Cambridge AS & A Level · Mathematics (9709) · Notes

Differentiation AS

Read online at pastopic.com/cambridge/mathematics-9709/notes/differentiation-p1

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Read first: Quadratics, Coordinate geometry

What you need to know

By the end of this topic you should be able to:

1. **Explain what "the gradient at a point" means.** Draw a chord from the point to a second point on the curve, then move the second point closer and closer. The chord gradients settle on one value, and that value is the gradient of the curve at the point. You need the idea, not a formal proof. Know the four ways of writing first and second derivatives: $f'(x)$, $f''(x)$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
2. **Differentiate any power of** x, including negative and fractional powers, one term at a time, and differentiate a bracket raised to a power with the chain rule, such as $(x^2+1)^4$ or $\frac{3}{4x-1}$.
3. **Put the derivative to work.** Find the gradient at a point, write the equations of tangents and normals, find where a graph goes up and where it goes down, and link the speeds at which two connected quantities change over time (for example, how fast a balloon's volume grows when you know how fast its radius grows).
4. **Find the points where the gradient is zero**, decide whether each is a maximum or a minimum (the second derivative is the usual test, but the sign of the gradient either side also works), and use these points to sketch a curve.

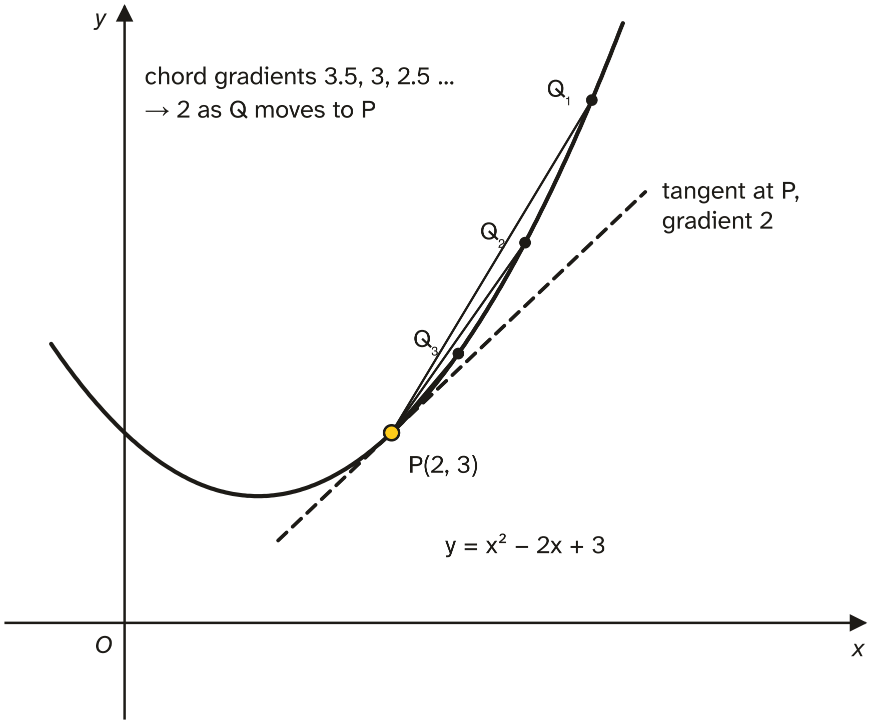
Points of inflexion are **not** in the syllabus. Differentiating e^x, $\ln x$, trigonometric functions, products and quotients comes later, in Paper 2 and Paper 3.

Understand it

Gradient of a curve

A straight line has the same steepness everywhere. A curve does not: it is steeper in some places than others. The **gradient of a curve at a point** is the gradient of the **tangent** there, the straight line that just touches the curve at that point and runs in the same direction as it.

You cannot find a tangent's gradient from two points on it, because you only know one point, P. So take a second point Q on the curve and work out the gradient of the chord PQ. Then slide Q towards P. The chords turn and get closer and closer to the tangent, and their gradients get closer and closer to one number. That number is the gradient at P.



For $y = x^2 - 2x + 3$ and $P(2, 3)$, take Q at $x = 2 + h$, where h is small. Then

$$\text{gradient of } PQ = \frac{\left((2+h)^2 - 2(2+h) + 3\right) - 3}{h} = \frac{2h + h^2}{h} = 2 + h.$$

With $h = 0.1$ the chord's gradient is 2.1; with $h = 0.01$ it is 2.01. As h shrinks to 0 the gradient gets closer to 2, so the gradient of the curve at P is 2.

The derivative

Doing this at every point gives a new function, the **derivative** or **gradient function**, written $\frac{dy}{dx}$ or $f'(x)$. You don't need chords each time, because the pattern for powers of x is simple: **multiply by the power, then take one off the power**.

