Cambridge AS & A Level · Mathematics 9709

Mathematics 9709 formulas and definitions

All 329 formulas, facts and definitions to learn, by syllabus topic. Formulas the exam gives you are left out.

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1 Pure Mathematics 1

1.1 Quadratics

Discriminant $b^2 - 4ac$: > 0 two distinct real roots, $= 0$ one repeated root, < 0 no real roots

Real roots (distinct or repeated): $b^2 - 4ac \geqslant 0$

$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

$y = a(x + p)^2 + q$ has vertex $(-p,\ q)$ and line of symmetry $x = -p$; a minimum if $a > 0$, a maximum if $a < 0$

Line of symmetry of $y = ax^2 + bx + c$: $x = -\dfrac{b}{2a}$

Line and curve: two points $\Leftrightarrow b^2 - 4ac > 0$; tangent $\Leftrightarrow b^2 - 4ac = 0$; do not meet $\Leftrightarrow b^2 - 4ac < 0$ (for the equation made by eliminating y)

For $\alpha < \beta$: $(x - \alpha)(x - \beta) < 0 \Leftrightarrow \alpha < x < \beta$, and $(x - \alpha)(x - \beta) > 0 \Leftrightarrow x < \alpha$ or $x > \beta$

1.2 Functions

$gf(x) = g\big(f(x)\big)$: f acts first

gf exists only if the range of f lies inside the domain of g

Only a one-one function has an inverse; $f^{-1}f(x) = ff^{-1}(x) = x$

Domain of f^{-1} = range of f; range of f^{-1} = domain of f

The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections in $y = x$

$y = f(x) + a$: translation $\begin{pmatrix} 0 \\ a \end{pmatrix}$

$y = f(x + a)$: translation $\begin{pmatrix} -a \\ 0 \end{pmatrix}$

$y = af(x)$: stretch parallel to the y-axis, scale factor a

$y = f(ax)$: stretch parallel to the x-axis, scale factor $\frac{1}{a}$

$y = -f(x)$: reflection in the x-axis; $y = f(-x)$: reflection in the y-axis

$a(x+p)^2 + q$ has its vertex at $(-p, q)$: the least value is q if $a > 0$, the greatest value is q if $a < 0$

1.3 Coordinate geometry

Gradient of AB: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Midpoint of AB: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Length of AB: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Line through (x_1, y_1) with gradient m: $y - y_1 = m(x - x_1)$

Parallel: $m_1 = m_2$. Perpendicular: $m_1 m_2 = -1$, so $m_2 = -\dfrac{1}{m_1}$

Circle, centre (a, b), radius r: $(x-a)^2 + (y-b)^2 = r^2$

$x^2 + y^2 + 2gx + 2fy + c = 0$: centre $(-g, -f)$, radius $\sqrt{g^2 + f^2 - c}$

Tangent $\perp$ radius at the point of contact; the normal passes through the centre

The perpendicular from the centre to a chord bisects the chord

Angle in a semicircle $= 90^\circ$; a right-angled triangle's hypotenuse is a diameter of its circle

Line and circle: substitute, then $b^2 - 4ac > 0$ two points, $= 0$ tangent, < 0 no meeting

1.4 Circular measure

π radians $= 180^\circ$; degrees $\times \frac{\pi}{180}$ = radians; radians $\times \frac{180}{\pi}$ = degrees

Perimeter of a sector $= 2r + r\theta$

Area of a triangle $= \frac{1}{2}ab\sin C$, so triangle $OAB = \frac{1}{2}r^2 \sin\theta$

Minor segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

Chord $AB = 2r\sin\frac{\theta}{2}$

Reflex angle $= 2\pi - \theta$; major sector $= \frac{1}{2}r^2(2\pi - \theta)$

Sine rule $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$; cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$

Exact values of $\sin$, $\cos$, $\tan$ at $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$

1.5 Trigonometry

Exact values of $\sin$, $\cos$, $\tan$ at $0, 30^\circ, 45^\circ, 60^\circ, 90^\circ$ (table above)

$\sin(180^\circ - \theta) = \sin\theta$, $\cos(360^\circ - \theta) = \cos\theta$, $\tan(180^\circ + \theta) = \tan\theta$

Signs: all positive (first quadrant), $\sin$ (second), $\tan$ (third), $\cos$ (fourth)

