for example $(2+\mathrm{i})(2-\mathrm{i})=5$.
3. **Separate** into real and imaginary parts, $x+\mathrm{i}y$, simplified.

Variations you will meet:

- **An unknown real constant and an argument condition.** "$\arg u=-\frac{\pi}{4}$" means $\operatorname{Im} u=-\operatorname{Re} u$ with $\operatorname{Re} u>0$; "u is purely imaginary with positive imaginary part" means $\arg u=\frac{\pi}{2}$. For $u=\frac{a+4\mathrm{i}}{2-\mathrm{i}}=\frac{(2a-4)+(a+8)\mathrm{i}}{5}$, $\arg u=\frac{\pi}{2}$ gives $a=2$ (and then $\operatorname{Im} u=2>0$). Check the quadrant of any value you find.
- **Expressions with powers**, such as $f(a)=a^3+4a$ with $a=1-y\mathrm{i}$: expand a^2 first, then $a^3=a^2\times a$.
- **Show u, u^*, $u+v$ or $u-v$ on a diagram** and describe the shape: use the vector meaning of adding and subtracting.

6. Square roots of a complex number (6 questions)

1. Write $(x+\mathrm{i}y)^2=a+b\mathrm{i}$, so $x^2-y^2=a$ and $2xy=b$.
2. **Eliminate** $y=\frac{b}{2x}$ to get a quartic in x (a quadratic in x^2): $x^4-ax^2-\frac{b^2}{4}=0$.
3. **Solve** for x^2 and keep only the **positive** value, since x is real.
4. Find $x=\pm\ldots$, the matching y from $2xy=b$ (keep the pairs matched: the signs of x and y multiply to the sign of b), and write the roots as $\pm(x+\mathrm{i}y)$ in exact form.

"By first forming a quartic equation" means you must show step 2; a calculator answer alone scores nothing.

Variation: if the number is given in polar form, halve the argument and square-root the modulus, and remember the second root (add or subtract π).

7. Quadratic equations with complex coefficients (5 questions)

1. **Use the quadratic formula** with the complex coefficients, substituting carefully: in $z^2+2\mathrm{i}z+3=0$, $b^2=(2\mathrm{i})^2=-4$, so $b^2-4ac=-4-12=-16$ and $\sqrt{-16}=4\mathrm{i}$, giving $z=\frac{-2\mathrm{i}\pm4\mathrm{i}}{2}$: $z=\mathrm{i}$ or $z=-3\mathrm{i}$.
2. If the discriminant is not real, find its square root with the method of type 6.
3. If the z^2 coefficient is complex, you can first multiply the whole equation by its conjugate to make it real, or divide at the end by multiplying by the conjugate.
4. Give each answer as $x+\mathrm{i}y$.

The two roots of a quadratic with complex coefficients are **not** usually conjugates of each other; that only holds when the coefficients are real.

8. Polynomials with real coefficients: conjugate roots (5 questions)

1. **Verify or use a given root** $a+b\mathrm{i}$: substitute it, working out z^2 and z^3 in the form $x+\mathrm{i}y$ first, and equate real and imaginary parts (this finds unknown coefficients).
2. **State the conjugate** $a-b\mathrm{i}$ as a second root, giving the reason (the coefficients are real).
3. **Form the real quadratic factor** $z^2-2az+(a^2+b^2)$.

4. **Divide** (long division or comparing coefficients) to find the other factor, and solve it for the remaining roots.

For example, if $2 - \mathrm{i}$ is a root of $z^3 + pz^2 + qz + 15 = 0$ with p, q real, then $2 + \mathrm{i}$ is also a root, the quadratic factor is $z^2 - 4z + 5$, and comparing constant terms the third factor is $z + 3$. So $z^3 + pz^2 + qz + 15 = (z^2 - 4z + 5)(z + 3) = z^3 - z^2 - 7z + 15$, giving $p = -1$, $q = -7$, and the roots are $2 \pm \mathrm{i}$ and -3.