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = nx^{n-1}$$

This works for every rational power, including negative powers and fractions. For example, $4x^3 \to 12x^2$, and $6\sqrt{x} = 6x^{1/2} \to 3x^{-1/2}$, and $\frac{5}{x^2} = 5x^{-2} \to -10x^{-3}$. A constant on its own has gradient 0, so it disappears. Differentiate a sum term by term: $x^5 - 2x^3 + 7 \to 5x^4 - 6x^2$.

$\frac{dy}{dx}$ is not a fraction; it is one symbol meaning "the rate at which y changes as x changes". Substituting a value of x into it gives the gradient of the curve at that point.

The chain rule

$(2x + 7)^5$ is a function inside a function: first work out $u = 2x + 7$, then take u^5. Each layer changes at its own rate, and the rates multiply:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

In practice: **differentiate the outside as if the bracket were a single letter, then multiply by the derivative of the inside**.

- $(2x+7)^5 \to 5(2x+7)^4 \times 2 = 10(2x+7)^4$
- $\dfrac{3}{4x-1} = 3(4x-1)^{-1} \to -3(4x-1)^{-2} \times 4 = -12(4x-1)^{-2}$
- $(x^2+1)^4 \to 4(x^2+1)^3 \times 2x = 8x(x^2+1)^3$

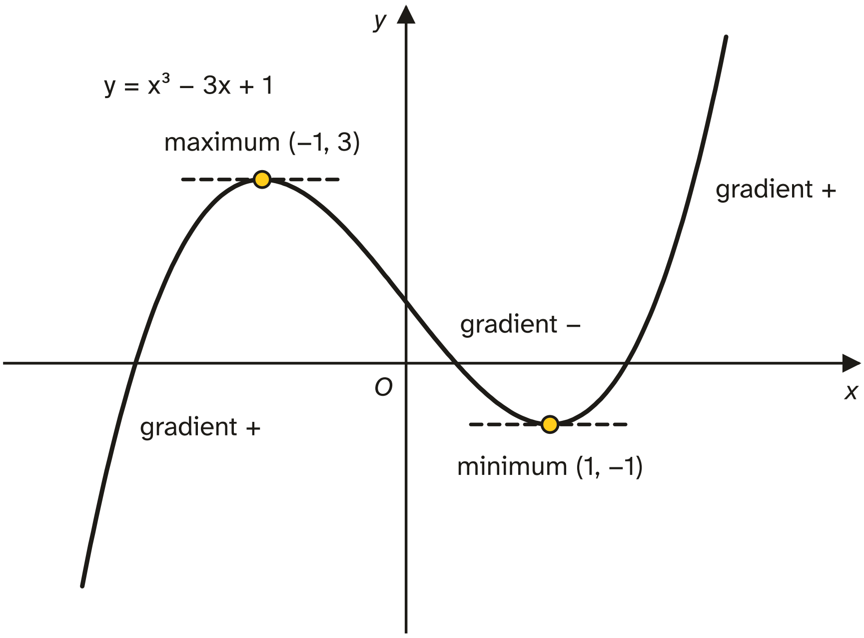
Forgetting the "$\times$ inside" is the most common error in this topic.

The second derivative

Differentiate $\frac{dy}{dx}$ again and you get $\frac{d^2y}{dx^2}$ (or $f''(x)$), the rate at which the gradient itself changes. If it is positive, the gradient is increasing, so the curve bends upwards like a cup; if it is negative, the gradient is decreasing and the curve bends downwards like a cap.

Stationary points

At the top of a hill or the bottom of a valley on a graph, the tangent is flat, so the gradient is 0. These are **stationary points**: find them by solving $\frac{dy}{dx} = 0$.



For $y = x^3 - 3x + 1$: $\frac{dy}{dx} = 3x^2 - 3 = 0$ gives $x = \pm 1$, so the stationary points are $(-1, 3)$ and $(1, -1)$.

To tell which is which, use the second derivative, $\frac{d^2y}{dx^2} = 6x$:

- at $x = -1$ it is $-6 < 0$: the curve bends down (a cap), so $(-1, 3)$ is a **maximum**;
- at $x = 1$ it is $6 > 0$: the curve bends up (a cup), so $(1, -1)$ is a **minimum**.

Another way is to check the sign of the gradient just either side. At a maximum it goes $+, 0, -$ (uphill, flat, downhill); at a minimum it goes $-, 0, +$. For $(1, -1)$: at $x = 0.5$ the gradient is -2.25 and at $x = 2$ it is 9, so it goes $-, 0, +$: a minimum.

