Cambridge AS & A Level · Mathematics (9709) · Notes

Forces and equilibrium AS

Read online at pastopic.com/cambridge/mathematics-9709/notes/forces-and-equilibrium

Last checked 28 September 2026

What you need to know

By the end of this topic you should be able to:

1. **Identify every force acting** on an object in a given situation, and show them on a clear force diagram.
2. **Treat forces as vectors.** Split a force into components in two perpendicular directions, and combine several forces into a single resultant (its size and its direction). Answers must be calculated; drawing to scale is not accepted.
3. **Use the equilibrium condition.** When a particle is in equilibrium the forces on it add to zero, which is the same as saying the components in any direction add to zero. Resolving in two directions is the expected method.
4. **Split a contact force** between two surfaces into two parts: the normal reaction (at right angles to the surface) and friction (along the surface).
5. **Use the "smooth" model**, where there is no friction, and know what it leaves out.
6. **Use friction and limiting equilibrium.** Know what "limiting friction", "limiting equilibrium" and "about to slip" mean, know the definition of the coefficient of friction μ, and use $F = \mu R$ when friction is limiting and $F \leqslant \mu R$ in general.
7. **Use Newton's third law**: if one object pushes or pulls on another, the second pushes or pulls back on the first with a force of the same size in the opposite direction.

Throughout Paper 4, take $g = 10 \text{ m s}^{-2}$ unless a question says otherwise. The triangle of forces and Lami's theorem are allowed but never needed; this note resolves forces, which always works.

Understand it

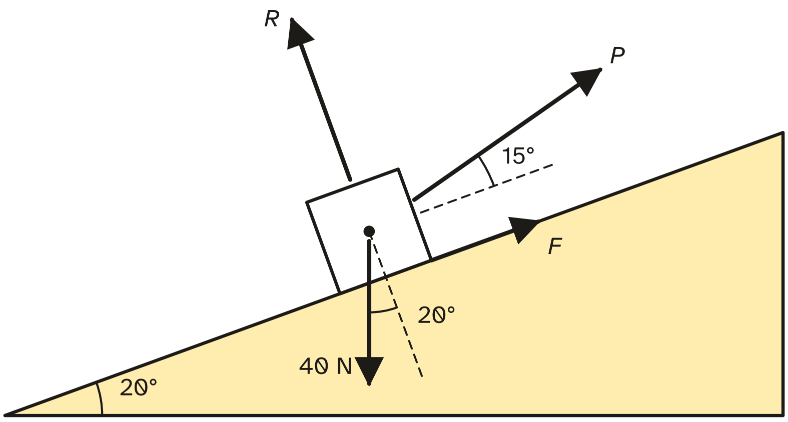
What a force diagram shows

A **force** is a push or a pull. It has a size (in newtons, N) and a direction, so it is a vector. Before writing any equation, draw the object as a dot or a small block and add an arrow for **every** force acting **on** it, labelled with its size or a letter. The forces you will meet are:

- **Weight** mg, straight down. A mass of 4 kg has weight $4 \times 10 = 40$ N.

- **Tension** T in a string, pulling **away** from the object along the string. A light string over a smooth pulley has the same tension on both sides.
- **Thrust** in a rod or strut, pushing **towards** the object. (If you label it as a tension and get a negative answer, the force is really a thrust.)
- **Normal reaction** R from a surface, at right angles to the surface, pushing the object away from it.
- **Friction** F along the surface, opposing the way the object moves or is about to move.
- **Applied forces**: pushes and pulls given in the question.

about to slide down, so friction acts up the slope



Components

Any force can be replaced by two forces at right angles, its **components**, that have the same effect. A force of size P at angle θ to a direction has component $P\cos\theta$ **along** that direction and $P\sin\theta$ **at right angles** to it.

A useful memory aid: the component next to the angle uses **cos**; the component across from the angle uses **sin**. A 20 N pull at 35° above the horizontal has horizontal component $20\cos 35^\circ = 16.4$ N and vertical component $20\sin 35^\circ = 11.5$ N.