Variation: a later part may ask for roots of a related equation, such as the same polynomial in z^2: solve for z^2 first, then take square roots of each value.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

By first forming a quartic equation in x, find the square roots of $21 - 20\mathrm{i}$ in the form $x + \mathrm{i}y$, where x and y are real and exact. **[5]**

Let $(x + \mathrm{i}y)^2 = 21 - 20\mathrm{i}$. Then $x^2 - y^2 + 2xy\,\mathrm{i} = 21 - 20\mathrm{i}$. **M1** for squaring and equating real and imaginary parts:

$$x^2 - y^2 = 21, \qquad 2xy = -20.$$

A1

From the second, $y = -\dfrac{10}{x}$, so $x^2 - \dfrac{100}{x^2} = 21$. **M1** for eliminating one variable:

$$x^4 - 21x^2 - 100 = 0 \;\Rightarrow\; (x^2 - 25)(x^2 + 4) = 0.$$

A1

x is real, so $x^2 = 25$, $x = \pm 5$, and $y = -\frac{10}{x} = \mp 2$. The square roots are $5 - 2\mathrm{i}$ and $-5 + 2\mathrm{i}$, that is $\pm(5 - 2\mathrm{i})$. **A1**

Check: $(5 - 2\mathrm{i})^2 = 25 - 20\mathrm{i} - 4 = 21 - 20\mathrm{i}$.

Example 2

Find all the complex numbers z which satisfy $z^2 + 4z^* = 0$. Give your answers in the form $x + \mathrm{i}y$. **[6]**

Substitute $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$. **B1**

$$x^2 + 2xy\,\mathrm{i} + \mathrm{i}^2y^2 + 4x - 4y\,\mathrm{i} = 0 \;\Rightarrow\; (x^2 - y^2 + 4x) + \mathrm{i}(2xy - 4y) = 0.$$

M1 for expanding with $\mathrm{i}^2 = -1$.

Equate real and imaginary parts to 0: **M1**

$$x^2 - y^2 + 4x = 0, \qquad 2y(x-2) = 0.$$

A1 for both.

The second equation gives $y = 0$ **or** $x = 2$ (factorise; don't divide by y). **M1** for solving both cases.

- $y = 0$: $x^2 + 4x = 0$, so $x = 0$ or $x = -4$.
- $x = 2$: $4 - y^2 + 8 = 0$, so $y^2 = 12$ and $y = \pm 2\sqrt{3}$.

$z = 0$, $z = -4$, $z = 2 + 2\sqrt{3}\,\mathrm{i}$, $z = 2 - 2\sqrt{3}\,\mathrm{i}$. **A1**

Example 3

The complex numbers u and v are given by $u = -2 + 2\mathrm{i}$ and $v = 1 + \sqrt{3}\,\mathrm{i}$.

(a) Express u and v in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving exact values. **[3]**

(b) Hence express uv and $\dfrac{u}{v}$ in the same form. **[3]**

(c) Find $\dfrac{u}{v}$ in the form $x + \mathrm{i}y$, and hence show that $\cos\dfrac{5\pi}{12} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$. **[4]**

(d) State the geometrical effect of multiplying a complex number by v. **[2]**

(a) $|u| = \sqrt{4+4} = 2\sqrt{2}$. u is in the second quadrant with $\alpha = \tan^{-1} 1 = \frac{\pi}{4}$, so $\arg u = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. $|v| = \sqrt{1+3} = 2$ and $\arg v = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

$$u = 2\sqrt{2}\,\mathrm{e}^{3\pi\mathrm{i}/4}, \qquad v = 2\,\mathrm{e}^{\pi\mathrm{i}/3}.$$

B1 for both moduli, **B1** for $\arg u$, **B1** for $\arg v$.

(b) uv: moduli multiply, $2\sqrt{2} \times 2 = 4\sqrt{2}$; arguments add, $\frac{3\pi}{4} + \frac{\pi}{3} = \frac{13\pi}{12}$, which is more than π, so subtract 2π: $-\frac{11\pi}{12}$. **M1**

$$uv = 4\sqrt{2}\,\mathrm{e}^{-11\pi\mathrm{i}/12}.$$

A1

$\dfrac{u}{v}$: $\dfrac{2\sqrt{2}}{2} = \sqrt{2}$ and $\dfrac{3\pi}{4} - \dfrac{\pi}{3} = \dfrac{5\pi}{12}$, so $\dfrac{u}{v} = \sqrt{2}\,\mathrm{e}^{5\pi\mathrm{i}/12}$. **A1**

(c) Multiply top and bottom by $v^* = 1 - \sqrt{3}\,\mathrm{i}$: **M1**

$$\frac{u}{v} = \frac{(-2+2\mathrm{i})(1-\sqrt{3}\,\mathrm{i})}{(1+\sqrt{3}\,\mathrm{i})(1-\sqrt{3}\,\mathrm{i})} = \frac{-2 + 2\sqrt{3}\,\mathrm{i} + 2\mathrm{i} + 2\sqrt{3}}{4} = \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2}\,\mathrm{i}.$$

A1

From (b), the real part is also $\sqrt{2}\cos\frac{5\pi}{12}$. **M1** for comparing the two forms:

$$\sqrt{2}\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2} \;\Rightarrow\; \cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

A1, the given answer, with the step of multiplying top and bottom by $\sqrt{2}$ shown.

