Cambridge AS & A Level · Mathematics (9709) · Notes

Differential equations

A2

Read online at pastopic.com/cambridge/mathematics-9709/notes/differential-equations

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Read first: Integration, Algebra

What you need to know

By the end of this topic you should be able to:

1. **Turn a sentence about a rate of change into a differential equation.** "The rate of change of x" means $\frac{dx}{dt}$, and "is proportional to" means "equals a constant k times". You may have to bring in that constant and then work out its value from information in the question.
2. **Solve a first order differential equation whose variables can be separated**, that is one you can rearrange so that everything with y is on one side and everything with x is on the other, then integrate both sides to get the general solution. Any integration method from the Paper 3 integration topic can be needed: partial fractions, integration by parts, trigonometric identities, the inverse tangent and so on.
3. **Use an initial condition** (a pair of values such as $y = 0$ when $x = 1$) to find the constant of integration, and so the particular solution.
4. **Interpret the solution** when the equation models a real situation: a time taken, a value after a certain time, what happens in the long run. You never need any specialist knowledge of the context; everything you need is in the question.

Only first order equations with separable variables are in Paper 3. Integrating factors and second order equations are not.

Understand it

What a differential equation is

A **differential equation** is an equation that contains a derivative, such as

$$\frac{dy}{dx} = y \cos x.$$

It does not tell you y directly; it tells you how fast y changes. **Solving** it means finding the relationship between y and x itself, with no derivative left. It is **first order** because only the first derivative appears.

Separating the variables

Put everything with y (including dy) on the left and everything with x (including dx) on the right, then integrate each side with respect to its own letter:

$$\frac{1}{y}\,dy = \cos x\,dx \quad \Rightarrow \quad \int \frac{1}{y}\,dy = \int \cos x\,dx \quad \Rightarrow \quad \ln|y| = \sin x + c.$$

Why is this allowed? Divide the equation by y to get $\frac{1}{y}\frac{dy}{dx} = \cos x$, then integrate both sides with respect to x. By the chain rule, $\int \frac{1}{y}\frac{dy}{dx}\,dx$ is the same as $\int \frac{1}{y}\,dy$. Writing "$\frac{1}{y}\,dy = \cos x\,dx$" is shorthand for exactly that.

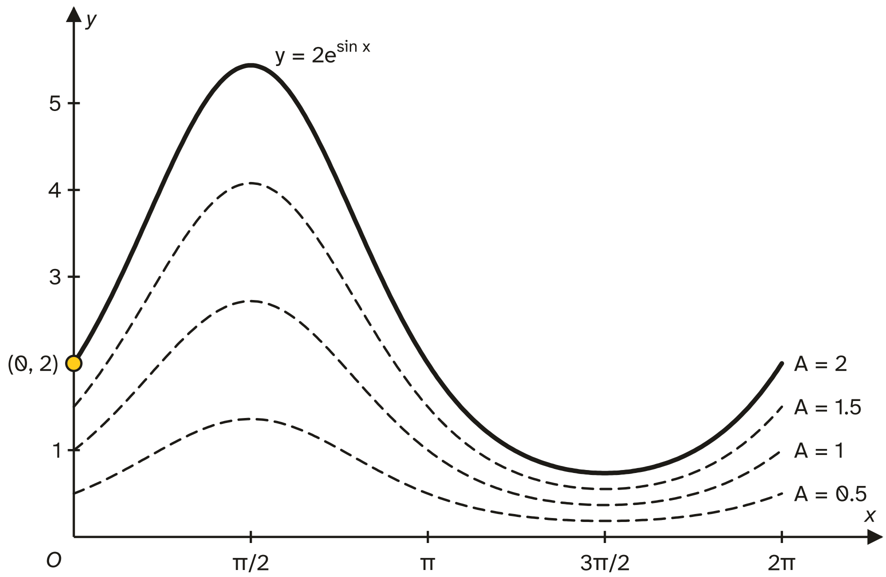
Only **one** constant is needed: a constant on each side would combine into one anyway.

General and particular solutions

$\ln|y| = \sin x + c$ is the **general solution**: every value of c gives a different curve, all with the same "gradient rule". Taking e to the power of both sides removes the logarithm:

$$|y| = e^{\sin x + c} = e^c\,e^{\sin x} \quad \Rightarrow \quad y = Ae^{\sin x},$$

where A stands for $\pm e^c$. An **initial condition** picks out one curve. If $y = 2$ when $x = 0$, then $2 = Ae^0 = A$, so the **particular solution** is $y = 2e^{\sin x}$.