On a slope at angle α, the weight splits into $mg\sin\alpha$ **down the slope** and $mg\cos\alpha$ **into the slope**. The diagram shows why: the angle between the weight and the perpendicular to the slope equals the slope angle. For the 40 N block on a 20° slope, that is $40\sin 20^\circ = 13.7$ N down the slope and $40\cos 20^\circ = 37.6$ N into it.

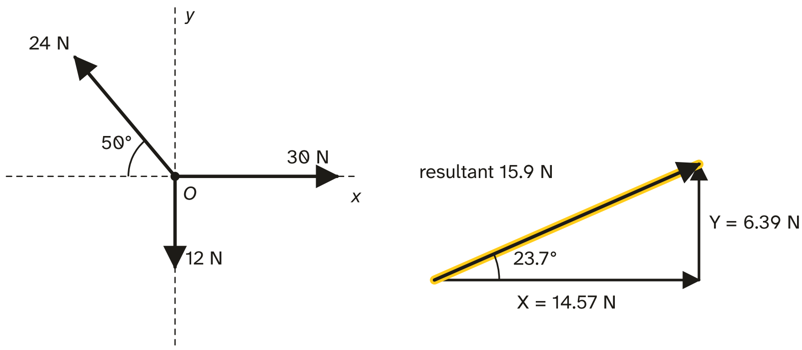
If a question gives $\tan\alpha = \frac{3}{4}$, draw a 3-4-5 right-angled triangle: $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$ exactly. Use these exact values rather than rounding the angle.

The resultant

The **resultant** is the single force with the same effect as all the forces together. Choose two perpendicular directions (usually x and y), add the components of every force in each direction to get X and Y, then

$$\text{magnitude} = \sqrt{X^2 + Y^2}, \qquad \tan\theta = \frac{|Y|}{|X|},$$

where θ is the angle the resultant makes with the x-direction. Always say which way it points, for example "23.7° above the positive x-axis"; the signs of X and Y tell you the quadrant.

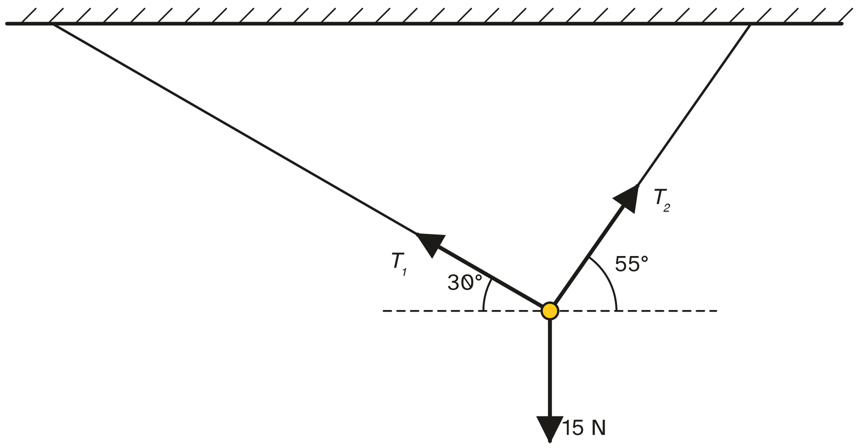


Equilibrium

A particle is in **equilibrium** when the resultant force on it is zero. It then stays at rest (or moves at constant velocity). Zero resultant means **the components in every direction add to zero**, so you can write one equation for each of two directions:

$$\text{sum of components in direction } 1 = 0, \qquad \text{sum of components in direction } 2 = 0.$$

Two equations can find two unknowns (two forces, or a force and an angle). Pick directions that make the algebra easy: horizontal and vertical for strings, along and perpendicular to the plane for slopes. Resolving perpendicular to the plane first usually gives R straight away.



Contact forces and friction

When two surfaces touch, each pushes on the other with a **contact force**. We split it into the **normal reaction** R (at right angles to the surface) and **friction** F (along it). The contact force is the resultant of the two: with $R = 30$ N and $F = 12$ N it is $\sqrt{30^2 + 12^2} = 32.3$ N, at $\tan^{-1}\frac{12}{30} = 21.8^\circ$ to the normal.

