Cambridge AS & A Level · Mathematics (9709) · Notes

Differentiation AS

Read online at pastopic.com/cambridge/mathematics-9709/notes/differentiation-p2

Last checked 28 September 2026

Read first: Differentiation, Logarithmic and exponential functions, Trigonometry

What you need to know

By the end of this topic you should be able to:

1. **Differentiate e^x, $\ln x$, $\sin x$, $\cos x$ and $\tan x$**, together with constant multiples, sums and differences of them, and functions built from them with the chain rule, such as e^{5x-1}, $\ln(x^2+3)$, $\cos 4x$ or $\tan^2 x$. Angles are in **radians**.
2. **Differentiate products and quotients** with the product rule and the quotient rule, for example $x^3 \sin x$ or $\dfrac{e^x}{x+4}$.
3. **Find and use $\frac{dy}{dx}$ for a curve given parametrically** (both x and y in terms of a third letter, such as t or θ) **or implicitly** (an equation mixing x and y that you don't rearrange into $y = \ldots$). Use it for gradients, tangents and normals.

Everything from the Paper 1 Differentiation note still applies: gradients, tangents and normals, stationary points and their nature. In Paper 2 you use those same methods with the new functions. Second derivatives of parametric or implicit curves are **not** needed.

Understand it

A reminder from Paper 1

$\frac{dy}{dx}$ is the gradient of the curve. Powers of x differentiate as $x^n \to nx^{n-1}$, and a function inside a function uses the **chain rule**: differentiate the outside, then multiply by the derivative of the inside,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

This topic adds five new functions and three new rules. The chain rule is used in almost every question.

The exponential function

$e \approx 2.718$ is the one number for which the graph of $y = e^x$ has gradient equal to its height at every point. So

$$\frac{d}{dx}\left(e^x\right) = e^x.$$

With the chain rule, $e^{\text{something}}$ differentiates to **itself times the derivative of the something**:

- $e^{4x} \to 4e^{4x}$
- $e^{1-3x} \to -3e^{1-3x}$
- $e^{x^2+1} \to 2x\,e^{x^2+1}$

e^x is always positive, so $e^x = 0$ has no solutions. You will use this fact again and again when solving $\frac{dy}{dx} = 0$.

The natural logarithm

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

With the chain rule, $\ln(\text{something})$ differentiates to **the derivative of the something, over the something**:

$$\frac{d}{dx}\ln\big(f(x)\big) = \frac{f'(x)}{f(x)}.$$

- $\ln(5x-2) \to \dfrac{5}{5x-2}$ (the 5 on top is easy to forget)
- $\ln(x^2+3) \to \dfrac{2x}{x^2+3}$
- $\ln 7x \to \dfrac{7}{7x} = \dfrac{1}{x}$, which makes sense because $\ln 7x = \ln 7 + \ln x$, and $\ln 7$ is a constant.

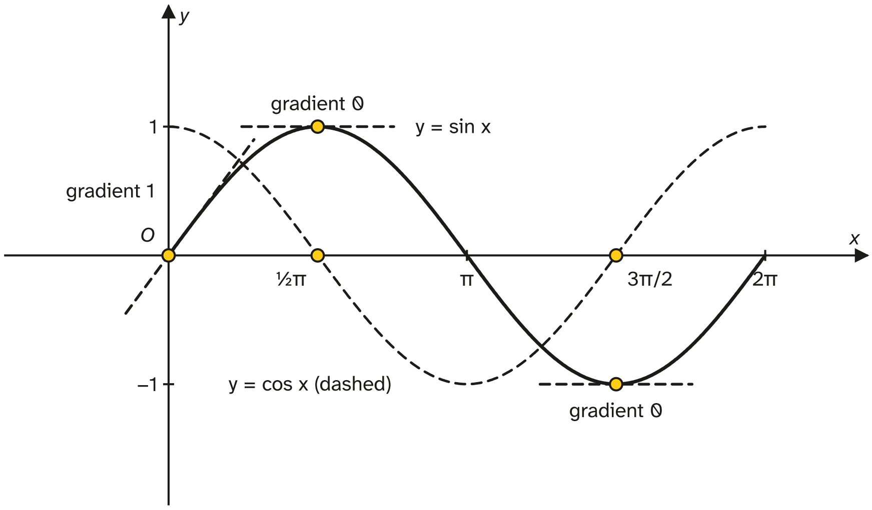
Use the laws of logarithms first when they make things simpler: $\ln\sqrt{x+1} = \frac{1}{2}\ln(x+1) \to \dfrac{1}{2(x+1)}$. But don't confuse $(\ln x)^2$ with $\ln x^2$. $\ln x^2 = 2\ln x$, which gives $\frac{2}{x}$. $(\ln x)^2$ is a bracket squared, so the chain rule gives $2\ln x \times \frac{1}{x} = \frac{2\ln x}{x}$.