You can find the constant straight after integrating (put $x = 0$, $y = 2$ into $\ln|y| = \sin x + c$ to get $c = \ln 2$) or after rearranging, but if you rearrange first, the constant must go with it correctly: $\ln y = \sin x + c$ becomes $y = Ae^{\sin x}$, **not** $y = e^{\sin x} + c$. The safe habit is to find the constant first.

Forming a differential equation

Real-life questions describe a rate in words. Translate each phrase:

In words	As an equation
the rate of change of N (over time t)	$\frac{dN}{dt}$
is proportional to N	$= kN$
decreases at a rate proportional to N	$\frac{dN}{dt} = -kN$, with $k > 0$
is inversely proportional to $\sqrt{t}$	$= \frac{k}{\sqrt{t}}$
is proportional to the product of x and $(1-x)$	$= kx(1-x)$
water flows in at 400 cm³ per second and leaks out at a rate proportional to h	$\frac{dV}{dt} = 400 - kh$

When the question is about volume but asks for an equation in the depth h, link the two rates with the chain rule from differentiation: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$. For a tank with vertical sides and base area A, $V = Ah$, so $\frac{dV}{dt} = A\frac{dh}{dt}$.

A curve can be described the same way: "the gradient at (x, y) is proportional to $\frac{y}{(x+1)^2}$" means $\frac{dy}{dx} = \frac{ky}{(x+1)^2}$.

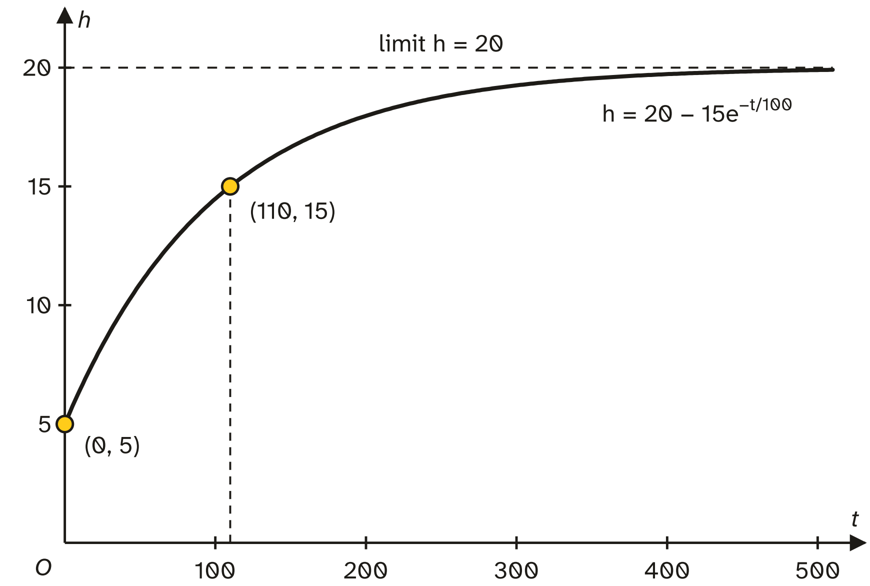
The integration you will need

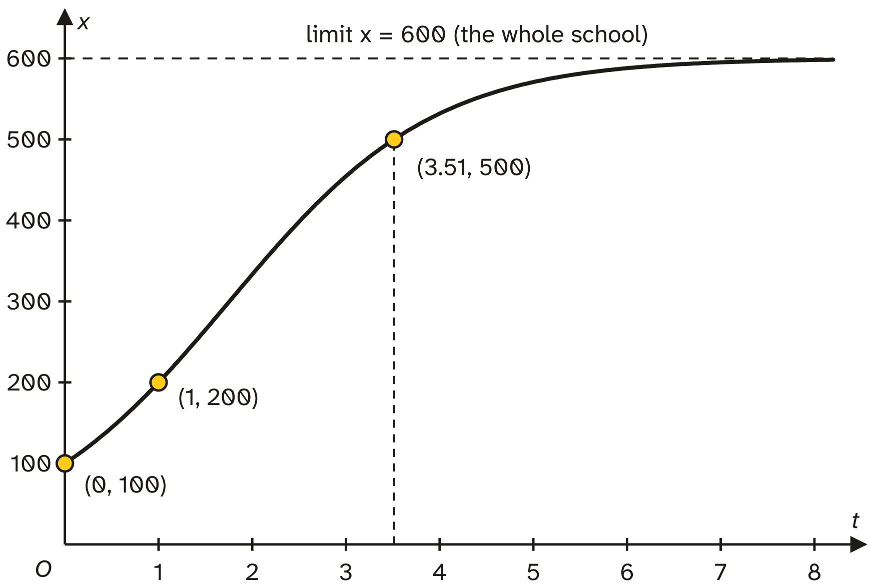
Separating is the easy part; most marks are lost in the integration. The methods come from the integration topic (P2 and P3). The ones these questions use most:

- **Logarithms:** $\int \frac{1}{y}\, dy = \ln|y|$, $\int \frac{1}{ax+b}\, dx = \frac{1}{a}\ln|ax+b|$, and $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)|$, so $\int \frac{x}{x^2+1}\, dx = \frac{1}{2}\ln(x^2+1)$.
- **Exponentials:** $\int e^{ky}\, dy = \frac{1}{k}e^{ky}$, so $\int e^{-2y}\, dy = -\frac{1}{2}e^{-2y}$. Dividing by e^{3y} is the same as multiplying by e^{-3y}, and $e^{y-x} = e^y e^{-x}$ splits into a y part and an x part.
- **Partial fractions:** for example $\frac{1}{x(600-x)} = \frac{1}{600}\left(\frac{1}{x} + \frac{1}{600-x}\right)$. Watch the sign: $\int \frac{1}{600-x}\, dx = -\ln|600-x|$.
- **Inverse tangent** (given in MF19): $\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$. A fraction like $\frac{2y+3}{y^2+9}$ splits into $\frac{2y}{y^2+9} + \frac{3}{y^2+9}$, which integrates to $\ln(y^2+9) + \tan^{-1}\frac{y}{3}$. Also given: $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$.
- **Integration by parts:** for products such as $x\sin 3x$ or ye^{-3y}.
- **Trigonometry:** $\int \sec^2 kx\, dx = \frac{1}{k}\tan kx$ (so $\frac{1}{\cos^2 3x}$ gives $\frac{1}{3}\tan 3x$); $\int \tan x\, dx = -\ln|\cos x|$; $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ before integrating; $\frac{\sin 2y}{\sin y} = 2\cos y$ from the double angle formula; $\frac{1}{1+\cos 2\theta} = \frac{1}{2\cos^2\theta} = \frac{1}{2}\sec^2\theta$.
- **Division first:** if the top of a fraction has a power at least as high as the bottom, divide (or split it) before integrating: $\frac{20-x}{x} = \frac{20}{x} - 1$.

What solutions look like

Solutions of models often **level off**. The depth of water in a tank that fills at a fixed rate but leaks faster as it gets deeper rises towards a limit; a rumour spreading through a school grows slowly, then fast, then slowly again as almost everyone has heard it. The exponential term tells you the limit: as $t \to \infty$, $e^{-t/100} \to 0$.





Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
Separating: $\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{1}{g(y)}\, dy = \int f(x)\, dx$	Learn it
One constant of integration, found from the initial condition	Learn it
$\ln	y
"Rate of change of x" $= \frac{dx}{dt}$; "proportional to" $= k\times$; "decreasing" means a negative rate	Learn it
Connected rates: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$	Learn it

Formula	Status
$\displaystyle\int \frac1x,dx = \ln	x
$\displaystyle\int \frac{1}{ax + b},dx = \frac1a\ln	ax + b
$\displaystyle\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$	Given
$\displaystyle\int \frac{1}{a^2 - x^2},dx = \frac{1}{2a}\ln\left	\frac{a + x}{a - x}\right
$\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$	Given
$\int \sin x\,dx = -\cos x$, $\int \cos x\,dx = \sin x$, $\int \sec^2 x\,dx = \tan x$	Given
$\int \tan x,dx = -\ln	\cos x
$\sin 2A = 2\sin A\cos A$, $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$	Given
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ (rearranged from the given $\cos 2A$)	Learn it
Vertical-sided tank with base area A: $V = Ah$; cylinder $V = \pi r^2 h$	Learn it
Sphere $V = \frac{4}{3}\pi r^3$, cone $V = \frac{1}{3}\pi r^2 h$	Given

How to do it

In the Paper 3 questions we have tagged for this topic (37 questions, 2021–2025), the types came up this often. 18 of the 37 questions mix two or three types, so the counts add up to more than 37.