Friction only grows as much as it needs to, up to a maximum:

$$F \leqslant \mu R,$$

where μ is the **coefficient of friction** between the two surfaces. When friction has reached its maximum, $F = \mu R$: friction is **limiting**, and an object at rest is in **limiting equilibrium**, "on the point of moving" or "about to slip". Once an object slides, friction stays at μR.

For a 5 kg box on a rough horizontal floor with $\mu = 0.4$: $R = 50$ N, so friction can be at most $0.4 \times 50 = 20$ N. A horizontal push of 12 N leaves it at rest with friction 12 N (not 20 N). A push of 25 N is more than friction can match, so the box moves.

A **smooth** surface is a model with no friction at all: the contact force is just R. It is simpler, but it ignores the force that slows real objects, so it only suits very slippery surfaces (such as ice or a well-oiled track).

Newton's third law

Forces come in pairs. If a book presses down on a table with a force of 6 N, the table pushes up on the book with 6 N. The two forces act on **different** objects, so only one of them appears on each force diagram. The same idea tells you that a string pulls equally hard on the objects at both of its ends.

Key facts and formulas

None of these is in the MF19 list of formulae, so you must learn them all.

Formula	Status
Weight $W = mg$, with $g = 10\ \mathrm{m\ s^{-2}}$	Learn it
Components of a force P at angle θ to a direction: $P\cos\theta$ along it, $P\sin\theta$ at right angles	Learn it
On a slope at angle α: weight gives $mg\sin\alpha$ down the slope and $mg\cos\alpha$ into the slope	Learn it
Resultant: $\sqrt{X^2+Y^2}$ at angle $\tan^{-1}\dfrac{	Y
Equilibrium: $\sum(\text{components}) = 0$ in each of two directions	Learn it
Friction: $F \leqslant \mu R$; limiting friction $F = \mu R$	Learn it
Limiting equilibrium on a slope with no other force along it: $\mu = \tan\alpha$	Learn it
Newton's third law: action and reaction are equal in size and opposite in direction	Learn it
$\tan\alpha = \frac{a}{b}$ gives $\sin\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$, $\cos\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$	Learn it

How to do it

In the Paper 4 questions we have tagged for this topic (58 questions, 2021–2025), the types came up this often. 3 of the 58 questions mix two types, so the counts add up to more than 58.

Question type	Questions
Particle held by strings, rods or rings	15
Particle on a rough or smooth inclined plane	14
Forces at a point in equilibrium: find unknown forces or angles	12
Resultant of forces at a point	11
Friction on a horizontal surface	5
Connected particles in equilibrium	4

Every type uses the same core method:

1. **Draw a force diagram** with every force and every angle.
2. **Choose two perpendicular directions** and resolve every force into them.
3. **Write one equation per direction** (sum $= 0$ for equilibrium).
4. **Add** $F = \mu R$ if friction is limiting.
5. **Solve**, keeping full calculator accuracy until the end, and give answers to 3 significant figures (angles to 1 decimal place).

1. Particle held by strings, rods or rings (15 questions)

1. Draw each tension along its string, **away** from the particle, and give each string its own letter (T_1, T_2). Different strings usually have different tensions.
2. Resolve horizontally and vertically. The weight only appears in the vertical equation.
3. Solve the pair of equations. A common route: make one tension the subject of the horizontal equation and substitute into the vertical one.

For example, a particle of weight 12 N hangs on a string held at 35° to the vertical by a horizontal force P. Vertically: $T\cos 35^\circ = 12$, so $T = 14.6$ N. Horizontally: $P = T\sin 35^\circ = 12\tan 35^\circ = 8.40$ N.

Variations you will meet:

- **Strings over smooth pulleys with blocks hanging from them**: the tension in each string equals the weight of the block hanging from it. Resolve at the knot where the strings meet.
- **A smooth ring threaded on one string**: the tension is the **same** on both sides of the ring, so there is only one unknown tension.
- **A string perpendicular to a force**, or a ring on a circular wire: resolving along and perpendicular to the string (or the radius) can remove one unknown at once.
- **Rods and struts** can push (thrust) as well as pull. Give the magnitude as a positive number.
- **"Find the greatest force for which the particle stays in this position"**: a string cannot push, so the limit is when one tension becomes zero. Set that tension to 0 in your equations.