Sine, cosine and tangent

In **radians**,

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x, \qquad \frac{d}{dx}(\tan x) = \sec^2 x.$$

The picture shows why the first one is true. Wherever the sine curve is flat, at its tops and bottoms, $\cos x$ is 0; where sine climbs steepest (through the origin, gradient 1), $\cos x = 1$.



These results only work in radians, so set your calculator to radians for every calculus question with trigonometry.

With the chain rule:

- $\sin 3x \to 3\cos 3x$
- $\cos(2x+1) \to -2\sin(2x+1)$
- $\tan 4x \to 4\sec^2 4x$
- $\sin^2 x = (\sin x)^2 \to 2\sin x\cos x$, which is $\sin 2x$
- $\cos^3 2x = (\cos 2x)^3 \to 3\cos^2 2x \times (-2\sin 2x) = -6\cos^2 2x \sin 2x$

Remember $\sec^2 x = \dfrac{1}{\cos^2 x}$ and $\sec^2 x = 1 + \tan^2 x$. You often need one of these to simplify an answer or to solve $\frac{dy}{dx} = 0$.

The product rule

When $y = uv$ is two functions of x multiplied together,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

In words: **first times the derivative of the second, plus second times the derivative of the first.** For $y = x^3 \sin x$, take $u = x^3$ and $v = \sin x$:

$$\frac{dy}{dx} = x^3\cos x + 3x^2\sin x.$$

Look out for products hidden inside a question, such as $t\ln t$, $e^{-x}\cos 2x$, or the term x^2y in an implicit equation.

The quotient rule

When $y = \dfrac{u}{v}$,

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

The order matters because of the minus sign: **bottom times derivative of top, minus top times derivative of bottom, all over bottom squared.** For $y = \dfrac{x+1}{x-2}$:

$$\frac{dy}{dx} = \frac{(x-2)(1) - (x+1)(1)}{(x-2)^2} = \frac{-3}{(x-2)^2}.$$

Put brackets round every factor before you expand. Examiners see many sign errors from missing brackets here. You can also write $\frac{u}{v}$ as uv^{-1} and use the product rule; either method earns the marks.

A fraction is zero only when its **numerator** is zero. So for stationary points of a quotient, set the numerator of $\frac{dy}{dx}$ equal to 0 and ignore the denominator.

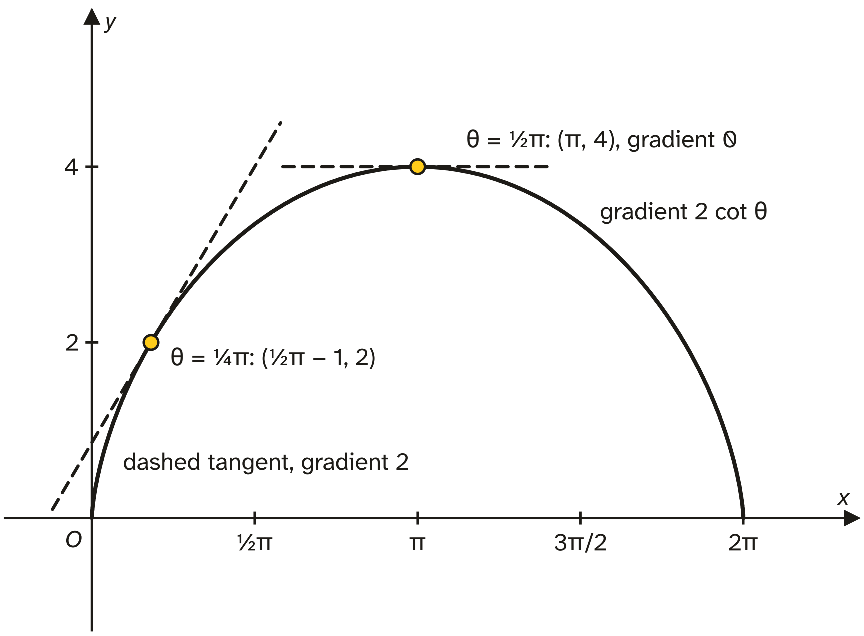
Parametric equations

Sometimes a curve is described by giving x and y separately in terms of a third variable, a **parameter**, such as t or θ. Each value of t gives one point (x, y). For example, $x = 3t^2$, $y = 2t^3$ gives the point $(3, 2)$ when $t = 1$ and $(12, 16)$ when $t = 2$.

To find the gradient, differentiate each one with respect to the parameter and divide:

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}.$$

This is the chain rule again, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$. Here $\frac{dx}{dt} = 6t$ and $\frac{dy}{dt} = 6t^2$, so $\frac{dy}{dx} = \frac{6t^2}{6t} = t$. The answer is in terms of t, so to find the gradient at a point you need **the value of t at that point**, not its x or y.