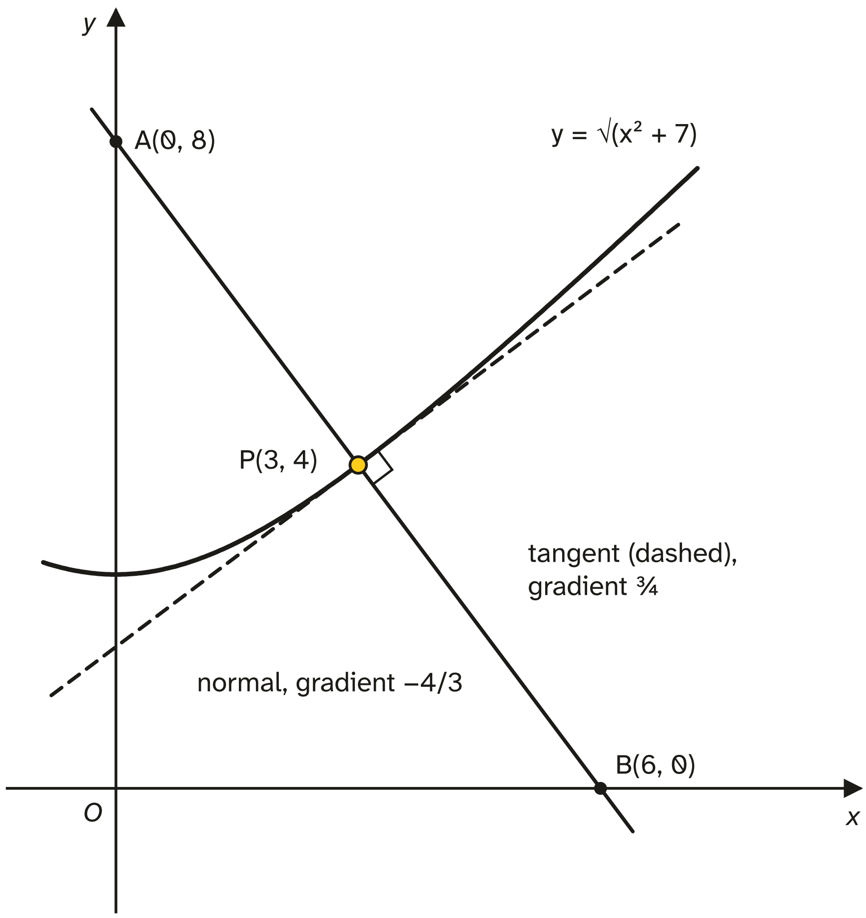
Knowing the stationary points and where the curve crosses the axes is enough to sketch it.

Increasing and decreasing

A function is **increasing** where its gradient is positive (the graph goes up from left to right) and **decreasing** where its gradient is negative. On the cubic above, y increases for $x < -1$ and $x > 1$ and decreases for $-1 < x < 1$. An **increasing function** is one whose gradient is positive for every x in its domain; a **decreasing function** has a negative gradient everywhere.

Tangents and normals

The tangent at a point has the curve's gradient m there. The **normal** is the line through the same point at right angles to the tangent, so its gradient is $-\frac{1}{m}$ (the two gradients multiply to -1).



Rates of change

$\frac{dy}{dx}$ says how fast y changes compared with x. When things change over time t, $\frac{dx}{dt}$ is how fast x changes per second (or per minute). If you know how fast one quantity changes and how two quantities are linked, the chain rule connects them:

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}.$$

For a square of side x cm, $A = x^2$, so $\frac{dA}{dx} = 2x$. If each side grows at 0.3 cm per second, then when $x = 10$ the area grows at $2(10) \times 0.3 = 6$ cm² per second. A quantity that is **decreasing** has a **negative** rate.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, for any rational n	Given
Constant multiples and sums: $\frac{d}{dx}\big(kf(x)\big) = kf'(x)$; differentiate term by term; a constant gives 0	Learn it
Chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	Learn it
$\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$	Learn it
Tangent at (x_1, y_1) with gradient m: $y - y_1 = m(x - x_1)$	Learn it
Normal gradient $= -\frac{1}{m}$	Learn it
Stationary point: $\frac{dy}{dx} = 0$	Learn it
$\frac{d^2y}{dx^2} < 0$: maximum; $\frac{d^2y}{dx^2} > 0$: minimum	Learn it
Increasing: $\frac{dy}{dx} > 0$; decreasing: $\frac{dy}{dx} < 0$	Learn it
Connected rates: $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$, and $\frac{dx}{dt} = \frac{dy}{dt} \div \frac{dy}{dx}$	Learn it
A line making angle θ with the positive x-axis has gradient $\tan\theta$	Learn it
Sphere: $V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$; cone: $V = \frac{1}{3}\pi r^2 h$	Given
Cylinder: $V = \pi r^2 h$, curved surface $2\pi rh$; circle: $A = \pi r^2$, circumference $2\pi r$; cube: $V = x^3$	Learn it

Rewrite before differentiating: $\sqrt{x} = x^{1/2}$, $\frac{1}{x^n} = x^{-n}$, $\frac{k}{\sqrt{x}} = kx^{-1/2}$, $\frac{k}{(ax+b)^n} = k(ax+b)^{-n}$. Multiply out simple products such as $x^2(x-3)$ first.