$\sin x$, $\cos x$ have period 360° (2π) and lie between -1 and 1; $\tan x$ has period 180° (π) and asymptotes at $x = 90^\circ, 270^\circ, \ldots$

$y = a\sin(bx) + c$: greatest $c + a$, least $c - a$ (for $a > 0$), period $\frac{360^\circ}{b}$ or $\frac{2\pi}{b}$

$180^\circ = \pi$ radians

1.6 Series

The term in b^r is $\binom{n}{r}a^{n-r}b^r$

$d = u_2 - u_1$ and $r = \dfrac{u_2}{u_1}$

a, b, c in AP: $2b = a + c$; a, b, c in GP: $b^2 = ac$

A GP converges (has a sum to infinity) only when $-1 < r < 1$

$u_n = S_n - S_{n-1}$; the sum of the terms after the kth is $S_\infty - S_k$

1.7 Differentiation

Constant multiples and sums: $\dfrac{d}{dx}\big(kf(x)\big) = kf'(x)$; differentiate term by term; a constant gives 0

Chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$

Tangent at (x_1, y_1) with gradient m: $y - y_1 = m(x - x_1)$

Normal gradient $= -\dfrac{1}{m}$

Stationary point: $\dfrac{dy}{dx} = 0$

$\dfrac{d^2y}{dx^2} < 0$: maximum; $\dfrac{d^2y}{dx^2} > 0$: minimum

Increasing: $\dfrac{dy}{dx} > 0$; decreasing: $\dfrac{dy}{dx} < 0$

Connected rates: $\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$, and $\dfrac{dx}{dt} = \dfrac{dy}{dt} \div \dfrac{dy}{dx}$

A line making angle θ with the positive x-axis has gradient $\tan\theta$

Cylinder: $V = \pi r^2 h$, curved surface $2\pi rh$; circle: $A = \pi r^2$, circumference $2\pi r$; cube: $V = x^3$

1.8 Integration

$\displaystyle\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + c \quad (n \neq -1)$

$\displaystyle\int kf(x) \, dx = k\int f(x) \, dx$, and integrate sums term by term

$\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an integral of f

Area under a curve: $\int_a^b y\,dx$; area next to the y-axis: $\int_c^d x\,dy$

Area between two graphs: $\int_a^b (y_{\text{top}} - y_{\text{bottom}})\,dx$

Volume about the x-axis: $V = \pi \int_a^b y^2\,dx$

Volume about the y-axis: $V = \pi \int_c^d x^2\,dy$

Washer: $V = \pi \int_a^b (R^2 - r^2)\,dx$

Volume of a cylinder $= \pi r^2 h$; area of a triangle $= \frac{1}{2}bh$; trapezium $= \frac{1}{2}(a+b)h$

2 Pure Mathematics 2

2.1 Algebra

$|x| = x$ if $x \geqslant 0$, and $|x| = -x$ if $x < 0$; so $|x| \geqslant 0$ always

$|a| = |b| \iff a^2 = b^2 \iff a = b$ or $a = -b$

$|x - a| < b \iff a - b < x < a + b$

$|x - a| > b \iff x < a - b$ or $x > a + b$

$y = |ax + b|$: a V with vertex at $\left(-\frac{b}{a}, 0\right)$, y-intercept $|b|$, arm gradients $\pm a$

$p(x) = \text{divisor} \times \text{quotient} + \text{remainder}$, with the remainder of lower degree than the divisor

Remainder theorem: dividing $p(x)$ by $(x - a)$ leaves $p(a)$; by $(ax + b)$ leaves $p\left(-\frac{b}{a}\right)$

Factor theorem: $(x - a)$ is a factor $\iff p(a) = 0$; $(ax + b)$ is a factor $\iff p\left(-\frac{b}{a}\right) = 0$

$b^2 - 4ac < 0$: the quadratic has no real roots

2.2 Logarithmic and exponential functions

$\log_a b = c \iff a^c = b$

$\log_a(xy) = \log_a x + \log_a y$

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

$\log_a(x^n) = n \log_a x$

$\log_a a = 1$, $\log_a 1 = 0$; in particular $\ln e = 1$, $\ln 1 = 0$

$\ln x = \log_e x$, with $e = 2.718\ldots$

$\ln(e^x) = x$ and $e^{\ln x} = x$ (for $x > 0$)