- **Lengths instead of angles** (for example a string of length 1.7 m whose end hangs 1.5 m below the ceiling): find $\sin$ and $\cos$ of the angle exactly from the lengths, here $\cos\theta = \frac{15}{17}$ and $\sin\theta = \frac{8}{17}$ with the vertical.

2. Particle on a rough or smooth inclined plane (14 questions)

1. Resolve **perpendicular** to the plane to get R. Include the component of any applied force that is not along the plane: a pull that lifts away from the plane makes R **smaller**; a horizontal push into the plane makes it **bigger**.
2. Decide which way friction acts. "About to slip **down**" means friction acts **up** the slope; "about to move **up**" means it acts **down** the slope.
3. Resolve **along** the plane.
4. Use $F = \mu R$ and solve.

Variations you will meet:

- **No force except weight** and the particle is about to slip: $F = mg\sin\alpha$ and $R = mg\cos\alpha$, so $\mu = \tan\alpha$.
- **Least and greatest force** that keeps the particle at rest: the least is when it is about to slip down (friction up the slope); the greatest is when it is about to move up (friction down the slope). Solve twice with F in opposite directions. Example 3 does this.
- **Find** μ: resolve both ways to get F and R as numbers, then $\mu = \frac{F}{R}$.
- **Unknown angle** θ with the force also at angle θ: after using $F = \mu R$ every term has $\sin\theta$ or $\cos\theta$. Collect them and divide by $\cos\theta$ to get $\tan\theta = k$.
- **"Show that $\mu = \dfrac{a\tan\theta - b}{c + \tan\theta}$"**: write $\mu = \dfrac{F}{R}$ in terms of $\sin\theta$ and $\cos\theta$, then divide the top and bottom by $\cos\theta$, and say that you did.
- **"Find the set of possible values of θ"**: a coefficient of friction must be positive, so set the numerator > 0 and solve for $\tan\theta$. Remember that θ is acute, so $\theta < 90^\circ$.
- **"Determine whether the particle moves"**: find the friction needed for equilibrium and compare it with the maximum μR. If the friction needed is smaller, it stays at rest.

3. Forces at a point in equilibrium: find unknown forces or angles (12 questions)

1. Resolve every force in two directions (along the axes, or along one of the forces) and set each sum to 0.
2. **Two unknowns P and θ in one force**: the equations give $P\cos\theta = a$ and $P\sin\theta = b$. Then $P = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$.
3. **Two unknown forces P and Q** at known angles: often one equation contains only Q; solve it first, then substitute into the other.
4. **Unknown angle with known forces**: one equation gives $\sin\theta$ (or $\cos\theta$) directly; use it, then find the remaining force from the other equation.

Variations you will meet:

- **Angles given as** $\sin\alpha = 0.6$ **or** $\tan\alpha = \frac{12}{35}$: use the exact $\sin$ and $\cos$ from a right-angled triangle.
- **Show-that equations** such as $\frac{a}{b-P} + \frac{c}{P+d} = 1$: they come from writing $\sin\theta$ and $\cos\theta$ from the two equations and using $\sin^2\theta + \cos^2\theta = 1$, or from dividing to get $\tan\theta$.
- **"Verify that** $P = \ldots$ **satisfies this equation"**: substitute it and show both sides are equal.

4. Resultant of forces at a point (11 questions)

1. Resolve every force along x and y (or along one given force and at right angles to it). Add them up, with signs, to get X and Y.
2. Magnitude $\sqrt{X^2 + Y^2}$; angle $\tan^{-1}\frac{|Y|}{|X|}$.
3. **Describe the direction** so it can only be drawn one way, for example "25.3° below the positive x-axis" or a sketch with the angle marked.

Variations you will meet:

- **"The resultant has no component in the y-direction"** (or acts along the x-axis): set $Y = 0$ to find the unknown, then the resultant is just X.
- **"The resultant is perpendicular to the force P"**: the components along P add to 0. Resolve along P and at right angles to it.
- **"The single extra force that produces equilibrium"**: it is equal in size to the resultant and points the **opposite** way.
- **One force replaced by P in the same direction** so that the resultant points along an axis: set the other component to 0 with P in place of the old value.