How to do it

In the Paper 1 questions we have tagged for this topic (78 questions, 2021–2025), the types came up this often. 26 of the 78 questions mix two types, so the counts add up to more than 78.

Question type	Questions
Gradients, tangents and normals	37
Stationary points and their nature	35
Connected rates of change	15
Increasing and decreasing functions	12
Gradient as the limit of chords	4
Practical maximum or minimum problem	1

1. Gradients, tangents and normals (37 questions)

1. **Differentiate.** Rewrite as powers, then use the power rule and the chain rule.
2. **Find the gradient at the point** by substituting its x-coordinate into $\frac{dy}{dx}$. If you only know x, find y from the curve's equation.
3. **Tangent:** $y - y_1 = m(x - x_1)$ with that gradient. **Normal:** use $-\frac{1}{m}$ instead.
4. Put the line in the form the question asks for, such as $y = mx + c$ or $ax + by + c = 0$ with integers.

For example, $y = x^3 - 4x$ at $x = 2$: $y = 0$ and $\frac{dy}{dx} = 3x^2 - 4 = 8$. The tangent is $y = 8(x - 2)$, that is $y = 8x - 16$. The normal is $y = -\frac{1}{8}(x - 2)$, that is $y = -\frac{1}{8}x + \frac{1}{4}$.

Variations you will meet:

- **Find the point where the gradient has a given value.** Set $\frac{dy}{dx}$ equal to that value and solve for x. For a normal with gradient m, the tangent there has gradient $-\frac{1}{m}$.
- **A given line is a tangent**, such as "$y = 7x + k$ is a tangent to the curve". Set $\frac{dy}{dx} = 7$ to find x, find y on the curve, then substitute into the line to get k. If the gradient is a letter m, the same steps give the point in terms of m.
- **Two tangents that meet**: find each tangent's equation, then solve them together. If you are told where they meet (for example the x-coordinate), substitute it to get an equation for the unknown constant.
- **Where the tangent or normal meets something else**: an axis (put $x = 0$ or $y = 0$), another line (solve the two equations together), or the curve again (substitute the line into the curve and solve).
- **Area of a triangle** made by a tangent, a normal and an axis: find where each line crosses the axis and where the two lines meet, then use $\frac{1}{2} \times \text{base} \times \text{height}$.

- **Angles.** A line that makes angle θ with the positive x-axis has gradient $\tan\theta$, so a gradient m gives $\theta = \tan^{-1} m$ (a negative angle when m is negative). To find the angle between two tangents, or a tangent and another line:

  1. find each line's angle with $\tan^{-1}$;
  2. subtract to get the difference;
  3. if the difference is more than 90°, take it away from 180°, because the angle between two lines is the **acute** one.

  For example, tangents with gradients 4 and $-\frac{1}{2}$ make angles $\tan^{-1} 4 = 75.96^\circ$ and $\tan^{-1}\left(-\frac{1}{2}\right) = -26.57^\circ$. The difference is 102.53°, which is more than 90°, so the angle between the tangents is $180^\circ - 102.53^\circ = 77.5^\circ$ (1 d.p.).

- **An unknown constant** k in the curve: the gradient at a known point gives an equation in k.

- **"Rate of change of the gradient"** means $\frac{d^2y}{dx^2}$ at that point.

2. Stationary points and their nature (35 questions)

1. Differentiate and set $\frac{dy}{dx} = 0$.
2. **Solve.** Common forms:
   - a quadratic: factorise or use the formula;
   - negative powers, such as $2x - \frac{16}{x^2} = 0$: multiply through by the power of x to get $2x^3 = 16$, so $x = 2$;
   - an even power, such as $x^4 = 16$: remember **both** $x = 2$ and $x = -2$, unless the domain rules one out;
   - two fractions with squared brackets, such as $\frac{A}{(\dots)^2} = \frac{B}{(\dots)^2}$: cross-multiply, then either expand to a quadratic or take square roots **with** $\pm$.
3. Substitute each x into the **curve's equation** (not into $\frac{dy}{dx}$) to find y.
4. **Determine the nature.** Work out $\frac{d^2y}{dx^2}$, substitute each x, state its value (or sign) and conclude: negative means maximum, positive means minimum. Alternatively check the sign of $\frac{dy}{dx}$ just either side, giving the values you used.