Question type	Questions
Solving a separable equation with a given condition	26
Forming a differential equation from a description	11
Solving a model and answering in context	11
Long-term behaviour, a greatest value or a sketch	7
Finding an unknown constant from a second condition	6

1. Solving a separable equation with a given condition (26 questions)

1. **Separate.** Divide or multiply so that all the y terms are with dy and all the x terms are with dx. Write both integral signs. Split exponentials first: $e^{2x-y} = e^{2x}e^{-y}$, so $e^y\,dy = e^{2x}\,dx$.
2. **Prepare each side** for integration: partial fractions, a trigonometric identity, dividing a fraction, or splitting it into two. If part (a) asked for partial fractions or a derivative, that is the hint for this step.
3. **Integrate both sides** and add **one** constant c.
4. **Find c at once** by substituting the given pair of values (in **radians** for trigonometric functions).

5. **Rearrange** into the form asked for ("y in terms of x", "t in terms of x"). Remove every $\ln$ by taking e to the power of both sides, after first combining the logarithms into one: $\ln(y+2) = \tan x + \ln 3 \Rightarrow y + 2 = 3e^{\tan x}$.
6. If a value is asked for, substitute it, exactly if the question says "exact".

For example, $\frac{dy}{dx} = 2xy$ with $y = 3$ when $x = 0$: $\int \frac{1}{y}\,dy = \int 2x\,dx$ gives $\ln y = x^2 + c$, and $c = \ln 3$, so $y = 3e^{x^2}$.

Variations you will meet:

- **Partial fractions** in x or in y, often as part (a): $\frac{2}{1-9y^2}$, $\frac{1}{(x+1)(3x+1)}$ and so on. Take care with the coefficient: $\int \frac{1}{3x+1}\,dx = \frac{1}{3}\ln|3x+1|$.
- **A fraction that splits into a logarithm and an inverse tangent**, such as $\frac{2y+3}{y^2+9}$ (see "The integration you will need").
- **Integration by parts** on one side, such as $\int x \sin 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x$, or $\int ye^{-3y}\,dy$.
- **Trigonometric equations:** $\sin^2 2\theta$ becomes $\frac{1}{2}(1-\cos 4\theta)$; $\frac{\sin 4y}{\sin 2y} = 2\cos 2y$; $\frac{1}{\cos^2 3y} = \sec^2 3y$ integrates to $\frac{1}{3}\tan 3y$.
- **Algebraic division** when the top's power is at least the bottom's, for example $\frac{x^3+\ldots}{x-3}$ = quotient $+$ $\frac{\text{remainder}}{x-3}$.
- **"Hence"**: part (a) asks you to differentiate something (for example, show that $\frac{d}{dx}\ln(\sin x) = \cot x$). Read it backwards: it is the integral you need in part (b), here $\int \cot x\,dx = \ln(\sin x)$. In the same way, showing $\frac{d}{dx}\tan^2 x = 2\tan x \sec^2 x$ tells you that $\int \tan x \sec^2 x\,dx = \frac{1}{2}\tan^2 x$.
- **A given point, then a value.** "Find x when $y = \ldots$" may end in a trigonometric equation: for $\sin 2y = 0.6$ with $0 < y < \frac{\pi}{2}$, $2y$ can be 0.6435 or $\pi - 0.6435$, so there are **two** answers, $y = 0.322$ and $y = 1.25$.
- **A curve described in words** (the gradient or normal at (x, y) satisfies some rule). Write the rule as $\frac{dy}{dx} = \ldots$ and solve. For a normal, remember its gradient is $-1 \div \frac{dy}{dx}$.

2. Forming a differential equation from a description (11 questions)

1. **Name the variables** and write the rate asked about, for example $\frac{dx}{dt}$.
2. **Translate the words** with the table in "Understand it". A decreasing quantity gets a minus sign. If the question gives the final equation ("show that . . ."), you must reach it exactly.
3. **Tanks and balloons:** write $\frac{dV}{dt} = (\text{rate in}) - (\text{rate out})$, then use the chain rule with V in terms of h or r : $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$.
4. **Find the constant of proportionality** if the question gives a rate at one instant: substitute the values and solve for k. For example, $\frac{dV}{dt} = 400 - kh$ and $V = 2000h$ give $2000\frac{dh}{dt} = 400 - kh$; if $\frac{dh}{dt} = 0.1$ when $h = 10$, then $200 = 400 - 10k$ and $k = 20$.
5. **Simplify** to the printed form, showing each step.