The tangent is **horizontal** where $\frac{dy}{dt} = 0$ (and $\frac{dx}{dt} \neq 0$). It is **vertical** where $\frac{dx}{dt} = 0$.

Implicit equations

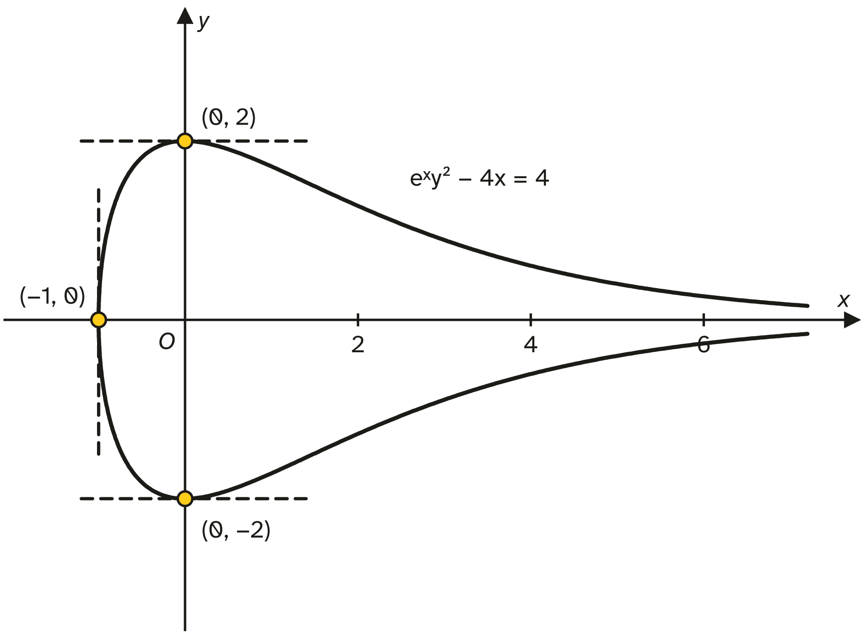
An equation such as $x^3 + y^3 = 9$ describes a curve, but it is awkward or impossible to rearrange into $y = \dots$. Instead, differentiate **every term** with respect to x, remembering that y is a function of x:

- a term in x only differentiates as usual: $x^3 \to 3x^2$;
- a term in y only uses the chain rule, so it gets a $\frac{dy}{dx}$ on the end: $y^3 \to 3y^2\frac{dy}{dx}$, $e^{2y} \to 2e^{2y}\frac{dy}{dx}$, $\ln y \to \frac{1}{y}\frac{dy}{dx}$, $\sin y \to \cos y\frac{dy}{dx}$;
- a term with both x and y multiplied is a **product**: $x^2y \to 2xy + x^2\frac{dy}{dx}$;
- a constant differentiates to 0, including a constant on the right-hand side.

Then collect the $\frac{dy}{dx}$ terms on one side and factorise. For $x^3 + y^3 = 9$:

$$3x^2 + 3y^2\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x^2}{y^2}.$$

At the point $(1, 2)$ the gradient is $-\frac{1}{4}$. The answer contains both x and y, so you need both coordinates of the point.



When $\frac{dy}{dx}$ is a fraction $\frac{N}{D}$:

- the tangent is **horizontal** (a stationary point) where the numerator $N = 0$;
- the tangent is **vertical** (parallel to the y-axis) where the denominator $D = 0$.

Either condition gives a link between x and y. Substitute that link into the curve's equation to find the actual points (Example 4 does this for the curve above).

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\frac{d}{dx}\left(e^x\right) = e^x, \quad \frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$	Given
$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x, \quad \frac{d}{dx}(\tan x) = \sec^2 x$ (radians)	Given
Product rule: $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$	Given
Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	Given
Parametric: $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$	Given
Chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	Learn it
$\frac{d}{dx}\, e^{f(x)} = f'(x)\, e^{f(x)}, \quad \frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$	Learn it
$\frac{d}{dx}\, \sin(ax+b) = a\cos(ax+b)$, and the same pattern for cos and tan	Learn it
Implicit: $\frac{d}{dx}\left(f(y)\right) = f'(y)\frac{dy}{dx}$, for example $\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$	Learn it
$\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$	Learn it
Tangent: $y - y_1 = m(x - x_1)$; normal gradient $-\frac{1}{m}$	Learn it
Stationary point: $\frac{dy}{dx} = 0$ (for a fraction, its numerator $= 0$)	Learn it
Tangent parallel to the y-axis: the denominator of $\frac{dy}{dx}$ is 0 ($\frac{dx}{dt} = 0$ for parametric curves)	Learn it
$\sin^2 x + \cos^2 x = 1,\ \sec^2 x = 1 + \tan^2 x,\ \sin 2x = 2\sin x\cos x,$ $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$	Given
$\sec x = \frac{1}{\cos x}$	Learn it
$e^x > 0$ for every x; $\ln e = 1$; $\ln 1 = 0$; $e^{\ln k} = k$	Learn it

