Cambridge AS & A Level · Mathematics (9709) · Notes

Differentiation A2

Read online at pastopic.com/cambridge/mathematics-9709/notes/differentiation-p3

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Read first: Differentiation, Trigonometry

What you need to know

By the end of this topic you should be able to:

1. **Differentiate e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$ and $\tan^{-1} x$**, together with constant multiples, sums and differences of them, and composites built from them with the chain rule, such as $e^{\sin 2x}$, $\ln(\cos x)$, $\tan^{-1}(3x)$ or $\cos^3 x$. Angles are in **radians**. Because $\sec x$, $\operatorname{cosec} x$ and $\cot x$ are quotients of these, you can differentiate them too (their derivatives are in the formula list).
2. **Differentiate products and quotients**, for example $x^3 \ln x$, $e^{-2x} \cos x$ or $\dfrac{\tan x}{1 + \sin x}$.
3. **Find and use $\frac{dy}{dx}$ for a curve given parametrically or implicitly**, including tangents and normals, and points where the tangent is parallel to an axis or to a given line.

With these you use the Paper 1 methods: gradients, tangents, normals, stationary points and their nature. Derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ are **not** needed, and neither are second derivatives of parametric or implicit curves. Compared with Paper 2, the new function is $\tan^{-1} x$; the rest is the same rules used on harder functions, often in radians with double angles, and often as the first part of an integration or iteration question.

Understand it

A reminder from Paper 2

The Paper 2 Differentiation note ("Read first") explains each rule slowly. In short:

Function	Derivative
$e^{f(x)}$	$f'(x)\, e^{f(x)}$
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$
$\sin f(x)$, $\cos f(x)$, $\tan f(x)$	$f'(x) \cos f(x)$, $-f'(x) \sin f(x)$, $f'(x) \sec^2 f(x)$
$\big(f(x)\big)^n$	$n\big(f(x)\big)^{n-1} f'(x)$

Function	Derivative
uv	$u\dfrac{dv}{dx}+v\dfrac{du}{dx}$
$\dfrac{u}{v}$	$\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

Every row with $f(x)$ in it is the **chain rule**: differentiate the outside, then multiply by the derivative of the inside. For example $e^{\sin 2x} \to 2\cos 2x\, e^{\sin 2x}$, and $\cos^3 x = (\cos x)^3 \to 3\cos^2 x \times (-\sin x)$.

- **Parametric:** $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$, and the answer is in terms of t.
- **Implicit:** differentiate every term with respect to x; a term in y picks up a factor $\frac{dy}{dx}$ ($y^3 \to 3y^2\frac{dy}{dx}$), a term such as xy needs the product rule ($xy \to y + x\frac{dy}{dx}$), and a constant gives 0. Then collect the $\frac{dy}{dx}$ terms and factorise.

The calculus results for $\sin$, $\cos$ and $\tan$ only hold in radians, so set your calculator to radians for the whole topic.

Secant, cosecant and cotangent

$\sec x = \dfrac{1}{\cos x}$, so write it as $(\cos x)^{-1}$ and use the chain rule:

$$\frac{d}{dx}(\cos x)^{-1} = -(\cos x)^{-2} \times (-\sin x) = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \sec x \tan x.$$

In the same way $\operatorname{cosec} x \to -\operatorname{cosec} x \cot x$, and the quotient rule on $\dfrac{\cos x}{\sin x}$ gives $\cot x \to -\operatorname{cosec}^2 x$. All three are in the formula list, and they take the chain rule like any other function:

- $\sec 3x \to 3\sec 3x \tan 3x$
- $\cot 2x \to -2\operatorname{cosec}^2 2x$
- $\sec^2 x = (\sec x)^2 \to 2\sec x \times \sec x \tan x = 2\sec^2 x \tan x$

A question may tell you how to start, for example "by writing $\sec^4 x$ as $\dfrac{1}{\cos^4 x}$". Then $(\cos x)^{-4} \to -4(\cos x)^{-5}(-\sin x) = 4\sin x \sec^5 x$.