Variations you will meet:

- **"Explain why"** (1 mark): say which factor comes from which phrase, for example "x is the fraction affected, $1 - x$ the fraction not affected, and the rate is k times their product".

- **"Write down the values of a and b"** in $\frac{dV}{dt} = a - bV$, for a water butt filling at 25 litres per minute and leaking $0.02V$ litres per minute: $\frac{dV}{dt} = 25 - 0.02V$, so $a = 25$, $b = 0.02$.
- **Proportional to $\sqrt{V}$ or h^2:** the rate out is $k\sqrt{V}$ or kh^2. Substitute V in terms of h before simplifying, and gather constants into one new constant if the question says so.

3. Solving a model and answering in context (11 questions)

1. **Separate and integrate** as in type 1. With $\frac{dh}{dt} = \frac{20-h}{100}$, keep it simple: $\int \frac{1}{20-h}\,dh = \int \frac{1}{100}\,dt$.
2. **Use the starting values** ($t = 0$ and the initial amount) for the constant. Don't swap in a later time as if it were $t = 0$.
3. **Answer the actual question:** a time ("find the time taken", "when will the tank be empty"), a value at a time, or t in terms of x. Give a **number with units** in context, rounded as asked: "$100 \ln 3$ seconds" should end as "110 seconds".
4. **Check it makes sense**: times positive, amounts inside their limits.

Variations you will meet:

- **Time to empty**: put the amount equal to 0. **Time for the whole area to be infected**: put x equal to the whole area.
- **Logistic ("S-shaped") models**, $\frac{dx}{dt} = kx(N - x)$: partial fractions give $\frac{1}{N} \ln \frac{x}{N-x} = kt + c$. See Example 4.
- **The solution feeds an iteration**: rearrange the solution into the form asked, for example $x = e^{\cdots}$, then iterate as in "Numerical solution of equations".

4. Long-term behaviour, a greatest value or a sketch (7 questions)

1. Get the solution into the form "$y = \ldots$" first.
2. **As $x \to \infty$ (or t becomes large):** use $e^{-x} \to 0$, $e^x \to \infty$ and $\frac{1}{x} \to 0$, and state the limit **exactly**. For example, $V = 400\left(1 - e^{-t/50}\right) \to 400$, and $y = 3 - 2e^{-x} \to 3$.
3. **Greatest value:** find where the varying part is largest. If $N = 100e^{2\sin(0.1t)}$, the greatest value is when $\sin(0.1t) = 1$, giving $N = 100e^2$.
4. **Sketch:** plot the key points (where the curve starts, crosses the axes, its highest point) and show any symmetry or limit.

Always give a reason, not a guess: "as $t \to \infty$, $e^{-t/50} \to 0$, so $V \to 400$".

5. Finding an unknown constant from a second condition (6 questions)

The equation contains k (or it comes from "proportional to"), and the question gives **two** pairs of values.

1. Separate and integrate, keeping k as a letter. Leave k on the x (or t) side; moving it across makes coefficient errors more likely.
2. Use the first pair (usually the start) to find c.

3. Use the second pair to find k. Keep it exact (for example $k = \frac{1}{2}\ln\frac{3}{2}$) until the last step.
4. Substitute both and answer the question.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

The variables x and y satisfy the differential equation

$$\cos^2 x\,\frac{dy}{dx} = y + 2,$$

for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and $y = 1$ when $x = 0$. Solve the differential equation, obtaining an expression for y in terms of x, and find the exact value of y when $x = \frac{\pi}{4}$. **[6]**

Separate:

$$\int \frac{1}{y+2}\,dy = \int \frac{1}{\cos^2 x}\,dx = \int \sec^2 x\,dx.$$

B1 for correct separation.

$$\ln(y+2) = \tan x + c.$$

B1 for $\ln(y+2)$, **B1** for $\tan x$.

At $x = 0$, $y = 1$: $\ln 3 = 0 + c$, so $c = \ln 3$. **M1** for using the condition to find the constant.

$$\ln(y+2) - \ln 3 = \tan x \;\Rightarrow\; \frac{y+2}{3} = e^{\tan x} \;\Rightarrow\; y = 3e^{\tan x} - 2.$$

A1

At $x = \frac{\pi}{4}$, $\tan x = 1$, so $y = 3e - 2$. **A1** (about 6.15, but the question asks for the exact value).

Example 2

The variables x and y satisfy the differential equation

$$x\frac{dy}{dx} = (y+1)(y+3),$$

for $x > 0$, and $y = 1$ when $x = 1$.