$e^x > 0$ for every x; $\ln x$ exists only for $x > 0$

$y = kx^n \Rightarrow \ln y = \ln k + n \ln x$

$y = k \times a^x \Rightarrow \ln y = \ln k + x \ln a$

Straight line: gradient $= \dfrac{Y_2 - Y_1}{X_2 - X_1}$, and $Y = mX + c$

2.3 Trigonometry

$\sec\theta = \dfrac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

$\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A)$ and $\sin^2 A \equiv \frac{1}{2}(1 - \cos 2A)$

$a\sin\theta \pm b\cos\theta$ or $a\cos\theta \pm b\sin\theta$ in R form: $R = \sqrt{a^2 + b^2}$, $\tan\alpha = \dfrac{b}{a}$

$R\sin(\theta \pm \alpha) + c$ and $R\cos(\theta \pm \alpha) + c$ lie between $c - R$ and $c + R$

sec and cosec: period 360° (2π), values never between -1 and 1; cot: period 180° (π)

Exact values of $\sin$, $\cos$, $\tan$ at 30°, 45°, 60° (Paper 1 note)

2.4 Differentiation

Chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{d}{dx}\, e^{f(x)} = f'(x)\, e^{f(x)}$, $\quad \dfrac{d}{dx} \ln f(x) = \dfrac{f'(x)}{f(x)}$

$\dfrac{d}{dx}\sin(ax + b) = a\cos(ax + b)$, and the same pattern for $\cos$ and $\tan$

Implicit: $\dfrac{d}{dx}\big(f(y)\big) = f'(y)\dfrac{dy}{dx}$, for example $\dfrac{d}{dx}\left(y^2\right) = 2y\dfrac{dy}{dx}$

$\dfrac{d}{dx}(xy) = y + x\dfrac{dy}{dx}$

Tangent: $y - y_1 = m(x - x_1)$; normal gradient $-\dfrac{1}{m}$

Stationary point: $\dfrac{dy}{dx} = 0$ (for a fraction, its numerator $= 0$)

Tangent parallel to the y-axis: the denominator of $\dfrac{dy}{dx}$ is 0 ($\dfrac{dx}{dt} = 0$ for parametric curves)

$\sec x = \dfrac{1}{\cos x}$

$e^x > 0$ for every x; $\ln e = 1$; $\ln 1 = 0$; $e^{\ln k} = k$

2.5 Integration

$$\int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + c$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c,\ \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c,$$
$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$$

$\cos^2 A = \frac{1}{2}(1+\cos 2A),\ \sin^2 A = \frac{1}{2}(1-\cos 2A)$

$\tan^2 A = \sec^2 A - 1,\ \sin A\cos A = \frac{1}{2}\sin 2A$

Trapezium rule: $\int_a^b y\,dx \approx \frac{1}{2}h\left[y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n\right],\ h = \frac{b-a}{n}$

Curve bends upwards: over-estimate; bends downwards: under-estimate

Area $= \int_a^b (y_{\text{top}} - y_{\text{bottom}})\,dx$; volume about the x-axis $= \pi\int_a^b y^2\,dx$

$\ln a + \ln b = \ln ab,\ \ln a - \ln b = \ln \frac{a}{b},\ k\ln a = \ln a^k,\ e^{\ln k} = k$

2.6 Numerical solution of equations

If f is continuous and $f(a)$, $f(b)$ have opposite signs, then $f(x) = 0$ has a root between a and b

Iterative formula: $x_{n+1} = F(x_n)$, starting from x_1

If $x_n \to \alpha$, then $\alpha = F(\alpha)$: the limit is a root of $x = F(x)$

Checking an answer $\alpha \approx c$: a sign change of $x - F(x)$ between $c-h$ and $c+h$, where h is half a unit in the last place (for example $h = 0.005$ for 2 decimal places)

$\ln e^k = k$, $e^{\ln k} = k$, $\ln a + \ln b = \ln ab$, $\ln a - \ln b = \ln \frac{a}{b}$, $k\ln a = \ln a^k$

$\sec x = \frac{1}{\cos x}$; $|x-a| = x - a$ when $x \geqslant a$ and $a - x$ when $x < a$

3 Pure Mathematics 3

3.1 Algebra

$(a+bx)^n = a^n\left(1 + \frac{b}{a}x\right)^n$, valid for $|x| < \left|\frac{a}{b}\right|$