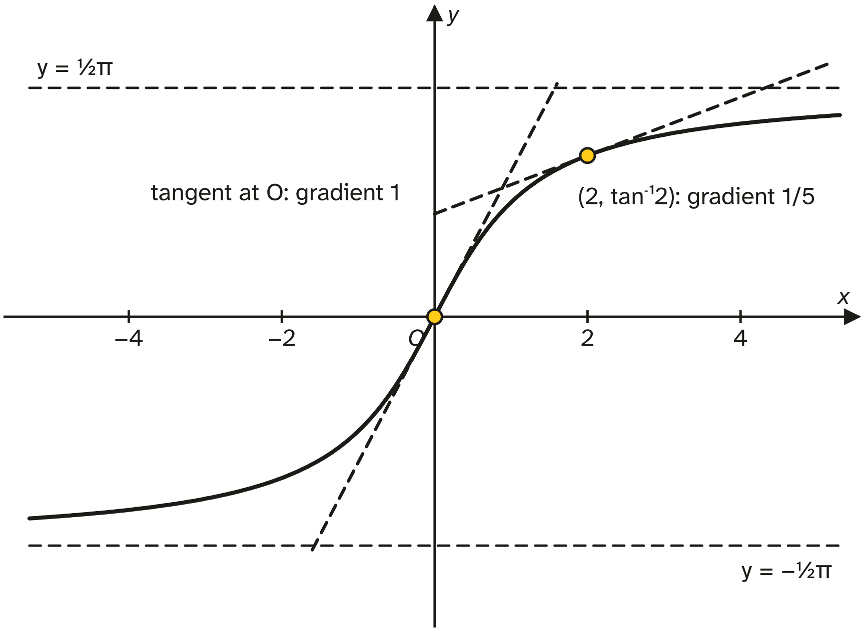
The inverse tangent

$y = \tan^{-1} x$ means $\tan y = x$, with y between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. Differentiate $\tan y = x$ implicitly:

$$\sec^2 y\,\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

So

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}.$$



The picture shows why the formula makes sense. The graph is steepest at the origin, where $\frac{1}{1+0^2} = 1$, and it flattens out towards its asymptotes, where $1 + x^2$ is large and the gradient is close to 0. The gradient is positive for every x.

With the chain rule, $\tan^{-1} f(x) \to \dfrac{f'(x)}{1 + \left(f(x)\right)^2}$:

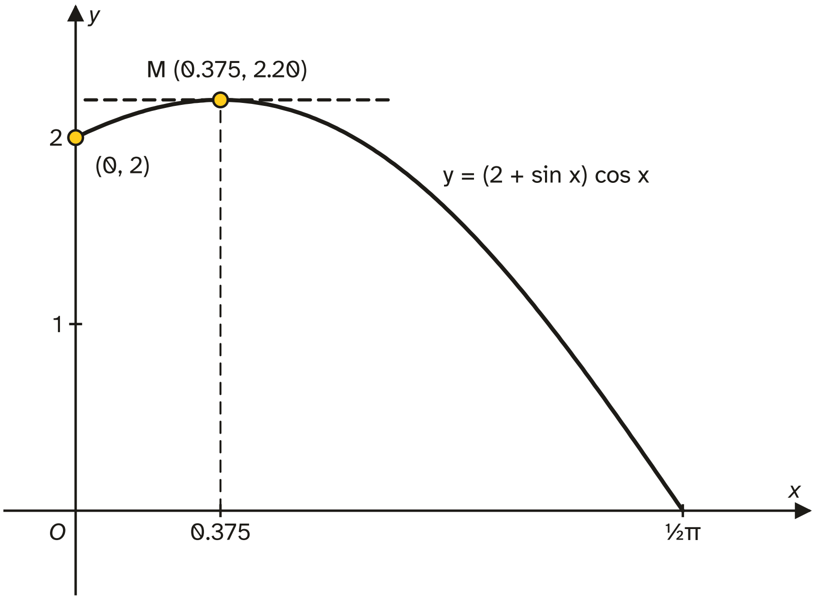
- $\tan^{-1} 3x \to \dfrac{3}{1 + 9x^2}$ (square the whole $3x$, not just the x)
- $\tan^{-1}\left(x^2\right) \to \dfrac{2x}{1 + x^4}$
- $\tan^{-1}\left(\dfrac{x}{2}\right) \to \dfrac{\frac{1}{2}}{1 + \frac{x^2}{4}} = \dfrac{2}{4 + x^2}$

If an equation contains $\tan^{-1}$, take $\tan$ of both sides to remove it: $\tan^{-1} a = \dfrac{a}{2}$ becomes $a = \tan \dfrac{a}{2}$.

Stationary points of harder functions

Paper 3 curves are usually products or quotients, so $\frac{dy}{dx}$ has several terms. The route is always the same: **differentiate, set equal to 0, simplify to one function, solve.** Simplifying is where most of the work is:

- **Take out factors that can't be zero**, such as $e^{\sin 2x}$, e^{-x} or $\sec^2 x$, and say so.
- **Never divide by a factor that can be zero.** Factorise it out and set it equal to 0 as well, or you lose solutions.
- **Use identities to get one trigonometric function**: $\sin 2x = 2 \sin x \cos x$, $\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$, $\cos^2 x = 1 - \sin^2 x$, $\sec^2 x = 1 + \tan^2 x$. The same formulas work for doubled angles: $\sin 4x = 2 \sin 2x \cos 2x$ and $\cos 4x = 1 - 2\sin^2 2x$. You often get a quadratic in $\sin x$ or $\tan x$ (or in $\sin 2x$).
- **Clear square roots and negative powers** by multiplying through, so that the equation has only positive whole powers. Mark schemes call this a "horizontal" equation.