(a) Express $\dfrac{2}{(y+1)(y+3)}$ in partial fractions. **[2]**

(b) Solve the differential equation, obtaining an expression for y in terms of x. **[6]**

(a) $\dfrac{2}{(y+1)(y+3)} = \dfrac{A}{y+1} + \dfrac{B}{y+3}$, so $2 = A(y+3) + B(y+1)$. Putting $y = -1$ gives $A = 1$; putting $y = -3$ gives $B = -1$. **M1**

$$\frac{2}{(y+1)(y+3)} = \frac{1}{y+1} - \frac{1}{y+3}.$$

A1

(b) Separate, multiplying both sides by 2 so that part (a) can be used:

$$\int \frac{2}{(y+1)(y+3)}\,dy = \int \frac{2}{x}\,dx.$$

B1

$$\ln(y+1) - \ln(y+3) = 2\ln x + c.$$

B1 for the y side (follow through from part (a)), **B1** for $2\ln x$.

At $x = 1, y = 1$: $\ln 2 - \ln 4 = 0 + c$, so $c = \ln \frac{1}{2} = -\ln 2$. **M1**

$$\ln \frac{y+1}{y+3} = \ln \frac{x^2}{2} \quad \Rightarrow \quad \frac{y+1}{y+3} = \frac{x^2}{2}.$$

M1 for removing the logarithms correctly. Cross-multiply and collect the y terms: $2y + 2 = x^2y + 3x^2$, so $y(2 - x^2) = 3x^2 - 2$ and

$$y = \frac{3x^2 - 2}{2 - x^2}.$$

A1. Check: $x = 1$ gives $y = \frac{1}{1} = 1$. ✓

Example 3

A tank with vertical sides has a horizontal base of area 2000 cm². Water is poured into the tank at a constant rate of 400 cm³ per second. Water leaks out through a hole in the base at a rate proportional to h, where h cm is the depth of the water at time t seconds. When $h = 10$, the depth is increasing at 0.1 cm per second.

(a) Show that $\dfrac{dh}{dt} = \dfrac{20 - h}{100}$. **[4]**

(b) Initially the depth is 5 cm. Solve the differential equation, obtaining an expression for h in terms of t. **[4]**

(c) Find the time taken for the depth to reach 15 cm, giving your answer correct to 3 significant figures. **[1]**

(d) State what happens to the depth of the water as t becomes large. **[1]**

(a) $\frac{dV}{dt} = 400 - kh$. **B1**

$V = 2000h$, so $\frac{dV}{dt} = 2000\frac{dh}{dt}$ and $\frac{dh}{dt} = \frac{400 - kh}{2000}$. **M1** for linking the rates correctly.

At $h = 10$: $0.1 = \frac{400 - 10k}{2000}$, so $200 = 400 - 10k$ and $k = 20$. **M1**

$$\frac{dh}{dt} = \frac{400 - 20h}{2000} = \frac{20 - h}{100}.$$

A1, the given answer from correct working.

(b)

$$\int \frac{1}{20 - h}\,dh = \int \frac{1}{100}\,dt \;\Rightarrow\; -\ln(20 - h) = \frac{t}{100} + c.$$

B1 for separating, **B1** for $-\ln(20 - h)$ with the minus sign.

At $t = 0$, $h = 5$: $c = -\ln 15$. **M1**

$$\ln\frac{15}{20 - h} = \frac{t}{100} \;\Rightarrow\; 20 - h = 15e^{-t/100} \;\Rightarrow\; h = 20 - 15e^{-t/100}.$$

A1

(c) $h = 15$: $\ln\frac{15}{5} = \frac{t}{100}$, so $t = 100\ln 3 = 110$ seconds (3 s.f.). **B1**

(d) As $t \to \infty$, $e^{-t/100} \to 0$, so $h \to 20$: the depth gets closer and closer to 20 cm. **B1**

The second diagram in "Understand it" shows this solution.

Example 4

A rumour is spreading through a school of 600 students. At time t days after the rumour starts, x students have heard it. The rate at which x increases is proportional to the product of the number of students who have heard the rumour and the number who have not. It is given that $x = 100$ when $t = 0$ and $x = 200$ when $t = 1$. Treat x and t as continuous variables.