5. Friction on a horizontal surface (5 questions)

1. Resolve **vertically** for R. A pull T at angle θ **above** the horizontal lifts: $R = mg - T\sin\theta$. A push at θ **below** the horizontal presses down: $R = mg + T\sin\theta$.
2. Resolve **horizontally**: $F = T\cos\theta$ for equilibrium.
3. On the point of moving, $F = \mu R$. Substitute R and solve for T (or μ).

A horizontal force alone gives $R = mg$ and friction equal to the force, so in limiting equilibrium $\mu = \frac{\text{force}}{mg}$. If the object then accelerates, the same R and $F = \mu R$ go into Newton's second law (the Newton's laws topic).

6. Connected particles in equilibrium (4 questions)

1. The string is light and the pulley smooth, so the tension T is the same all along the string. For a hanging particle at rest, $T = mg$.
2. For the particle on the table or slope, find R and resolve along the surface with T in it.
3. **Range of masses for equilibrium**: friction can act either way. The hanging mass is **largest** when the other particle is about to be pulled up (friction down the slope) and **smallest** when it is about to slide down (friction up the slope). Use $F = \mu R$ in each case.
4. **Find μ in limiting equilibrium**: find T from the hanging particle, then F and R, then $\mu = \frac{F}{R}$.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **DM1** method mark that needs the M mark before it.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

Three coplanar forces act at a point O: a force of 30 N along the positive x-axis, a force of 24 N at 50° above the negative x-axis, and a force of 12 N along the negative y-axis.

(a) Find the magnitude and direction of the resultant of the three forces. **[5]**

(b) The 12 N force is replaced by a force of P N in the same direction. The resultant now acts along the positive x-axis. Find P. **[2]**

(a) Resolve along x and y. **M1** for resolving every force in one direction.

$$X = 30 - 24\cos 50^\circ = 14.573\ldots, \qquad Y = 24\sin 50^\circ - 12 = 6.385\ldots$$

A1 A1

Magnitude $= \sqrt{14.573^2 + 6.385^2} = 15.9$ N. **M1**

$\theta = \tan^{-1}\dfrac{6.385}{14.573} = 23.7^\circ$, so the resultant is 15.9 N at 23.7° above the positive x-axis. **A1** (both values and a clear direction)

(b) No y-component: $24\sin 50^\circ - P = 0$. **M1** So $P = 18.4$. **A1**

Example 2

A particle of mass 1.5 kg hangs in equilibrium from two light inextensible strings attached to a horizontal ceiling. The strings make angles of 30° and 55° with the horizontal, and their tensions are T_1 N and T_2 N respectively. Find T_1 and T_2. **[5]**

The weight is $1.5 \times 10 = 15$ N (the third diagram in "Understand it").

Horizontally: $T_1\cos 30^\circ = T_2\cos 55^\circ$. **M1 A1**

Vertically: $T_1\sin 30^\circ + T_2\sin 55^\circ = 15$. **A1**

From the first, $T_1 = \dfrac{T_2 \cos 55^\circ}{\cos 30^\circ} = 0.6623\,T_2$. Substitute: $0.3312\,T_2 + 0.8192\,T_2 = 15$, so $T_2 = 13.0$. **M1** for solving the pair.

Then $T_1 = 0.6623 \times 13.04 = 8.64$. **A1** for both: $T_1 = 8.64$ N, $T_2 = 13.0$ N.

Example 3

A block of mass 4 kg rests on a rough plane inclined at 20° to the horizontal. The coefficient of friction between the block and the plane is 0.3. A force of magnitude P N acts on the block at 15° above a line of greatest slope, in the vertical plane containing that line. Find the least and greatest values of P for which the block stays in equilibrium. **[8]**

The first diagram in "Understand it" shows the least case.