Variations you will meet:

- **Unknown constants from a stationary point.** "The curve has a stationary point at $(2, 12)$" gives two facts: $\frac{dy}{dx} = 0$ when $x = 2$, and the point is on the curve. For $y = ax^2 + \frac{b}{x}$: $\frac{dy}{dx} = 2ax - \frac{b}{x^2}$, so $4a - \frac{b}{4} = 0$, and $4a + \frac{b}{2} = 12$. Solving gives $a = 1, b = 16$.
- **Stationary point in terms of k.** Solve $\frac{dy}{dx} = 0$ as usual, treating k as a number.
- **"The curve has a minimum (or maximum) at $x = a$; find the possible values of k."** Find $\frac{d^2y}{dx^2}$, substitute $x = a$, and solve the inequality: > 0 for a minimum, < 0 for a maximum. For example, if $\frac{dy}{dx} = x^3 - kx + (k-1)$, then $x = 1$ is always stationary, and $\frac{d^2y}{dx^2} = 3x^2 - k = 3 - k$ there. It is a minimum when $3 - k > 0$, that is $k < 3$.

- **"Show that there are no other stationary points."** Solve $\frac{dy}{dx} = 0$ completely and show that it has only the root you already know. For example, $\frac{dy}{dx} = 16 - \frac{2}{x^3} = 0$ gives $x^3 = \frac{1}{8}$, and a number has only one real cube root, so $x = \frac{1}{2}$ is the only stationary point. If you get a quadratic factor, show its discriminant is negative: $x^3 - x^2 + x - 6 = (x-2)(x^2 + x + 3)$, and $x^2 + x + 3$ has $b^2 - 4ac = -11 < 0$, so $x = 2$ is the only root. Finish with a sentence such as "so this is the only stationary point".
- **Only $\frac{dy}{dx}$ is given** (the curve itself comes from integration). You can still find the x-values of the stationary points and their nature from $\frac{dy}{dx}$ alone.
- **"Verify that $x = \ldots$ is a stationary point"**: substitute it into $\frac{dy}{dx}$ and show you get 0.
- **Sketching or the range.** A minimum point is the lowest point of that part of the graph, so its y-value can be the least value of a function.

3. Connected rates of change (15 questions)

1. **Name the rates.** Write down the rate you are given and the rate you want, with t for time, for example $\frac{dV}{dt} = 50$ and $\frac{dr}{dt} = ?$. A decreasing quantity has a negative rate.
2. **Link them** with the chain rule, for example $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$. The middle derivative comes from the formula that connects the two quantities (the curve's equation, or a volume or area formula).
3. **Differentiate** that formula and substitute the value at the instant asked about.
4. **Solve** for the unknown rate and give units if the question has them.

A point moving along a curve. For $y = x^3 - 2x$, if x increases at 0.5 units per second, then at $x = 2$: $\frac{dy}{dx} = 10$, so $\frac{dy}{dt} = 10 \times 0.5 = 5$ units per second. If instead y decreases at 3 units per second when $x = 1$: $\frac{dy}{dx} = 1$, so $\frac{dx}{dt} = \frac{-3}{1} = -3$, and x is decreasing at 3 units per second.

Variations you will meet:

- **"x and y change at the same rate"** means $\frac{dy}{dt} = \frac{dx}{dt}$, so $\frac{dy}{dx} = 1$. Solve that for x.
- **Working backwards.** You are given the rate you would normally find, such as $\frac{dh}{dt}$ at some instant. Use it to find $\frac{dV}{dh}$, solve for h, then find what the question asks (for example V).
- **Moving along a straight line** (for example, a point that leaves the curve and moves along the normal): $\frac{dy}{dx}$ is simply the line's gradient, so $\frac{dy}{dt} = \text{gradient} \times \frac{dx}{dt}$.
- **Units.** Convert at the end if asked, for example m per hour to cm per minute: multiply by 100 and divide by 60.
- **Both rates given, point unknown.** Divide them to get the gradient: if y increases twice as fast as x, then $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = 2$. Set $\frac{dy}{dx} = 2$ and solve for x.

4. Increasing and decreasing functions (12 questions)

1. Find $f'(x)$.

2. **"Find the set of values of x for which y decreases (or f is decreasing)"**: solve the inequality $f'(x) < 0$ (or $f'(x) > 0$ for increasing). For a quadratic $f'(x)$, find its roots and use a sketch of the parabola to pick the right region, for example $-3 < x < 2$.