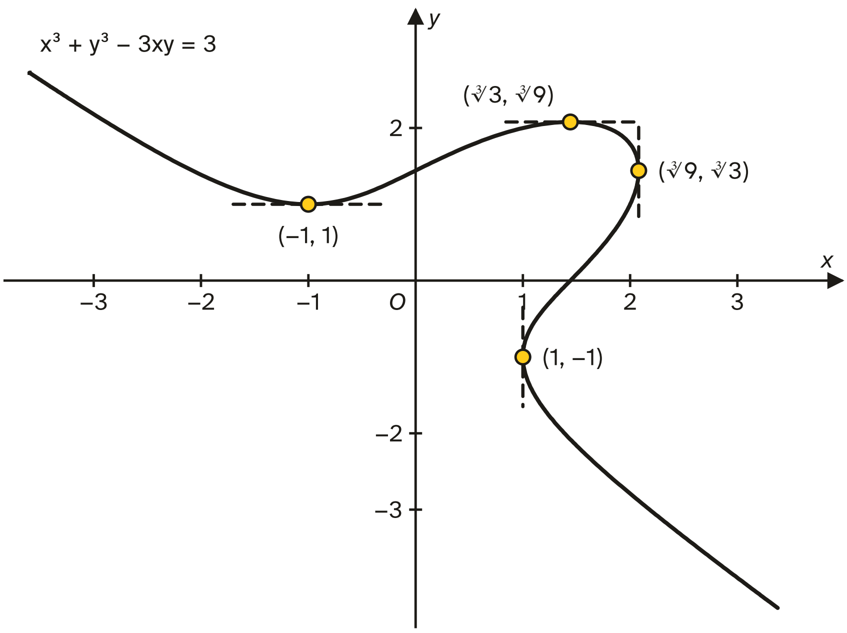
Example 1 finds M on this curve. Its equation $\frac{dy}{dx} = 0$ becomes a quadratic in $\sin x$ with one root between 0 and 1.

Tangents parallel to the axes

When $\frac{dy}{dx} = \dfrac{N}{D}$ (after implicit differentiation, or $\frac{dy}{dt} \div \frac{dx}{dt}$ for a parametric curve):

- the tangent is **parallel to the x-axis** (a stationary point) where the numerator $N = 0$;
- the tangent is **parallel to the y-axis** where the denominator $D = 0$.

The normal is perpendicular to the tangent, so "the **normal** is parallel to the y**-axis**" means the tangent is parallel to the x-axis: $N = 0$. "The normal is parallel to the x-axis" means $D = 0$.



For an implicit curve, $N = 0$ or $D = 0$ gives a link between x and y, not a point. Substitute that link into the **curve's equation** to find the points (Example 4 does this for the curve above).

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\frac{d}{dx}(x^n) = nx^{n-1},\ \frac{d}{dx}(e^x) = e^x,\ \frac{d}{dx}(\ln x) = \frac{1}{x}$	Given
$\frac{d}{dx}(\sin x) = \cos x,\ \frac{d}{dx}(\cos x) = -\sin x,\ \frac{d}{dx}(\tan x) = \sec^2 x$ (radians)	Given
$\frac{d}{dx}(\sec x) = \sec x \tan x,\ \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x,\ \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$	Given
$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	Given
Product rule: $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$	Given
Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	Given
Parametric: $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$	Given
Chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	Learn it
$\frac{d}{dx} e^{f(x)} = f'(x)\, e^{f(x)},\ \frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$	Learn it
$\frac{d}{dx} \tan^{-1} f(x) = \frac{f'(x)}{1 + \left(f(x)\right)^2}$, for example $\frac{d}{dx} \tan^{-1}(ax) = \frac{a}{1 + a^2x^2}$	Learn it
Implicit: $\frac{d}{dx}\left(g(y)\right) = g'(y)\frac{dy}{dx},\ \frac{d}{dx}(xy) = y + x\frac{dy}{dx}$	Learn it
Tangent: $y - y_1 = m(x - x_1)$; normal gradient $-\frac{1}{m}$	Learn it
$\frac{dy}{dx} = \frac{N}{D}$: tangent parallel to the x-axis where $N = 0$, parallel to the y-axis where $D = 0$	Learn it
$\sec x = \frac{1}{\cos x},\ \operatorname{cosec} x = \frac{1}{\sin x},\ \cot x = \frac{\cos x}{\sin x}$	Learn it
$\sin^2 x + \cos^2 x = 1,\ 1 + \tan^2 x = \sec^2 x,\ 1 + \cot^2 x = \operatorname{cosec}^2 x$	Given
$\sin 2x = 2\sin x \cos x,\ \cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$, and the $\sin(A \pm B)$, $\cos(A \pm B)$ formulas	Given

Formula	Status
$-1 \leqslant \sin x \leqslant 1,\ -1 \leqslant \cos x \leqslant 1,\ e^x > 0$	Learn it

How to do it

In the Paper 3 questions we have tagged for this topic (72 questions, 2021–2025, all different), the types came up this often. Each question is counted once, under its main type. 32 of the 72 go on into another topic: an area, volume or integral (20), an iteration from Numerical solution of equations (11), or a differential equation (1).