How to do it

In the Paper 2 questions we have tagged for this topic (79 questions, 2021–2025; many appear in two or three versions of the same exam, so there are 54 different ones), the types came up this often. 7 of the 79 questions mix two types, so the counts add up to more than 79. 28 of the 79 go on into another topic: an iteration from Numerical solution of equations, an area or integral from Integration, or the factor theorem from Algebra.

Question type	Questions
Stationary points of $y = f(x)$	27
Parametric equations	21
Gradients, tangents and normals of $y = f(x)$	17
Implicit equations	16
Find or show a derivative	5

1. Stationary points of $y = f(x)$ (27 questions)

1. **Differentiate**, choosing the rule: product, quotient or chain rule, plus the five standard derivatives.
2. **Set $\frac{dy}{dx} = 0$ and simplify.**
    - For a quotient, set the **numerator** to 0.
    - Take out any common factor that can never be 0, such as e^{-x}, e^{2x} or $\frac{1}{x}$, and **say** it can't be zero: "$e^{-x} > 0$, so $x^2 - 4x + 3 = 0$."
    - With a square root or negative power, multiply through to clear it. For example, $y = e^{-x}\sqrt{2x+5}$ gives $\frac{dy}{dx} = -e^{-x}(2x+5)^{1/2} + e^{-x}(2x+5)^{-1/2} = 0$. Divide by e^{-x} and multiply by $(2x+5)^{1/2}$: $-(2x+5)+1=0$, so $x=-2$.
3. **Solve.** Common forms:
    - $e^{kx} = c$: take logs, $x = \frac{1}{k}\ln c$ (exact);
    - a quadratic in e^x or in $\ln x$: let $u = e^x$ (or $u = \ln x$), solve, and reject $e^x \leqslant 0$;
    - trigonometric: use an identity to get one ratio. For example $\sin 2x - \sin x = 0$ becomes $\sin x(2\cos x - 1) = 0$; or $a + b\sec^2 kx = 0$ gives $\cos^2 kx$. Give angles in radians and find **every** solution in the interval.
4. **Find** y by substituting each x into the curve's equation. Give exact values if asked ("exact" means e, $\ln$, surds and π stay in).
5. **Nature, if asked:** the sign of $\frac{d^2y}{dx^2}$, or the sign of $\frac{dy}{dx}$ either side, exactly as in Paper 1.

Variations you will meet:

- **"Show that the x-coordinate satisfies $x = \ldots$"**, followed by an iteration. Set $\frac{dy}{dx} = 0$, clear fractions, and rearrange step by step until you reach the printed equation. For example, $\frac{d}{dx}\big(x\ln(2x+1)\big) - 1 = \ln(2x+1) + \frac{2x}{2x+1} - 1 = 0$ rearranges to $\ln(2x+1) = \frac{1}{2x+1}$. The iteration itself belongs to the Numerical solution topic.
- **"Show by calculation that the stationary point lies between a and b"**: work out $\frac{dy}{dx}$ (or its numerator) at a and at b and show the **sign changes**. Values of y itself prove nothing here.
- **"Show that the curve has only one stationary point."** Solve completely and explain why the other factors give nothing: $e^x \neq 0$, $e^x + 4 > 0$, a quadratic with negative discriminant, or a root outside the domain.

- **Maximum or minimum points marked on a diagram**, sometimes with an area to find afterwards by integration. Find both coordinates, and check which root matches the point on the diagram.
- **Several points in a range such as $0 \leqslant x \leqslant 2\pi$**: list them all, and no extra ones.

2. Parametric equations (21 questions)

1. Differentiate: find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ separately. Each may need the product, quotient or chain rule.
2. **Divide:** $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$, the y derivative on top. Simplify, clearing any fraction inside a fraction.
3. **Find the value of** t at the point you want. Then substitute it into $\frac{dy}{dx}$, and into x and y if you need the coordinates.
4. Finish as the question asks: a gradient, a tangent or a normal.