3. **"Determine whether f is increasing, decreasing or neither"**: show that $f'(x)$ has the same sign for **every** x in the domain, and say why:
   - complete the square, for example $f'(x) = 3x^2 + 6x + 5 = 3(x+1)^2 + 2$, which is always at least 2, so f is increasing;
   - or look at the terms: a positive number over a squared bracket is always positive, and a sum of positive terms is positive;
   - or use the discriminant: if $f'(x)$ is a quadratic with a positive x^2 coefficient and $b^2 - 4ac < 0$, it is always positive.

   Substituting a few values of x is **not** enough on its own. If $f'(x)$ changes sign in the domain, the answer is "neither".

   - If you are told what $f'(x) = 0$ simplifies to, show that equation has **no real roots** (discriminant < 0). Then $f'(x)$ is never 0, so it cannot change sign: work out $f'(x)$ at one value in the domain to find which sign it has, and conclude. For example, $f'(x) = 0$ leading to $x^2 - 4x + 7 = 0$ (discriminant -12) and $f'(1) > 0$ means f is increasing.
4. **"The function is decreasing for $x < a$; find the largest a"**: the gradient changes sign at a stationary point, so solve $f'(x) = 0$ and pick the root that matches the domain.

5. Gradient as the limit of chords (4 questions)

With a small h:

1. Write the two points: P at $x = a$ and Q at $x = a + h$, with y-values from the curve.
2. Gradient of $PQ = \dfrac{y_Q - y_P}{(a+h) - a}$. Expand, simplify and cancel the h in the denominator.
3. As $h \to 0$, the chord gradient tends to the gradient of the curve at P. State this in words and give the value.

With a table of values: find any missing y-value by substituting into $f(x)$ and give it to the number of decimal places asked. Find each chord's gradient with $\frac{y_2 - y_1}{x_2 - x_1}$, again to the stated accuracy. Then say what the gradients are tending to as the second point gets closer to the first: that value is $f'(a)$. Mention **chords** and the **limit**; don't talk about stationary points or increasing functions.

6. Practical maximum or minimum problem (1 question)

1. **Write the quantity to be made largest or smallest** (volume, area, cost) using the shape's formulas. It usually has two variables. Count the faces carefully: an open cylinder (no top) has surface area $\pi r^2 + 2\pi r h$; an open box with a square base has $x^2 + 4xh$.
2. **Use the fixed fact** (a given surface area, volume or length) to write one variable in terms of the other, and substitute. This is often a "show that", so show each step.
3. **Differentiate**, set the derivative to 0 and solve. Reject values that make no sense (negative lengths).
4. **Show it is a maximum or minimum** with the second derivative if asked, then substitute back to find the largest or smallest value, exactly if the question says "exact".

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

A curve has equation $y = \sqrt{x^2 + 7}$.

(a) Find the equation of the tangent to the curve at the point where $x = 3$. Give your answer in the form $ax + by + c = 0$. **[4]**

(b) The normal to the curve at this point meets the y-axis at A and the x-axis at B. Find the area of triangle OAB, where O is the origin. **[4]**

(a) $y = (x^2 + 7)^{1/2}$, so

$$\frac{dy}{dx} = \frac{1}{2}(x^2 + 7)^{-1/2} \times 2x = \frac{x}{\sqrt{x^2 + 7}}.$$

B1 for $\frac{1}{2}(x^2 + 7)^{-1/2}$, **B1** for $\times 2x$ (the chain rule).

At $x = 3$: $y = \sqrt{16} = 4$ and $\frac{dy}{dx} = \frac{3}{4}$. **M1** for substituting $x = 3$ into the derivative and forming the line:

$$y - 4 = \frac{3}{4}(x - 3) \;\Rightarrow\; 4y - 16 = 3x - 9 \;\Rightarrow\; 3x - 4y + 7 = 0.$$

A1

(b) Normal gradient $= -1 \div \frac{3}{4} = -\frac{4}{3}$. **M1** for using $-\frac{1}{m}$.

$$y - 4 = -\frac{4}{3}(x - 3) \;\Rightarrow\; y = -\frac{4}{3}x + 8.$$

A1

At $x = 0$: $y = 8$, so $A(0, 8)$. At $y = 0$: $x = 6$, so $B(6, 0)$. **M1** for both intercepts.

Area of $OAB = \frac{1}{2} \times 6 \times 8 = 24$. **A1**

The third diagram in "Understand it" shows this curve, tangent and normal.

Example 2

A curve has equation $y = 2x^3 + 3x^2 - 36x + 10$.

(a) Find the coordinates of the stationary points of the curve. **[4]**

(b) Determine the nature of each stationary point. **[3]**

(c) Find the set of values of x for which y decreases as x increases. **[2]**

(a) $\dfrac{dy}{dx} = 6x^2 + 6x - 36$. **B1**

$6x^2 + 6x - 36 = 0 \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x+3)(x-2) = 0$, so $x = -3$ or $x = 2$. **M1** for setting the derivative to 0 and solving.