Question type	Questions
Stationary points of $y = f(x)$	36
Implicit equations	14
Parametric equations	14
Gradients and tangents of $y = f(x)$	6
Find or show a derivative	2

1. Stationary points of $y = f(x)$ (36 questions)

1. **Differentiate**, choosing the product, quotient or chain rule. Write the derivative out in full before simplifying.
2. **Set $\frac{dy}{dx} = 0$.** For a quotient, set the **numerator** to 0. Take out common factors; any that can't be 0 (such as e^{2x}, $e^{\sin x}$ or $\sec^2 x$) can be divided away, and say why.
3. **Get one function.** Use identities to reach an equation in just $\sin x$, $\cos x$ or $\tan x$ (or $\ln x$, or e^x), with no square roots or negative powers.
4. **Solve**, in radians, giving **every** solution in the interval and **no** others. Reject values that are impossible ($\sin x = 1.4$, $e^x = -2$) or outside the domain.
5. **Find** y from the curve's equation if the coordinates are asked for, exactly if the question says "exact".

For example, $y = e^{-x}\cos x$ gives $\frac{dy}{dx} = -e^{-x}\cos x - e^{-x}\sin x = -e^{-x}(\cos x + \sin x)$. Since $e^{-x} > 0$, $\sin x = -\cos x$, so $\tan x = -1$ and, for $0 \leqslant x \leqslant 2\pi$, $x = \frac{3}{4}\pi$ or $\frac{7}{4}\pi$.

Variations you will meet:

- **A maximum or minimum point marked on a diagram**, often followed by an area or volume by integration. Find the root that matches the point on the diagram, and ignore roots that are endpoints of the interval or belong to other points. If you are told the interval that contains M, use it to choose the root.

- **Trigonometric products** such as $\sin x \cos 3x$, $\cos^2 x \sin 2x$ or $\sin^3 x\sqrt{\cos x}$. Rewrite with double angles, then collect into one ratio: for example $\cos x \cos 2x - \sin x \sin 2x$ is $\cos 3x$ by the compound-angle formula, and $\cos x(1 - 3\sin^2 x) = 0$ gives $\cos x = 0$ or $\sin^2 x = \frac{1}{3}$.
- **A power of $\tan x$ times an exponential**, such as $e^{kx}\tan^2 x$. Use $\sec^2 x = 1 + \tan^2 x$ to get a polynomial in $\tan x$. Factorise out $\tan x$ rather than dividing by it, because $\tan x = 0$ (so $x = 0$) is a stationary point too.
- **"Show that the derivative can be written as $\sec^2 x\,(a + b\sin 2x)e^{kx}$"**, then find the x-coordinates. Show each step of the rewriting; the form tells you where you are heading.
- **Exact answers with logarithms and exponentials.** From $x^5 \ln x$: $5x^4 \ln x + x^4 = 0$, and $x^4 \neq 0$ for $x > 0$, so $\ln x = -\frac{1}{5}$ and $x = e^{-1/5}$. From a quadratic in e^x: let $u = e^x$, solve, reject $u \leqslant 0$, then $x = \ln u$.
- **Square roots** such as $(x+3)\sqrt{5-x}$: the product rule gives $\sqrt{5-x} - \frac{x+3}{2\sqrt{5-x}}$. Multiply through by $2\sqrt{5-x}$ to get a linear or quadratic equation. If you square both sides instead, check each answer in the original equation.
- **An inverse tangent**, such as $y = \dfrac{\tan^{-1} x}{1+x}$ for $x > 0$: the quotient rule gives the numerator $\dfrac{1+x}{1+x^2} - \tan^{-1} x$, so at the turning point $\tan^{-1} a = \dfrac{1+a}{1+a^2}$. Take tan of both sides: $a = \tan\left(\dfrac{1+a}{1+a^2}\right)$, which has a root between 1 and 1.5.
- **"Show that the x-coordinate satisfies $a = \ldots$"**, followed by an iteration. Set the derivative to 0 and rearrange step by step to the printed equation, in terms of the letter the question uses. The sign-change check and the iteration belong to Numerical solution of equations; do them in radians.
- **"Show that the curve has no stationary points."** Show that the numerator can never be 0. For example, if $\frac{dy}{dx} = 0$ leads to $\sin^3 x = -\frac{3}{2}$, then $\sin x = -1.14\ldots < -1$, which is impossible because $-1 \leqslant \sin x \leqslant 1$. Say this in words.
- **A constant in the equation**, such as $y = xe^{-ax}$: treat a as a number and give the answer in terms of a.
- **The nature of the point**: use the sign of $\frac{d^2y}{dx^2}$, or the sign of $\frac{dy}{dx}$ either side, and state which one you worked out and what it shows.