Variations you will meet:

- **"Show that $\frac{dy}{dx} = \ldots$"** in a given form. Expect to use identities: $\sec^2\theta = \frac{1}{\cos^2\theta}$, $\sin^2\theta = 1 - \cos^2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = 1 - 2\sin^2\theta$. Example 3 shows one.
- **The point is described, not given by t.** "Where the curve crosses the y-axis": solve $x = 0$ for t. "Crosses the x-axis": solve $y = 0$ (often it factorises, such as $\sin\theta(2\cos\theta - 1) = 0$). "The point where $x = \ln 4$": use log laws, such as $\ln(t+5) - \ln t = \ln 4 \Rightarrow \frac{t+5}{t} = 4$. Use the given range of t to reject values.
- **A point given by its coordinates**, such as $(a, \ln 100)$: use the coordinate you know to find t first.
- **Stationary points:** solve $\frac{dy}{dt} = 0$ (for example, $\tan 2t = k$ from $\sin$ and $\cos$ terms), then find x and y.
- **The gradient equals a given value**, such as $\frac{dy}{dx} = 2$: form the equation in t and solve it. It may become a polynomial to factorise with the factor theorem (then show the other factor has no real roots, so that point is the only one), or an equation to solve by iteration.
- **An unknown constant** such as $x = k\tan t$: the gradient of the normal (or tangent) at a known point gives an equation for k.
- **A line meets the curve:** substitute $x(t)$ and $y(t)$ into the line's equation and solve for t.
- **Increasing or decreasing:** if $\frac{dy}{dx}$ is positive for every allowed t (for example a square times a positive constant), y increases as x increases, so the curve is an increasing function.

3. Gradients, tangents and normals of $y = f(x)$ (17 questions)

1. Differentiate with the right rule.
2. Find the x-coordinate of the point. If the point is described (on an axis, or marked on a diagram), solve for it first.
3. Substitute into $\frac{dy}{dx}$ for the gradient m. For a normal use $-\frac{1}{m}$.
4. Use $y - y_1 = m(x - x_1)$ and put the line in the form asked for.

Variations you will meet:

- **Where the curve crosses an axis.** $x = 0$ is easy. For $y = 0$ you must solve: $\ln x = 0$ gives $x = 1$; $(\ln x)^2 - 3\ln x = 0$ gives $\ln x = 0$ or 3, so $x = 1$ or e^3; $3e^{-x} - e^x = 0$ gives $e^{2x} = 3$, so $x = \frac{1}{2}\ln 3$. Don't divide by $\ln x$, or you lose the root $x = 1$.

- **Exact gradients** such as $-3e^{-4}$ or $1+\frac{\pi}{2}$: keep e, $\ln$ and π, and use $\ln e = 1$, $e^{\ln 2} = 2$.
- **The gradient has a given value**, such as "the gradient at P is 6": set $\frac{dy}{dx} = 6$ and rearrange, often into an equation for iteration.
- **Form of the answer:** "$ax + by + c = 0$ where a, b, c are integers" means clear all fractions; "$y = mx + c$ where m and c are exact" means keep surds or e.

4. Implicit equations (16 questions)

1. **Differentiate every term** with respect to x. Terms in y get $\times\frac{dy}{dx}$; products such as $x \ln y$ or $e^{2x}y$ use the product rule; constants (on either side) become 0.
2. **Collect** the $\frac{dy}{dx}$ terms on one side, factorise, and divide.
3. **Substitute** both coordinates of the point. If you know only one coordinate, find the other from the curve's equation first.
4. Finish: gradient, tangent or normal.

For example, $x^2 + 4xy - y^2 = 11$ at $(2, 1)$:

$$2x + 4y + 4x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0 \ \Rightarrow\ \frac{dy}{dx} = \frac{2x+4y}{2y-4x} = \frac{8}{-6} = -\frac{4}{3}.$$

Variations you will meet:

- **"Find the gradient at the point where $y = 0$"**: substitute $y = 0$ into the curve's equation to get x, then both into $\frac{dy}{dx}$.
- **Tangent or normal at a point**, in a stated form, sometimes with exact values such as e^2.
- **Stationary points:** set the **numerator** of $\frac{dy}{dx}$ to 0. This gives a link such as $y = 3x$ or $e^x = 4$; substitute it into the curve's equation to find the points.
- **"Show that the curve has no stationary points"**: do the same, and show the result is impossible, for example a negative number equal to a positive one, or $e^{2x} = 0$.
- **Tangent parallel to the y-axis:** set the **denominator** to 0 and substitute into the curve's equation. If the denominator is a sum of things that can't be negative, such as $3x^2 + 2e^y$, it is never 0, so there is no such tangent.
- **"Show that there is no point on the curve with $y = k$"**: this needs no calculus. Substitute $y = k$ into the curve's equation and show the equation in x has no real roots (a negative discriminant, or a completed square).