At $x = -3$: $y = -54 + 27 + 108 + 10 = 91$. At $x = 2$: $y = 16 + 12 - 72 + 10 = -34$. **A1 A1**

The stationary points are $(-3, 91)$ and $(2, -34)$.

(b) $\dfrac{d^2y}{dx^2} = 12x + 6$. **B1**

At $x = -3$: $\frac{d^2y}{dx^2} = -30 < 0$, so $(-3, 91)$ is a **maximum**. At $x = 2$: $\frac{d^2y}{dx^2} = 30 > 0$, so $(2, -34)$ is a **minimum**. **M1** for substituting both x-values, **A1** for both conclusions with the signs stated.

(c) y decreases where $\frac{dy}{dx} < 0$: $6(x+3)(x-2) < 0$. The parabola is below the axis between its roots, so $-3 < x < 2$. **M1 A1**

Example 3

A spherical balloon is being inflated so that its volume increases at a constant rate of 50 cm³ per second.

(a) Find the rate at which the radius is increasing when the radius is 5 cm. **[3]**

(b) Find the rate at which the surface area is increasing at the same instant. **[3]**

(a) $V = \frac{4}{3}\pi r^3$ (given in MF19), so $\dfrac{dV}{dr} = 4\pi r^2 = 100\pi$ when $r = 5$. **B1**

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} \quad \Rightarrow \quad 50 = 100\pi \times \frac{dr}{dt}.$$

M1 for the chain rule with the right rates in the right places.

$$\frac{dr}{dt} = \frac{1}{2\pi} = 0.159 \text{ cm per second (3 s.f.)}$$

A1

(b) $A = 4\pi r^2$, so $\dfrac{dA}{dr} = 8\pi r = 40\pi$ when $r = 5$. **B1**

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 40\pi \times \frac{1}{2\pi} = 20 \text{ cm}^2 \text{ per second.}$$

M1 A1. Keeping $\frac{1}{2\pi}$ exact makes the π cancel; rounding 0.159 first would give $19.98\ldots$

Example 4

An open box (it has no lid) has a square base of side x cm and height h cm. Its volume is 256 cm³.

(a) Show that the external surface area, S cm², of the box is given by $S = x^2 + \dfrac{1024}{x}$. **[3]**

(b) Find the value of x for which S is stationary, and find this value of S. **[4]**

(c) Determine whether this value of S is a maximum or a minimum. **[2]**

(a) Volume: $x^2h = 256$, so $h = \dfrac{256}{x^2}$. **B1**

The box has a base (x^2) and four sides, each xh, but no lid: $S = x^2 + 4xh$. **M1** for substituting h:

$$S = x^2 + 4x \times \frac{256}{x^2} = x^2 + \frac{1024}{x}.$$

A1, the given answer reached with every step shown.

(b) $S = x^2 + 1024x^{-1}$, so $\dfrac{dS}{dx} = 2x - 1024x^{-2}$. **B1**

$2x - \dfrac{1024}{x^2} = 0 \Rightarrow 2x^3 = 1024 \Rightarrow x^3 = 512 \Rightarrow x = 8$. **M1 A1**

$S = 64 + \dfrac{1024}{8} = 192$. **A1**

(c) $\dfrac{d^2S}{dx^2} = 2 + 2048x^{-3} = 2 + 4 = 6 > 0$ at $x = 8$, so $S = 192$ is a **minimum**. **M1 A1**

(The box that uses the least material is 8 cm by 8 cm by 4 cm.)

Common mistakes

- **Leaving out the chain rule.** $(5x-2)^4$ differentiates to $4(5x-2)^3 \times 5$, not $4(5x-2)^3$. Examiners see this every year, and the same error then spoils the second derivative too.
- **Getting negative and fractional powers wrong.** $\frac{d}{dx}\, x^{-2} = -2x^{-3}$ (the power goes down to -3, not up to -1), and $\frac{d}{dx}\, x^{1/2} = \frac{1}{2}x^{-1/2}$. Rewrite every term as a power before you start.
- **Using the tangent gradient for a normal, or the other way round.** Read the question again before writing the line. The normal needs $-\frac{1}{m}$.
- **Substituting into the wrong thing.** The y-coordinate comes from the curve's equation; the gradient comes from $\frac{dy}{dx}$. Putting x into $\frac{dy}{dx}$ and calling it y is a common slip.
- **Losing a root.** $x^4 = \frac{81}{16}$ gives $x = \pm\frac{3}{2}$; $(\ldots)^2 = (\ldots)^2$ needs $\pm$ when you square-root. Check the domain before throwing one away.
- **Stating the nature without evidence.** "It's a minimum" earns nothing on its own. Give the value (or sign) of $\frac{d^2y}{dx^2}$ at that point, or the gradients either side with the x-values used.
- **Not recognising a rates question.** Any "rate at which ... is increasing" needs differentiation and the chain rule, even when the question never says "differentiate".