2. Implicit equations (14 questions)

1. **Differentiate every term** with respect to x, both sides. Each y term gets $\times\frac{dy}{dx}$; each product of x and y terms needs the product rule; constants (including $\ln 6$ or e^2) give 0.
2. **Collect** the $\frac{dy}{dx}$ terms on one side, factorise and divide.
3. **Substitute** both coordinates of the point, or use the numerator or denominator as below.

Variations you will meet:

- **"Show that $\frac{dy}{dx} = \ldots$"**: finish with exactly the printed form. If your signs come out the other way round, multiply the top and bottom by -1 and show it.
- **The gradient at a point given by one coordinate.** Find the other coordinate from the curve's equation first. "Where $x = 0$" may give two values of y (two gradients to find); "where $y = 1$" in a curve with e^{2x} and e^x gives a quadratic in e^x, with the negative root rejected.

- **Logarithms in the equation.** $\ln(x+y) \to \dfrac{1+\frac{dy}{dx}}{x+y}$ and $x\ln 3y \to \ln 3y + \dfrac{x}{y}\dfrac{dy}{dx}$ (because $\ln 3y = \ln 3 + \ln y$). Exact answers keep the logarithms; simplify so that there are no fractions inside fractions.
- **Tangent parallel to the x-axis, or normal parallel to the y-axis**: numerator $= 0$. Solve for the link, such as $x(x-3y)=0$, so $x=0$ **or** $x=3y$; use **both** factors. Substitute each into the curve's equation and solve.
- **Tangent parallel to the y-axis, or normal parallel to the x-axis**: denominator $= 0$, then the same steps.
- **"Show that the curve has no tangent parallel to the x-axis"**: numerator $= 0$ gives a link; substituting it into the curve gives an equation with no real roots (for a quadratic, show its discriminant is negative).
- **Tangent parallel to a given line**, such as $2y = x+5$: set $\frac{dy}{dx} = \frac{1}{2}$ to get a link between x and y, then substitute into the curve's equation (not into the line's).
- **A constant a in the equation**: $ay^2 \to 2ay\frac{dy}{dx}$; a is not a variable, so it is not a product.
- **The angle between two tangents**: find both gradients, then use $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$ with $\tan A$ and $\tan B$ equal to the gradients. For the acute angle, take the positive value.

3. Parametric equations (14 questions)

1. Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ separately, each with its own rule.
2. **Divide**: $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$, and simplify, clearing fractions inside fractions.
3. **Find** t at the point, then substitute into $\frac{dy}{dx}$, and into x and y for the coordinates.
4. Form the tangent or normal in the form asked for.

Variations you will meet:

- **"Show that $\frac{dy}{dx}$ can be written as $k\sec 2t$"** (or some other single term): expect identities such as $\sin 2t = 2\sin t\cos t$, $\sec t = \frac{1}{\cos t}$ or $\operatorname{cosec}^2 t = 1+\cot^2 t$. Show the steps and give the constant.
- **"Obtain a simplified expression for $\frac{dy}{dx}$"**: factorise fully, for example $t - \dfrac{2}{t+1} = \dfrac{t^2+t-2}{t+1} = \dfrac{(t+2)(t-1)}{t+1}$.
- **Derivatives from the formula list**: $x = \dfrac{4}{\cos 2t} = 4\sec 2t$ gives $\dfrac{dx}{dt} = 8\sec 2t\tan 2t$.
- **A point given by its coordinates**: use the easier coordinate to find t. For example, if $x = e^{2t}$ and the point is $(e^2, \ldots)$, then $t=1$.
- **A point described in words**: on the y-axis ($x=0$), on the x-axis ($y=0$), or where the normal has a given gradient (set $-\frac{1}{dy/dx}$ equal to it and solve for t).
- **"Show that the normal passes through a given point"**: find the normal's equation, then substitute the point. Don't use the printed point to find the line.
- **"Show that the gradient is always positive"**: write $\frac{dy}{dx}$ as a single fraction and explain why each part is positive for every allowed t (for example, $1+2t>0$ because $\ln(1+2t)$ must exist).
- **A stationary point**: $\frac{dy}{dt} = 0$, find t, then y (exactly if asked). Reject values of t outside the given range.

4. Gradients and tangents of $y = f(x)$ (6 questions)

1. Differentiate and substitute the x-coordinate to find the gradient; find y from the curve.
2. Form the tangent or normal, or solve "gradient $=$ a given value".