Perpendicular to the plane: $R + P\sin 15^\circ = 40\cos 20^\circ$, so $R = 40\cos 20^\circ - P\sin 15^\circ$. **M1 A1**

Least P: the block is about to slide down, so friction acts up the slope.

$$P\cos 15^\circ + F = 40\sin 20^\circ$$

M1 A1. With $F = 0.3R$: **DM1**

$$P\cos 15^\circ + 0.3(40\cos 20^\circ - P\sin 15^\circ) = 40\sin 20^\circ \ \Rightarrow\ P = \frac{40\sin 20^\circ - 12\cos 20^\circ}{\cos 15^\circ - 0.3\sin 15^\circ} = 2.71.$$

A1

Greatest P: the block is about to move up, so friction acts down the slope. **M1**

$$P\cos 15^\circ = 40\sin 20^\circ + 0.3(40\cos 20^\circ - P\sin 15^\circ) \ \Rightarrow\ P = \frac{40\sin 20^\circ + 12\cos 20^\circ}{\cos 15^\circ + 0.3\sin 15^\circ} = 23.9.$$

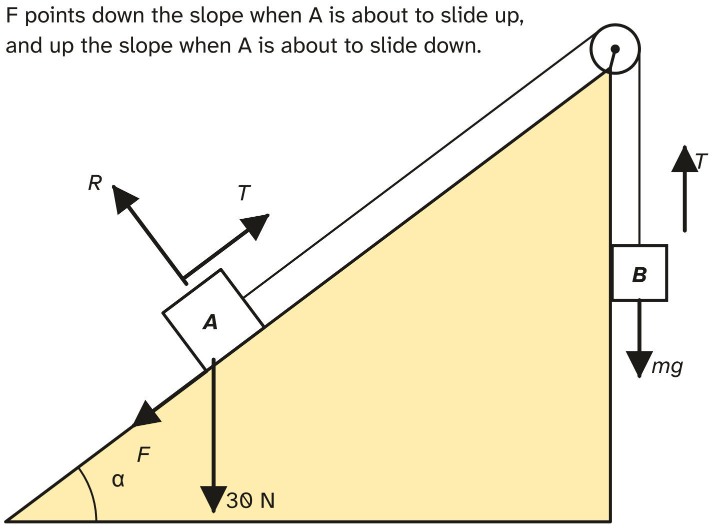
A1. So $2.71 \leqslant P \leqslant 23.9$.

Example 4

Particle A, of mass 3 kg, rests on a rough plane inclined at angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. A light inextensible string, parallel to a line of greatest slope, joins A to particle B of mass m kg. The string passes over a small smooth pulley at the top of the plane, and B hangs freely. The coefficient of friction between A and the plane is 0.5.

(a) Given that the system is in equilibrium, find the set of possible values of m. **[6]**

(b) Given that $m = 2.4$, find the friction force on A and state its direction. **[2]**



$\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$.

(a) For B: $T = mg = 10m$. **B1**

For A, perpendicular to the plane: $R = 30 \times \frac{4}{5} = 24$. **B1** So the greatest friction is $0.5 \times 24 = 12$ N. The weight component down the slope is $30 \times \frac{3}{5} = 18$ N.

About to move up (friction down the slope): $T = 18 + 12 = 30$. About to slide down (friction up the slope): $T = 18 - 12 = 6$. **M1 A1** for resolving along the slope in both cases.

$6 \leqslant 10m \leqslant 30$, so $0.6 \leqslant m \leqslant 3$. **M1 A1**

(b) $T = 24$. Along the slope, $24 = 18 + F$, so $F = 6$ N. **M1** It acts **down** the slope (the string pulls harder than the weight component). **A1** Since $6 < 12$, friction is not limiting, which agrees with (a).

Common mistakes

- **Mixing up sin and cos.** Examiners report this every year. Mark the angle on your diagram and use cos for the side next to it, sin for the side across from it.
- **Missing a component in R.** A slanted pull or a horizontal push changes the normal reaction. Writing $R = mg$ or $R = mg\cos\alpha$ when another force has a component at right angles to the surface is the most common error in friction questions.
- **Friction the wrong way.** Decide first which way the object is about to move; friction opposes it. For "least and greatest", friction changes direction between the two cases.
- **Using $F = \mu R$ when friction is not limiting.** Only use it when the question says limiting, about to slip or on the point of moving (or the object is sliding). Otherwise find F from resolving.
- **Assuming all strings have the same tension.** Two different strings have different tensions. Only one string over a smooth pulley, or through a smooth ring, keeps the same tension.
- **Tension instead of thrust.** A strut can push. If your "tension" comes out negative, give its magnitude and call it a thrust.