5. Find or show a derivative (5 questions)

1. Rewrite if it helps: $\tan^2 \frac{1}{2}x = \left(\tan \frac{1}{2}x\right)^2$, $\sin^2 3x = (\sin 3x)^2$.
2. Differentiate with the chain rule (and the product rule if needed), then substitute the value asked for.
3. **"Show that"**: rearrange with identities to the given form, showing each step. For example, $\frac{d}{dx}\left(\tan^3 x\right) = 3\tan^2 x \sec^2 x = 3\tan^2 x(1+\tan^2 x) = 3\tan^2 x + 3\tan^4 x$.

These are usually part (a) of an integration question: the result tells you what to integrate in part (b).

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

A curve has equation $y = \dfrac{3x}{x^2+5}$.

(a) Find the equation of the normal to the curve at the point where $x = 1$. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers. **[5]**

(b) Find the exact coordinates of the stationary points of the curve. **[3]**

(a) Quotient rule with $u = 3x$ and $v = x^2 + 5$:

$$\frac{dy}{dx} = \frac{(x^2+5)(3) - 3x(2x)}{(x^2+5)^2} = \frac{15-3x^2}{(x^2+5)^2}.$$

M1 for using the quotient rule, **A1** for the correct derivative.

At $x = 1$: $y = \frac{3}{6} = \frac{1}{2}$ and $\frac{dy}{dx} = \frac{12}{36} = \frac{1}{3}$. **A1**

The normal has gradient -3. **M1** for using $-\frac{1}{m}$ through $\left(1, \frac{1}{2}\right)$:

$$y - \frac{1}{2} = -3(x-1) \;\Rightarrow\; 2y - 1 = -6x + 6 \;\Rightarrow\; 6x + 2y - 7 = 0.$$

A1

(b) The numerator is zero: $15 - 3x^2 = 0$, so $x^2 = 5$. **M1**

$x = \sqrt{5}$ or $x = -\sqrt{5}$. **A1**

At $x = \sqrt{5}$: $y = \dfrac{3\sqrt{5}}{10}$. At $x = -\sqrt{5}$: $y = -\dfrac{3\sqrt{5}}{10}$. **A1**

The stationary points are $\left(\sqrt{5}, \frac{3\sqrt{5}}{10}\right)$ and $\left(-\sqrt{5}, -\frac{3\sqrt{5}}{10}\right)$.

Example 2

A curve has equation $y = (x^2-3)e^x$.

(a) Find the exact coordinates of the stationary points of the curve. **[5]**

(b) Determine the nature of each stationary point. **[3]**

(a) Product rule with $u = x^2 - 3$ and $v = e^x$:

$$\frac{dy}{dx} = 2x\,e^x + (x^2-3)e^x = e^x(x^2+2x-3).$$

M1 for the product rule, **A1** for the correct derivative.

$e^x > 0$ for every x, so $x^2+2x-3=0$, that is $(x+3)(x-1)=0$. **M1** for taking out e^x and solving the quadratic.

$x=-3$ or $x=1$. **A1**

At $x=-3$: $y=6e^{-3}$. At $x=1$: $y=-2e$. **A1**

The stationary points are $\left(-3, 6e^{-3}\right)$ and $(1,-2e)$.

(b) Differentiate $e^x(x^2+2x-3)$ with the product rule again:

$$\frac{d^2y}{dx^2} = e^x(2x+2) + e^x(x^2+2x-3) = e^x(x^2+4x-1).$$

M1 A1

At $x=-3$: $\frac{d^2y}{dx^2} = e^{-3}(9-12-1) = -4e^{-3} < 0$, so $\left(-3, 6e^{-3}\right)$ is a **maximum**. At $x=1$: $\frac{d^2y}{dx^2} = 4e > 0$, so $(1,-2e)$ is a **minimum**. **A1** for both, with the values.

Example 3

A curve has parametric equations

$$x = 2\theta - \sin 2\theta, \quad y = 4\sin^2\theta, \quad \text{for } 0 < \theta < \pi.$$

(a) Show that $\dfrac{dy}{dx} = 2\cot\theta$. **[4]**

(b) Find the equation of the tangent to the curve at the point where $\theta = \frac{1}{4}\pi$. Give your answer in the form $y = mx + c$, where m and c are exact. **[3]**

(c) Find the exact coordinates of the point on the curve at which the tangent is parallel to the x-axis. **[3]**

(a) $\dfrac{dx}{d\theta} = 2 - 2\cos 2\theta$. **B1**

$y = 4(\sin\theta)^2$, so $\dfrac{dy}{d\theta} = 8\sin\theta\cos\theta$ by the chain rule. **B1**

Divide, and use $\cos 2\theta = 1 - 2\sin^2\theta$ in the denominator: $2 - 2\cos 2\theta = 2 - 2 + 4\sin^2\theta = 4\sin^2\theta$. **M1**

$$\frac{dy}{dx} = \frac{8\sin\theta\cos\theta}{4\sin^2\theta} = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta.$$

A1, the given answer with every step shown.