Variations you will meet:

- **The gradient equals a number**: set $\frac{dy}{dx}$ equal to it and solve. It may give a quadratic in x (then find both points exactly), a quadratic in e^{2x}, or, with $\tan^{-1}$, something like $\dfrac{3}{1+9x^2} = \dfrac{1}{2}$, so $9x^2 = 5$ and $x = \pm\dfrac{\sqrt{5}}{3}$.
- **"Show that a satisfies $a = \ldots$"** where the gradient is a given value, followed by an iteration: set $\frac{dy}{dx}$ equal to the value and rearrange to the printed form.
- **"Express $\frac{dy}{dx}$ in terms of $\tan x$"**: replace $\sec^2 x$ by $1 + \tan^2 x$. "Verify that the gradient has a given value at a given point" needs the substitution and the arithmetic shown, not just the answer.
- **A tangent at an exact point** such as $x = \frac{1}{2}\pi$: keep π in the coordinates and give the line in the form asked.

5. Find or show a derivative (2 questions)

1. **Given $\tan y = kx$, show that $\frac{dy}{dx} = \dfrac{k}{1+k^2x^2}$**: differentiate implicitly, $\sec^2 y \frac{dy}{dx} = k$; replace $\sec^2 y$ by $1 + \tan^2 y = 1 + k^2x^2$. This is the proof in "Understand it" with kx in place of x.
2. **"By writing $\sec^n \theta$ as $\dfrac{1}{\cos^n \theta}$, show that ..."**: use the chain rule on $(\cos\theta)^{-n}$ and rewrite in the printed form.

These are usually part (a) of an integration or differential equation question: the result is what you integrate in part (b).

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

The diagram in "Understand it" shows the curve $y = (2 + \sin x)\cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M. Find the x-coordinate of M, giving your answer correct to 3 significant figures. **[5]**

Product rule with $u = 2 + \sin x$ and $v = \cos x$:

$$\frac{dy}{dx} = \cos x \times \cos x + (2 + \sin x)(-\sin x) = \cos^2 x - 2\sin x - \sin^2 x.$$

M1 for the product rule, **A1** for the correct derivative.

Set it to 0 and use $\cos^2 x = 1 - \sin^2 x$ to get one function:

$$1 - \sin^2 x - 2\sin x - \sin^2 x = 0 \;\Rightarrow\; 2\sin^2 x + 2\sin x - 1 = 0.$$

M1 for equating to zero and using an identity, **A1** for the correct quadratic in $\sin x$.

$$\sin x = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-1 \pm \sqrt{3}}{2}.$$

$\frac{-1-\sqrt{3}}{2} < -1$ is impossible, so $\sin x = \frac{\sqrt{3}-1}{2} = 0.3660\ldots$ and $x = 0.375$ (3 s.f.), in radians. **A1**

Example 2

A curve has equation $y = 2\tan^{-1} x - \ln\left(1 + x^2\right)$.

(a) Find the exact coordinates of the stationary point of the curve. **[4]**

(b) Determine whether this point is a maximum or a minimum. **[3]**

(a)

$$\frac{dy}{dx} = \frac{2}{1+x^2} - \frac{2x}{1+x^2}.$$

B1 for $\frac{2}{1+x^2}$, **B1** for $-\frac{2x}{1+x^2}$.

$\dfrac{dy}{dx} = \dfrac{2(1-x)}{1+x^2} = 0$, so $x = 1$. Then $y = 2\tan^{-1} 1 - \ln 2 = 2 \times \frac{1}{4}\pi - \ln 2$. **M1** for setting the derivative to zero and solving.

The stationary point is $\left(1,\ \frac{1}{2}\pi - \ln 2\right)$. **A1**

(b) Quotient rule on $\dfrac{2-2x}{1+x^2}$:

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)(-2) - (2-2x)(2x)}{(1+x^2)^2} = \frac{2x^2 - 4x - 2}{(1+x^2)^2}.$$

M1 for the quotient rule on their $\frac{dy}{dx}$, **A1** for a correct second derivative.

At $x = 1$: $\dfrac{d^2y}{dx^2} = \dfrac{2-4-2}{4} = -1 < 0$, so the point is a **maximum**. **A1**

(Checking the sign of $\frac{dy}{dx}$ also works: at $x = 0$ it is $2 > 0$ and at $x = 2$ it is $-\frac{2}{5} < 0$, so $+, 0, -$: a maximum.)

Example 3

The parametric equations of a curve are

$$x = \operatorname{cosec} t, \qquad y = t + \cot t, \qquad \text{for } 0 < t < \pi.$$

(a) Show that $\dfrac{dy}{dx} = \cos t$. **[5]**

(b) Find the equation of the normal to the curve at the point where $t = \frac{1}{6}\pi$. Give your answer in the form $y = mx + c$, where m and c are exact. **[4]**

(a) From the formula list, $\dfrac{dx}{dt} = -\operatorname{cosec} t \cot t$. **B1**

$\dfrac{dy}{dt} = 1 - \operatorname{cosec}^2 t$. **B1**

Using $\operatorname{cosec}^2 t = 1 + \cot^2 t$, $\dfrac{dy}{dt} = -\cot^2 t$. **M1**

$$\frac{dy}{dx} = \frac{-\cot^2 t}{-\operatorname{cosec} t \cot t} = \frac{\cot t}{\operatorname{cosec} t} = \frac{\cos t}{\sin t} \times \sin t = \cos t.$$

M1 for dividing the right way round, **A1** for the given answer with every step shown.