(d) $v = 2\mathrm{e}^{\pi \mathrm{i}/3}$: an **enlargement** with scale factor 2, centre O (**B1**), and a **rotation** through $\frac{\pi}{3}$ anticlockwise about O (**B1**). The third diagram in "Understand it" shows this for $z = 3 + \mathrm{i}$.

Example 4

(a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both $|z - 3 - 4\mathrm{i}| \leqslant 2$ and $|z| \leqslant |z - 6|$. **[4]**

(b) Find the greatest value of $\arg z$ for points in this region. **[3]**

(c) Find the greatest value of $|z|$ for points in this region, in exact form. **[2]**

(a) $|z - (3 + 4\mathrm{i})| \leqslant 2$: a circle with centre $3 + 4\mathrm{i}$ and radius 2, and its inside. **B1** centre, **B1** radius.

$|z - 0| \leqslant |z - 6|$: closer to 0 than to 6. The perpendicular bisector of 0 and 6 is the vertical line $\mathrm{Re}\, z = 3$ **B1**, and the region is the side containing 0: $\mathrm{Re}\, z \leqslant 3$.

Shade the left half of the disc. **B1** (The fifth diagram in "Understand it" shows this region.)

(b) The greatest argument is where a line from O touches the circle on its upper left. The centre is at distance $d = |3 + 4\mathrm{i}| = 5$ with $\arg(3 + 4\mathrm{i}) = \tan^{-1}\frac{4}{3}$. In the right-angled triangle O, T, centre, the angle at O is $\sin^{-1}\frac{2}{5}$. **B1** for one of these angles.

$$\arg z_{\max} = \tan^{-1}\frac{4}{3} + \sin^{-1}\frac{2}{5} = 0.9273 + 0.4115 = 1.34 \text{ radians } (3 \text{ s.f.}).$$

M1 A1. The tangent point T has real part $\sqrt{21}\cos(1.339) = 1.05 < 3$, so it is in the region.

(c) On the whole circle the furthest point from O would be on the line through the centre, at distance $5 + 2 = 7$, but that point, $4.2 + 5.6\mathrm{i}$, has real part 4.2, so it is outside the region. Around the circle the distance from O falls steadily either side of that point, so on the left half the furthest point is one of the corners where the line meets the circle: $3 + 6\mathrm{i}$ (distance $\sqrt{45}$) rather than $3 + 2\mathrm{i}$ (distance $\sqrt{13}$). **M1**

$$|z|_{\max} = \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}.$$

A1

Common mistakes

- **Unequal scales on the axes.** Examiners see this every year: a circle drawn on unequal scales is really an ellipse, and angles and perpendicular bisectors come out wrong. Use the same scale and label it.

- **The wrong centre.** $|z+4-\mathrm{i}|$ is $|z-(-4+\mathrm{i})|$, centre $-4+\mathrm{i}$. Change **both** signs.
- **A full line instead of a half-line** for $\arg(z-a)=\alpha$, or a half-line starting at O instead of at a, or at the wrong angle.
- **The argument in the wrong quadrant.** $\tan^{-1}\frac{y}{x}$ from a calculator is only right in the first and fourth quadrants. Sketch the point and adjust. Points below the real axis have negative arguments.
- **Degrees instead of radians**, or a decimal when "exact" is asked ($\frac{3\pi}{4}$, not 2.36 or 135°).
- **Equating the imaginary part to the wrong thing.** In $z+3\mathrm{i}z^*=4$ the right side has imaginary part 0, not 4. Also, the imaginary part of $x+\mathrm{i}y$ is y, not $\mathrm{i}y$.
- **Cancelling a factor and losing a root.** From $2y(x-2)=0$ you get $y=0$ **or** $x=2$; dividing by y throws away real solutions.
- **Working with z and z^* directly** in an equation that mixes them. Substituting $x+\mathrm{i}y$ straight away is usually much easier.
- **Square roots: unmatched signs or extra answers.** For $15-8\mathrm{i}$, from $x=\pm4$ and $y=\mp1$ the roots are $\pm(4-\mathrm{i})$, not all four sign combinations. Reject the negative value of x^2 (it would make x non-real).
- **4 instead of $4\mathrm{i}$ in the quadratic formula.** With complex coefficients, b^2 includes i^2: $(-4\mathrm{i})^2=-16$. And don't assume the two roots are conjugates.
- **Using a calculator to find roots of a polynomial** with no working. It scores nothing; show substitution, the quadratic factor and the division.
- **Stopping at the diagram.** For "greatest arg", mark the tangent on the diagram, then use the right-angled triangle; check that the point you use is really in the shaded region.