(b) At $\theta = \frac{1}{4}\pi$: $x = \frac{1}{2}\pi - \sin\frac{1}{2}\pi = \frac{1}{2}\pi - 1$ and $y = 4\left(\frac{1}{\sqrt{2}}\right)^2 = 2$. **B1**

Gradient $= 2\cot\frac{1}{4}\pi = 2$. **M1** for the line through the point with this gradient:

$$y - 2 = 2\left(x - \tfrac{1}{2}\pi + 1\right) \;\Rightarrow\; y = 2x + 4 - \pi.$$

A1

(c) Parallel to the x-axis means $\frac{dy}{dx} = 0$: $2\cot\theta = 0$, so $\cos\theta = 0$. **M1**

In $0 < \theta < \pi$ this gives $\theta = \frac{1}{2}\pi$. **A1**

$x = \pi - \sin\pi = \pi$ and $y = 4\sin^2\frac{1}{2}\pi = 4$, so the point is $(\pi, 4)$. **A1**

The second diagram in "Understand it" shows this curve, the tangent from part (b) and the top of the arch from part (c).

Example 4

A curve has equation $e^x y^2 - 4x = 4$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{4 - e^x y^2}{2e^x y}$. **[4]**

(b) Find the coordinates of the stationary points of the curve. **[4]**

(c) Find the coordinates of the point on the curve at which the tangent is parallel to the y-axis. **[2]**

(a) Differentiate every term with respect to x. The first term is a product, $e^x \times y^2$. **M1** for the product rule:

$$\frac{d}{dx}\left(e^x y^2\right) = e^x y^2 + e^x \times 2y\frac{dy}{dx}.$$

A1

The other terms: $-4x \to -4$, and the constant $4 \to 0$. **B1**

$$e^x y^2 + 2e^x y\frac{dy}{dx} - 4 = 0 \;\Rightarrow\; 2e^x y\frac{dy}{dx} = 4 - e^x y^2 \;\Rightarrow\; \frac{dy}{dx} = \frac{4 - e^x y^2}{2e^x y}.$$

A1, the given answer.

(b) Stationary points: the numerator is 0, so $e^x y^2 = 4$. **M1**

Substitute into the curve's equation: $4 - 4x = 4$. **M1**

So $x = 0$. **A1** Then $e^0 y^2 = 4$ gives $y = \pm 2$. The stationary points are $(0, 2)$ and $(0, -2)$. **A1**

(c) Parallel to the y-axis: the denominator is 0, so $2e^x y = 0$. Since $e^x > 0$, $y = 0$. **M1**

Substitute: $0 - 4x = 4$, so $x = -1$. The point is $(-1, 0)$. **A1**

The third diagram in "Understand it" shows this curve with all three points.

Common mistakes

- **Losing the chain-rule factor.** $\ln(3t+1)$ differentiates to $\frac{3}{3t+1}$, not $\frac{1}{3t+1}$; $\tan 3x \to 3\sec^2 3x$, so $5\tan 3x \to 15\sec^2 3x$; $\sin 4t \to 4\cos 4t$. Examiners report these missing numbers almost every session.
- **Not spotting a product.** $t\ln t$, x^2e^{3x}, $e^{-t}\sin 3t$ and $2e^x y$ are all products. Differentiating only one part loses nearly every mark.
- **Treating $(\ln x)^2$ as $2\ln x$.** That law is for $\ln x^2$. $(\ln x)^2$ needs the chain rule: $\frac{2\ln x}{x}$.
- **Dividing by something that could be zero.** Dividing $(\ln x)^2 - 5\ln x = 0$ by $\ln x$ loses $x = 1$. Factorise instead.
- **Bracket and sign errors in the quotient rule.** Write the rule out with brackets round every factor, and keep "bottom times derivative of top" first.
- **Implicit slips:** forgetting that the constant on the right differentiates to 0 (it must not stay in the equation), leaving $\frac{dy}{dx}$ off a y term, or starting the line with "$\frac{dy}{dx} =$" before differentiating, which then gets mixed into the working.
- **Stopping at the tangent's gradient** when the question asks for the normal. Read the last line of the question again.
- **Degrees instead of radians.** Calculus with trigonometry is always in radians. If an iteration or a gradient gives strange numbers, check the calculator mode.
- **Ignoring the given range.** Solving $\sin\theta = -\frac{1}{4}$ when $\pi \leqslant \theta \leqslant \frac{3}{2}\pi$ needs the angle in that range, not the calculator's first answer. Likewise check whether t or x must be negative for the point on the diagram.
- **Decimals when "exact" is asked.** $x = \ln 3$ and $y = 4e^{-2}$ are exact; 1.10 and 0.541 are not.
- **Using the curve instead of the derivative** to show a stationary point lies in an interval. The sign change must be in $\frac{dy}{dx}$.
- **Not explaining why a factor gives no solution.** In "show that there is only one stationary point" you must say $e^x \neq 0$ and, for example, $e^x + 4 > 0$.
- **Using the wrong value for the gradient.** "The gradient is 1" means set $\frac{dy}{dx} = 1$, not 0.