(b) At $t = \frac{1}{6}\pi$: $x = \operatorname{cosec} \frac{1}{6}\pi = 2$ and $y = \frac{1}{6}\pi + \cot \frac{1}{6}\pi = \frac{1}{6}\pi + \sqrt{3}$. **B1**

The tangent's gradient is $\cos \frac{1}{6}\pi = \frac{\sqrt{3}}{2}$, so the normal's gradient is $-\dfrac{2}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$. **M1** for $-\frac{1}{m}$.

$$y - \left(\tfrac{1}{6}\pi + \sqrt{3}\right) = -\frac{2\sqrt{3}}{3}(x - 2).$$

M1 for a line through their point with their normal gradient.

$$y = -\frac{2\sqrt{3}}{3}x + \frac{7\sqrt{3}}{3} + \frac{1}{6}\pi.$$

A1, since $\frac{4\sqrt{3}}{3} + \sqrt{3} = \frac{7\sqrt{3}}{3}$.

Example 4

The equation of a curve is $x^3 + y^3 - 3xy = 3$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{x^2 - y}{x - y^2}$. **[4]**

(b) Find the exact coordinates of the points on the curve at which the tangent is parallel to the x-axis. **[5]**

(a) $y^3 \to 3y^2\dfrac{dy}{dx}$. **B1**

$-3xy \to -3y - 3x\dfrac{dy}{dx}$ (product rule). **B1**

The constant 3 gives 0:

$$3x^2 + 3y^2\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0 \;\Rightarrow\; (3y^2 - 3x)\frac{dy}{dx} = 3y - 3x^2.$$

M1 for differentiating every term, setting the result to 0 and collecting the $\frac{dy}{dx}$ terms.

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x} = \frac{x^2 - y}{x - y^2},$$

multiplying the top and bottom by -1. **A1**

(b) The tangent is parallel to the x-axis where the numerator is 0: $y = x^2$. **M1**

Substitute into the curve's equation: $x^3 + x^6 - 3x^3 = 3$, that is $x^6 - 2x^3 - 3 = 0$. **M1**

This is a quadratic in x^3: $(x^3 - 3)(x^3 + 1) = 0$, so $x^3 = 3$ or $x^3 = -1$. **A1**

$x = -1$ gives $y = 1$; $x = \sqrt[3]{3}$ gives $y = \sqrt[3]{9}$. The points are $(-1, 1)$ **A1** and $\left(\sqrt[3]{3}, \sqrt[3]{9}\right)$ **A1**.

The third diagram in "Understand it" shows the curve. By the same method, $x = y^2$ (the denominator $= 0$) gives the two points where the tangent is parallel to the y-axis, $(1, -1)$ and $\left(\sqrt[3]{9}, \sqrt[3]{3}\right)$.

Common mistakes

- **Leaving out the chain rule inside a product.** Examiners report the inner derivative lost (for example $\cos 2x \to -\sin 2x$), $\cos^3 x$ differentiated with no chain rule at all, and $\ln(\sin 2t) \to \frac{1}{\sin 2t}$. Differentiate each factor on its own line first, then put the product together.
- **Quotient rule errors**: the terms in the numerator reversed, a sign slip, or the denominator not squared (or left out). Put brackets round every factor, and write u, v, u' and v' first.
- **Dividing by a factor that can be zero.** Dividing by $\tan x$ throws away $x = 0$; ignoring a factor x in the numerator of $\frac{dy}{dx}$ throws away a whole point. Factorise and set each factor to 0.
- **Working in degrees.** An answer in degrees loses the accuracy mark after calculus with trigonometry, and an iteration run in degrees goes wrong. Check radian mode before you start.
- **Implicit slips**: writing "$\frac{dy}{dx} =$" in front of the whole differentiated equation, forgetting that the constant on the right-hand side differentiates to 0, missing the $1 + \frac{dy}{dx}$ from $\ln(x + y)$, or treating a constant a as a variable.
- **Mixing up the conditions.** Horizontal tangent (or vertical normal) is numerator $= 0$; vertical tangent (or horizontal normal) is denominator $= 0$. Examiners see students answer for the wrong axis, or substitute the line's equation into the curve instead of the link that comes from $\frac{dy}{dx}$.
- **Mixing up letters in parametric questions**: writing x for t (or t for θ). Keep the parameter's letter throughout; for a normal, use $-\frac{1}{dy/dx}$, not the tangent's gradient.
- **Using the printed answer to prove itself.** In "show that the normal passes through a given point", find the normal first, then check the point.
- **Keeping impossible roots**, such as a negative value of e^x, $\sin x > 1$, or t outside the range, and **giving decimals** when the question says "exact".
- **Not enough working in a "show that".** Examiners repeatedly say that given answers need every step, including the identity you used and where you set the derivative equal to 0.
- **Deciding the nature without saying what was calculated.** If you use values either side, say whether they are values of $\frac{dy}{dx}$ (or of y) and what they show.