- **Putting the rates the wrong way round**, for example $\frac{dx}{dt} = \frac{dy}{dx} \times \frac{dy}{dt}$. Write the chain so that the letters "cancel": $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.
- **Ignoring the sign of a decreasing rate.** "Decreasing at 5 units per second" means $\frac{dy}{dt} = -5$.
- **Proving "increasing" by trying numbers.** Showing $f'(1) > 0$ and $f'(5) > 0$ proves nothing about other values. Give a reason that covers every x: a completed square, a squared bracket, or the discriminant.
- **Rounding too early.** Keep exact values (surds, π, fractions) until the last line.

How to get the marks

- **Show the derivative.** Each correct term usually earns a mark on its own, and the method marks that follow need to see your $\frac{dy}{dx}$. Write it out before substituting.
- **Tangents and normals:** write the gradient you used, then $y - y_1 = m(x - x_1)$ with the numbers in. Then give the form asked for; "integers" means clear every fraction.
- **Stationary points:** "find the stationary points" means **both** coordinates. "Determine the nature" means a calculation and a conclusion, for each point.
- **Rates:** write the chain rule with the actual rates, for example $50 = 100\pi \times \frac{dr}{dt}$. The method mark is for linking the right rates correctly.
- **"Show that"** means the answer is given. Every step must be visible, and your last line must match the printed answer exactly.
- **"Hence"** means use the part before, for example the r you just found gives the maximum volume.
- **"Exact"** means surds, fractions and π, no decimals. Otherwise give non-exact answers to 3 significant figures unless the question asks for something else, for example "correct to 4 decimal places" in a chord question.
- **"State"** (for example what chord gradients suggest) needs just the answer, but it must be the right idea: the value the gradients approach.
- **"Determine whether ... increasing, decreasing or neither"** needs a reason that works for every x in the domain, then a clear conclusion.
- **Method marks follow through.** A wrong gradient used correctly in a line equation still earns the line mark, so keep going.

Quick check

1. The points $P(1, 3)$ and $Q(1 + h,\, y_Q)$ lie on the curve $y = 2x^2 + x$. Find the gradient of the chord PQ in terms of h, and deduce the gradient of the curve at P.
2. Find $\dfrac{dy}{dx}$ when $y = \dfrac{6}{\sqrt{5x + 4}}$.
3. Show that $f(x) = x^5 + 4x^3 + x$ is an increasing function for all real x.
4. Each edge of a cube is increasing at 0.2 cm per second. Find the rate at which the total surface area of the cube is increasing when the edges are 5 cm long.

5. The line $y = 7x + k$ is a tangent to the curve $y = x^3 - 5x + 1$ at a point where $x > 0$. Find the value of k.

Answers

1. $y_Q = 2(1+h)^2 + (1+h) = 3 + 5h + 2h^2$. Gradient $= \dfrac{5h+2h^2}{h} = 5 + 2h$. As $h \to 0$ this tends to 5, so the gradient at P is 5.
2. $y = 6(5x+4)^{-1/2}$, so $\dfrac{dy}{dx} = 6 \times \left(-\frac{1}{2}\right)(5x+4)^{-3/2} \times 5 = -15(5x+4)^{-3/2}$.
3. $f'(x) = 5x^4 + 12x^2 + 1$. Since $x^4 \geqslant 0$ and $x^2 \geqslant 0$, $f'(x) \geqslant 1 > 0$ for every x, so f is increasing.
4. $A = 6x^2$, so $\dfrac{dA}{dx} = 12x = 60$ when $x = 5$. $\dfrac{dA}{dt} = \dfrac{dA}{dx} \times \dfrac{dx}{dt} = 60 \times 0.2 = 12$ cm² per second.
5. $\dfrac{dy}{dx} = 3x^2 - 5 = 7$, so $x^2 = 4$ and $x = 2$ (as $x > 0$). Then $y = 8 - 10 + 1 = -1$, and $-1 = 14 + k$, so $k = -15$.

Past questions to try

- **9709/11/M/J/24 Q4**: gradients of chords from a table of values.
- **9709/12/F/M/24 Q5**: the equation of a normal, with the chain rule.
- **9709/13/O/N/24 Q11**: a stationary point, then two tangents that meet.
- **9709/11/M/J/23 Q9**: rates of change for water filling a tank.