(a) Write down a differential equation relating x, t and a constant k. **[1]**

(b) Solve the differential equation and show that $x = \frac{600}{1 + 5(0.4)^t}$. **[7]**

(c) Find the time taken until 500 students have heard the rumour. **[1]**

(d) State what happens to x as t becomes large. **[1]**

(a) $\frac{dx}{dt} = kx(600 - x)$. **B1**

(b) $\int \frac{1}{x(600 - x)}\,dx = \int k\,dt$. **B1**

Partial fractions: $\dfrac{1}{x(600-x)} = \dfrac{1}{600}\left(\dfrac{1}{x} + \dfrac{1}{600-x}\right)$. **M1 A1**

$$\frac{1}{600}\big(\ln x - \ln(600-x)\big) = kt + c.$$

A1. Multiply by 600 and write $K = 600k$: $\ln\dfrac{x}{600-x} = Kt + C$.

At $t = 0$, $x = 100$: $C = \ln\frac{100}{500} = -\ln 5$. **M1**

At $t = 1$, $x = 200$: $\ln\frac{200}{400} = K - \ln 5$, so $K = \ln 5 - \ln 2 = \ln 2.5$. **M1** (so $k = \frac{\ln 2.5}{600} = 0.00153$ to 3 s.f.)

Then $\ln\dfrac{x}{600-x} = t\ln 2.5 - \ln 5$, so $\dfrac{x}{600-x} = \dfrac{2.5^t}{5}$. Rearranging, $5x = 600(2.5^t) - x(2.5^t)$, so

$$x = \frac{600(2.5^t)}{5 + 2.5^t} = \frac{600}{1 + 5(2.5)^{-t}} = \frac{600}{1 + 5(0.4)^t},$$

since $\frac{1}{2.5} = 0.4$. **A1**, the given answer.

(c) $x = 500$: $\dfrac{500}{100} = \dfrac{2.5^t}{5}$, so $2.5^t = 25$ and $t = \dfrac{\ln 25}{\ln 2.5} = 3.51$ days (3 s.f.). **B1**

(d) As $t \to \infty$, $(0.4)^t \to 0$, so $x \to 600$: in the long run every student hears the rumour. **B1**

The third diagram in "Understand it" shows this S-shaped curve.

Common mistakes

- **Not separating first.** Substituting the given values into the differential equation before integrating gets nowhere. Separate, integrate, then use the values.
- **Leaving out the constant of integration**, or adding it only after rearranging. Without it you can't use the initial condition, and several marks go.
- **Wrong form of the constant after removing logarithms.** From $\ln y = x^2 + c$ the next step is $y = Ae^{x^2}$, not $y = e^{x^2} + c$. Find c **before** you rearrange.
- **Coefficient slips in the integration**: $\int e^{-3y}\,dy = -\frac{1}{3}e^{-3y}$, not $-3e^{-3y}$; $\int \frac{1}{3x+1}\,dx = \frac{1}{3}\ln(3x+1)$; $\int \frac{1}{4+9y^2}\,dy = \frac{1}{6}\tan^{-1}\frac{3y}{2}$ (write it as $\frac{1}{9}\cdot\frac{1}{\frac{4}{9}+y^2}$ first).
- **Sign errors** with $\int \frac{1}{a-x}\,dx = -\ln|a-x|$, $\int \tan x\,dx = -\ln|\cos x|$ and $\int \sin x\,dx = -\cos x$. Examiners see these every year.
- **Not spotting the identity.** $\sin^2 2x$ and $\cos^2\theta$ can't be integrated until you use the double angle formula, which is in the formula list. Also, $\frac{\tan x}{\cos x}$ is $\sec x\tan x$, whose integral is $\sec x$.
- **Treating e^{y-x} as one thing.** Write it as $e^y e^{-x}$ so the variables can be separated.
- **Using degrees.** Every trigonometric value in calculus is in radians, including when you find the constant.
- **Wrong use of logarithm laws.** $\ln(A+B)$ is not $\ln A + \ln B$. Combine logarithms into one with the laws, then remove the $\ln$.
- **Stopping too early.** "Obtain an expression for y" means $y = \ldots$ with no $\ln y$ left; a solution still written as $\ln(\ldots) = \ldots$ loses the last mark.

- **Missing the proportionality constant**, or assuming it is 1. "Proportional to" always brings in a k, which must then be found or carried through.
- **Answers out of context.** In a model, give the time as a number with units (110 seconds rather than $100\ln 3$), and use $t = 0$ for "initially".
- **Guessing the long-term value.** "y tends to infinity" without a reason earns nothing. Show which term tends to 0.