- **Rounding too early.** Using a rounded angle (such as 36.9° for $\tan\alpha = \frac{3}{4}$) or a rounded intermediate value can shift the final answer. Use exact $\sin$ and $\cos$ values when they are given and keep full accuracy until the end.
- **Direction of the resultant not clear.** An angle on its own is not enough. Say what it is measured from, and which side.
- **Wrong conclusion in "does it move?"** questions. If the friction needed is less than the maximum, the particle stays at rest; it does not "move the other way".
- **Too few figures.** $\mu = 0.05$ is not 3 significant figures; write 0.0500.

How to get the marks

- **Resolving marks** (usually **M1**) need an equation with the right number of terms: every force that has a component in that direction must appear, resolved if it needs to be. Sign errors and a sin/cos slip may still earn the method mark but lose the accuracy mark.
- **Write each resolved equation separately** and say which direction it is ("perpendicular to the plane:", "horizontally:") before solving.
- **Friction:** the mark for $F = \mu R$ needs R from your perpendicular equation, not from a guess.
- **"Show that"** (for example a given μ or a formula for μ): every step must be seen, including the step where you divide by $\cos\theta$. The last line must match the printed answer.
- **"Find the magnitude and direction"**: both are needed for the final mark, and the direction must be describable one way only.
- **"Set of values"**: give an inequality, such as $0.6 \leqslant m \leqslant 3$, with strict or non-strict signs as the situation needs.
- **Accuracy:** 3 significant figures for forces and μ, 1 decimal place for angles in degrees, unless the question asks otherwise. Exact answers (fractions, surds) are also accepted.
- **Draw the diagram when asked** ("draw a diagram showing the forces"): every force with an arrow and a label; the mark needs all of them and no extra ones.

Quick check

1. Three coplanar forces act at a point and are in equilibrium: 20 N along the positive x-axis, 14 N at 60° above the positive x-axis, and X N at θ° below the negative x-axis. Find X and θ.
2. A box of mass 8 kg rests on rough horizontal ground, where $\mu = 0.4$. A rope pulls on it with tension T N at 25° above the horizontal, and the box is on the point of moving. Find T.
3. A particle of mass 2 kg rests on a rough plane inclined at 30° to the horizontal, where $\mu = 0.7$. Show that it stays at rest, and find the friction force on it.
4. A smooth ring of mass 0.5 kg is threaded on a light string whose ends are fixed. The ring is held in equilibrium by a horizontal force of X N, with the two parts of the string making angles of 40° and 70° with the vertical on opposite sides of the ring. Find the tension in the string and X.

Answers

1. Horizontally: $X\cos\theta = 20 + 14\cos 60^\circ = 27$. Vertically: $X\sin\theta = 14\sin 60^\circ = 12.12$. So $X = \sqrt{27^2 + 12.12^2} = 29.6$ and $\theta = \tan^{-1}\dfrac{12.12}{27} = 24.2$.
2. Vertically: $R = 80 - T\sin 25^\circ$. Horizontally: $F = T\cos 25^\circ$. Limiting: $T\cos 25^\circ = 0.4(80 - T\sin 25^\circ)$, so $T = \dfrac{32}{\cos 25^\circ + 0.4\sin 25^\circ} = 29.8$.
3. $R = 20\cos 30^\circ = 17.32$ N, so friction can be up to $0.7 \times 17.32 = 12.1$ N. To stop it sliding, friction must balance $20\sin 30^\circ = 10$ N. Since $10 < 12.1$, it stays at rest, and the friction force is 10 N up the slope.
4. The tension T is the same on both sides of a smooth ring. Vertically: $T\cos 40^\circ + T\cos 70^\circ = 5$, so $T = 4.51$ N. Horizontally: $X = T\sin 70^\circ - T\sin 40^\circ = 1.34$ N.

Past questions to try

- **9709/42/M/J/25 Q3**: a string and a force at right angles to it.
- **9709/42/O/N/24 Q4**: two particles held by three strings.
- **9709/41/O/N/24 Q6**: the least horizontal force on a rough slope.
- **9709/43/O/N/24 Q2**: the extra force that gives equilibrium.