How to get the marks

- **Show the derivative before you use it.** Most questions give a method mark for using the right rule (product, quotient or implicit) and accuracy marks for each correct part, even if the answer later goes wrong.
- **Parametric:** write $\frac{dx}{dt}$ and $\frac{dy}{dt}$ separately, then the division. Don't simplify both at once in your head.
- **Implicit:** write the whole differentiated equation, "$\ldots = 0$", before rearranging. Mark schemes award the product-rule term, the chain-rule term and the equation separately.
- **"Show that"** answers are printed on the paper, so the marks are for the steps. Show the identity you use and every line of rearranging; the last line must match exactly.
- **"Exact"** means e, $\ln$, surds, fractions and π in the answer. Simplify with $\ln e = 1$ and $e^{\ln k} = k$.
- **"Hence"** means use the previous part, such as the derivative you were asked to show.

- **Accuracy:** non-exact answers to 3 significant figures unless told otherwise; angles in radians (for example $x = 0.742$), unless the question uses degrees.
- **Stationary points:** "find the coordinates" means both x and y; give every point in the range and no extra ones.
- **Lines:** give the form asked for, such as integer coefficients or $y = mx + c$. A correct line in the wrong form usually loses the last mark.
- **Method marks follow through.** A wrong gradient, used correctly in a tangent or normal, still earns the method mark, so keep going.

Quick check

1. Differentiate: (a) $y = x^3 e^{-2x}$; (b) $y = \dfrac{e^x}{2x+1}$.
2. Find the exact gradient of the curve $y = x \tan x$ at the point where $x = \frac{1}{4}\pi$.
3. A curve has parametric equations $x = 2\ln(t+1)$, $y = t^2 - 4t$, for $t > -1$. Find $\dfrac{dy}{dx}$ in terms of t, and find the exact coordinates of the stationary point.
4. Find the equation of the normal to the curve $x^2 + 3xy + y^2 = 11$ at the point $(1, 2)$. Give your answer in the form $ax + by + c = 0$ with integers a, b, c.
5. Find the exact x-coordinates of the stationary points of the curve $y = x - 2\ln(x^2 + 1)$.

Answers

1. (a) Product rule: $\frac{dy}{dx} = 3x^2e^{-2x} + x^3(-2e^{-2x}) = x^2e^{-2x}(3-2x)$. (b) Quotient rule: $\frac{dy}{dx} = \frac{(2x+1)e^x - e^x(2)}{(2x+1)^2} = \frac{e^x(2x-1)}{(2x+1)^2}$.
2. $\frac{dy}{dx} = \tan x + x\sec^2 x$. At $x = \frac{1}{4}\pi$: $\tan\frac{1}{4}\pi = 1$ and $\sec^2\frac{1}{4}\pi = 2$, so the gradient is $1 + \frac{1}{4}\pi \times 2 = 1 + \frac{1}{2}\pi$.
3. $\frac{dx}{dt} = \frac{2}{t+1}$ and $\frac{dy}{dt} = 2t - 4$, so $\frac{dy}{dx} = (2t-4) \div \frac{2}{t+1} = (t-2)(t+1)$. It is 0 when $t = 2$ ($t = -1$ is not allowed). Then $x = 2\ln 3$ and $y = -4$: the point is $(2\ln 3, -4)$.
4. $2x + 3y + 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{2x+3y}{3x+2y} = -\frac{8}{7}$ at $(1, 2)$. The normal has gradient $\frac{7}{8}$: $y - 2 = \frac{7}{8}(x-1)$, so $7x - 8y + 9 = 0$.
5. $\frac{dy}{dx} = 1 - \frac{4x}{x^2+1} = 0$, so $x^2 + 1 = 4x$, that is $x^2 - 4x + 1 = 0$. So $x = 2 - \sqrt{3}$ or $x = 2 + \sqrt{3}$.

Past questions to try

- **9709/23/M/J/24 Q4**: the normal to a parametric curve, with trigonometric derivatives.
- **9709/21/M/J/22 Q4**: the normal to an implicit curve, with integer coefficients.
- **9709/21/M/J/22 Q6**: stationary points of a quotient that factorises in e^x.
- **9709/22/F/M/22 Q7**: an implicit curve with no stationary points and a tangent parallel to the y-axis.