How to get the marks

- **Separation is the first mark.** Write it with both integral signs, dy and dx in place. It can't be awarded if dx or dy ends up on the wrong side.
- **Each correct integral usually earns its own mark**, so show the result of each side before combining them.
- **The constant mark** needs a constant used with the right terms: "$x = 0, y = 1 \Rightarrow c = \ldots$". Say which values you are using.
- **"Solve the differential equation"** means reach the particular solution, not just the general one. **"Obtain an expression for y in terms of x"** means y as the subject with no logarithm around it. "Obtain a relation" allows any correct form, and "simplified" means combined into as few terms as possible.
- **"Show that"** (forming the equation): every step visible, especially the chain rule and the value of k, and the final line must match the printed equation exactly.
- **"Exact"** means e, $\ln$, π and surds, no decimals. Otherwise give answers to 3 significant figures, or as asked ("to the nearest year").
- **"Hence"** means use the previous part: a derivative in part (a) is the integral you need in part (b).
- **"State"** (for long-term behaviour) needs a clear value with a brief reason.
- **Modulus signs in $\ln|y|$** are usually condoned when the quantity is positive, but keep them when it might be negative.
- **Follow-through.** If your partial fractions in part (a) are wrong, using them correctly in part (b) still earns method marks.

Quick check

1. The temperature of a cup of coffee is θ °C at time t minutes. It cools at a rate proportional to the difference between its temperature and the room temperature of 20 °C. Write down a differential equation for θ.
2. Solve $\dfrac{dy}{dx} = 6x^2y$, given that $y = 2$ when $x = 0$, obtaining y in terms of x.
3. The variables satisfy $e^y\dfrac{dy}{dx} = x\cos x$, and $y = 0$ when $x = 0$. Solve the differential equation, obtaining y in terms of x, and find y when $x = \frac{\pi}{2}$ correct to 3 significant figures.
4. The variables satisfy $y\dfrac{dy}{dx} = \sin^2 x$, with $y > 0$, and $y = 2$ when $x = 0$. Find y in terms of x and find the value of y when $x = \pi$ correct to 3 significant figures.

5. The variables satisfy $\frac{dy}{dx} = e^{-x}(y+1)$, and $y = 0$ when $x = 0$. Solve the differential equation and state the exact value that y approaches as x tends to infinity.

Answers

1. $\dfrac{d\theta}{dt} = -k(\theta - 20)$, where k is a positive constant. The minus sign is there because the coffee is cooling (θ decreases while $\theta > 20$).
2. $\displaystyle\int \frac{1}{y}\,dy = \int 6x^2\,dx$, so $\ln y = 2x^3 + c$. At $x = 0$: $c = \ln 2$. So $\ln \frac{y}{2} = 2x^3$ and $y = 2e^{2x^3}$.
3. $\int e^y\,dy = \int x\cos x\,dx$. By parts ($u = x$, $\frac{dv}{dx} = \cos x$): $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x$. So $e^y = x\sin x + \cos x + c$; at $x = 0$: $1 = 0 + 1 + c$, so $c = 0$. Hence $y = \ln(x\sin x + \cos x)$. At $x = \frac{\pi}{2}$: $y = \ln\frac{\pi}{2} = 0.452$.
4. $\int y\,dy = \int \sin^2 x\,dx = \int \frac{1}{2}(1 - \cos 2x)\,dx$, so $\frac{1}{2}y^2 = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$. At $x = 0$: $2 = c$. So $y^2 = x - \frac{1}{2}\sin 2x + 4$ and, as $y > 0$, $y = \sqrt{x - \frac{1}{2}\sin 2x + 4}$. At $x = \pi$: $y = \sqrt{\pi + 4} = 2.67$.
5. $\int \frac{1}{y+1}\,dy = \int e^{-x}\,dx$, so $\ln(y+1) = -e^{-x} + c$. At $x = 0$: $0 = -1 + c$, so $c = 1$. Then $y = e^{1-e^{-x}} - 1$. As $x \to \infty$, $e^{-x} \to 0$, so $y \to e - 1$.

Past questions to try

- **9709/33/M/J/23 Q8**: a fraction that splits into a logarithm and an inverse tangent, then a value of x.
- **9709/33/O/N/22 Q10**: forming and solving a model of a leaking pool, and its long-term behaviour.
- **9709/31/O/N/24 Q10**: a tank emptying at a rate proportional to $\sqrt{V}$, with two conditions and the time to empty.
- **9709/31/M/J/24 Q11**: a disease spreading through plants, solved with partial fractions.