How to get the marks

- **Show the algebra for $\times$ and $\div$.** The syllabus asks for full working: write the conjugate you multiply by and the product it gives, for example $(2-5\mathrm{i})(2+5\mathrm{i})=29$.
- **Equating parts earns a method mark on its own.** Write the two real equations clearly, one labelled real and one imaginary.
- **Final answers in the form asked**: $x+\mathrm{i}y$ with x and y simplified, not coordinates or "$x=3$, $y=-2$" separately. For polar form, $r>0$ and $-\pi<\theta\leqslant\pi$ unless told otherwise.
- **"Exact"** means surds, fractions and multiples of π. Otherwise give angles in radians to 3 significant figures unless the question asks for degrees.
- **Argand sketches**: marks go to the centre, the radius, the key point (like the midpoint or the start of a half-line), each line, and finally the correct shading, which depends on everything before it. Show a scale on both axes (numbers or dashes) and make it equal.
- **"Show that" and proofs** (conjugate rules, equilateral triangles, exact trig values) need every step and a concluding sentence.
- **"Hence"** means use the part before: a root found earlier, a polar form from part (a), or the diagram you drew.

- **Geometric effects**: name both transformations with their details: for dividing by $\frac{4}{3}e^{\pi i/5}$, "enlargement, scale factor $\frac{3}{4}$" and "rotation through $\frac{\pi}{5}$ clockwise", centre O. Vague words like "shrink" or "turn" lose the mark.
- **Give every answer that the conditions allow**, and only those: two square roots, all solutions of an equation (apply conditions like $\operatorname{Im} z > 0$ at the end).

Quick check

1. Express $\dfrac{3+4i}{1-2i}$ in the form $x + iy$.
2. Given that $3 - i$ is a root of $z^3 - 5z^2 + 4z + 10 = 0$, find the other two roots.
3. For points on the circle $|z - 6 + 8i| = 3$, find the least and greatest values of $|z|$.
4. $z = 4e^{i\pi/6}$ and $w = 2e^{-2\pi i/3}$. Find zw and $\dfrac{z}{w}$ in the form $re^{i\theta}$ with $-\pi < \theta \leqslant \pi$.
5. Solve $z^2 - (3 + i)z + 2 + 2i = 0$, giving your answers in the form $x + iy$.

Answers

1. Multiply top and bottom by $1+2\mathrm{i}$: $\dfrac{(3+4\mathrm{i})(1+2\mathrm{i})}{1+4} = \dfrac{3+6\mathrm{i}+4\mathrm{i}-8}{5} = \dfrac{-5+10\mathrm{i}}{5} = -1+2\mathrm{i}$.
2. The coefficients are real, so $3+\mathrm{i}$ is also a root. Quadratic factor: $z^2-6z+10$. Then $z^3-5z^2+4z+10 = (z^2-6z+10)(z+1)$, so the other roots are $3+\mathrm{i}$ and -1.
3. The centre is $6-8\mathrm{i}$, at distance $\sqrt{36+64} = 10$ from O. Least $|z| = 10-3 = 7$, greatest $|z| = 10+3 = 13$.
4. $zw = 8\mathrm{e}^{\mathrm{i}(\pi/6-2\pi/3)} = 8\mathrm{e}^{-\pi\mathrm{i}/2}$ (which is $-8\mathrm{i}$). $\dfrac{z}{w} = 2\mathrm{e}^{\mathrm{i}(\pi/6+2\pi/3)} = 2\mathrm{e}^{5\pi\mathrm{i}/6}$.
5. Discriminant: $(3+\mathrm{i})^2-4(2+2\mathrm{i}) = 9+6\mathrm{i}-1-8-8\mathrm{i} = -2\mathrm{i}$. Its square roots: $(x+\mathrm{i}y)^2 = -2\mathrm{i}$ gives $x^2=y^2$ and $xy=-1$, so $\pm(1-\mathrm{i})$. Then $z = \dfrac{3+\mathrm{i}\pm(1-\mathrm{i})}{2}$: $z=2$ or $z=1+\mathrm{i}$.

Past questions to try

- **9709/31/M/J/25 Q3**: a fraction that is real, together with a given modulus.
- **9709/31/M/J/25 Q6**: multiplying in exponential form and the geometric effect on a diagram.
- **9709/33/M/J/24 Q6**: a circle and a half-line region, then the greatest argument.
- **9709/31/O/N/21 Q10**: a polynomial with a given complex root, then a region and the least imaginary part.