How to get the marks

- **Show the derivative before you use it.** The first marks are usually for the product, quotient or chain rule applied correctly, and later method marks need to see it. A correct answer with no working can score nothing.
- **"Show that"** means the answer is printed. Every step must be visible and correct, and the last line must match the printed form exactly (for example $\frac{dy}{dx} = \cos t$ in Example 3, not an equivalent).
- **"Hence"** means use the previous part: the given $\frac{dy}{dx}$, or the form you just showed.
- **"Exact"** means keep π, e, $\ln$ and surds. $x = e^{-1/5}$ earns the mark; 0.819 does not.
- **"Simplified"** means one fraction with no fractions inside it, common factors cancelled and, where it helps, the denominator factorised.
- **Accuracy.** When the question states an accuracy (3 significant figures, 3 decimal places), give exactly that: some mark schemes accept only that accuracy. Otherwise give non-exact answers to 3 significant figures. Angles are in radians.
- **All the roots, and only them.** "Find the x-coordinates of the stationary points in $0 \leqslant x \leqslant \pi$" loses the last mark if one is missing or an extra one is included.
- **"Determine the nature"** needs a calculation (the value or sign of $\frac{d^2y}{dx^2}$, or the gradient either side) and a conclusion.
- **Tangents and normals:** state the point and the gradient you used, write $y - y_1 = m(x - x_1)$ with the numbers in, then give the form asked for (integer coefficients, or exact m and c).
- **Method marks follow through.** A slip in the derivative still leaves marks for setting it to zero and solving, so keep going.

Quick check

1. Find $\dfrac{dy}{dx}$ when (a) $y = \tan^{-1}(5x)$; (b) $y = x^2 \operatorname{cosec} x$.
2. Find the exact gradient of the curve $x \ln y + y^2 = 4$ at the point where it crosses the y-axis.
3. A curve has parametric equations $x = \ln\left(t^2 + 1\right)$, $y = \tan^{-1} t$, for $t > 0$. Find the equation of the tangent at the point where $t = 1$, giving your answer in the form $y = mx + c$ with m and c exact.
4. Show that the curve $y = \dfrac{e^{2x}}{e^x + 3}$ has no stationary points.
5. Find the exact x-coordinates of the stationary points of $y = e^{-x} \sin x$ for $0 \leqslant x \leqslant 2\pi$.

Answers

1. (a) $\frac{dy}{dx} = \frac{5}{1+25x^2}$. (b) Product rule: $\frac{dy}{dx} = 2x \operatorname{cosec} x - x^2 \operatorname{cosec} x \cot x$.
2. At $x = 0$: $y^2 = 4$, and $y > 0$ because of $\ln y$, so $y = 2$. Differentiating, $\ln y + \frac{x}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$. At $(0, 2)$: $\ln 2 + 4\frac{dy}{dx} = 0$, so the gradient is $-\frac{1}{4}\ln 2$.
3. $\frac{dx}{dt} = \frac{2t}{t^2+1}$ and $\frac{dy}{dt} = \frac{1}{1+t^2}$, so $\frac{dy}{dx} = \frac{1}{2t} = \frac{1}{2}$ at $t = 1$. The point is $\left(\ln 2, \frac{1}{4}\pi\right)$, so $y - \frac{1}{4}\pi = \frac{1}{2}(x - \ln 2)$, that is $y = \frac{1}{2}x + \frac{1}{4}\pi - \frac{1}{2}\ln 2$.
4. Quotient rule: $\frac{dy}{dx} = \frac{(e^x+3)2e^{2x} - e^{2x}e^x}{(e^x+3)^2} = \frac{e^{2x}\,(e^x+6)}{(e^x+3)^2}$. Since $e^{2x} > 0$ and $e^x + 6 > 0$ for every x, the numerator is never 0, so there are no stationary points.
5. $\frac{dy}{dx} = e^{-x}(\cos x - \sin x)$. As $e^{-x} > 0$, $\tan x = 1$, so $x = \frac{1}{4}\pi$ or $x = \frac{5}{4}\pi$.

Past questions to try

- **9709/33/M/J/25 Q5**: implicit differentiation with e^{-x}, then two gradients where $x = 0$.
- **9709/35/M/J/25 Q6**: a parametric curve with $\sec 3t$ and $\tan 3t$, then an exact normal.
- **9709/32/F/M/24 Q6**: an implicit curve with no tangent parallel to the x-axis.
- **9709/32/M/J/21 Q8**: stationary points of an exponential times $\tan^2 x$, without losing $x = 0$.