The expansion of $(1+\text{term})^n$ is valid when $|\text{term}| < 1$; for a sum or product of expansions, use the strictest condition

Partial fraction forms (the table in "Understand it"), with a constant in front when top and bottom have the same degree

$p(x) \equiv \text{divisor} \times \text{quotient} + \text{remainder}$, with the remainder of lower degree than the divisor

Remainder theorem: dividing $p(x)$ by $(ax+b)$ leaves $p\left(-\frac{b}{a}\right)$

Factor theorem: $(ax+b)$ is a factor $\iff p\left(-\frac{b}{a}\right) = 0$

$|a| = |b| \iff a^2 = b^2$; $|x - a| < b \iff a - b < x < a + b$

3.2 Logarithmic and exponential functions

$\log_a b = c \iff a^c = b$

$\log_a(pq) = \log_a p + \log_a q$

$\log_a\left(\frac{p}{q}\right) = \log_a p - \log_a q$

$\log_a(p^n) = n \log_a p$, so $\frac{1}{2}\ln p = \ln \sqrt{p}$ and $-\ln p = \ln \frac{1}{p}$

$\log_a a = 1$, $\log_a 1 = 0$; $\ln e = 1$, $\ln 1 = 0$; $n = \log_a a^n$

$\ln(e^x) = x$ and $e^{\ln x} = x$ (for $x > 0$)

$e^x > 0$ and $a^x > 0$ for every x; $\ln x$ exists only for $x > 0$

$e^{2x} = (e^x)^2$, $e^{-x} = \dfrac{1}{e^x}$, $a^{x+n} = a^n \times a^x$

$y = Ae^{kx}$: growth if $k > 0$, decay if $k < 0$; doubling (or halving) time $\dfrac{\ln 2}{|k|}$

$y = Ax^m \Rightarrow \ln y = \ln A + m \ln x$

$y = Ab^x \Rightarrow \ln y = \ln A + x \ln b$

Straight line: gradient $= \dfrac{Y_2 - Y_1}{X_2 - X_1}$, and $Y = mX + c$

3.3 Trigonometry

$\sec\theta = \dfrac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

$\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A)$ and $\sin^2 A \equiv \frac{1}{2}(1 - \cos 2A)$

$(\cos\theta \pm \sin\theta)^2 \equiv 1 \pm \sin 2\theta$

R form: $R = \sqrt{a^2 + b^2}$, $\tan\alpha = \dfrac{b}{a}$ (from matching $R\cos\alpha$ and $R\sin\alpha$)

$R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$ lie between $-R$ and R

$\sec$ and cosec: period 2π, never between -1 and 1; $\cot$: period π

Exact values of $\sin$, $\cos$, $\tan$ at $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ (30°, 45°, 60°)

3.4 Differentiation

Chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{d}{dx}e^{f(x)} = f'(x)\,e^{f(x)}$, $\dfrac{d}{dx}\ln f(x) = \dfrac{f'(x)}{f(x)}$

$\frac{d}{dx}\tan^{-1} f(x) = \frac{f'(x)}{1+\big(f(x)\big)^2}$, for example $\frac{d}{dx}\tan^{-1}(ax) = \frac{a}{1+a^2x^2}$

Implicit: $\frac{d}{dx}\big(g(y)\big) = g'(y)\frac{dy}{dx}$, $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$

Tangent: $y - y_1 = m(x - x_1)$; normal gradient $-\frac{1}{m}$

$\frac{dy}{dx} = \frac{N}{D}$: tangent parallel to the x-axis where $N = 0$, parallel to the y-axis where $D = 0$

$\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$, $\cot x = \frac{\cos x}{\sin x}$

$-1 \leqslant \sin x \leqslant 1$, $-1 \leqslant \cos x \leqslant 1$, $e^x > 0$

3.5 Integration

$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}$, $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|$

$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)$, $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)$,
$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)$

$\int \frac{1}{(ax+b)^2}\,dx = -\frac{1}{a(ax+b)}$

$\int \tan x\,dx = \ln|\sec x| = -\ln|\cos x|$

$\int f'(x)\big(f(x)\big)^n\,dx = \frac{\big(f(x)\big)^{n+1}}{n+1}\ (n \neq -1)$

$\int_a^b u\frac{dv}{dx}\,dx = \Big[uv\Big]_a^b - \int_a^b v\frac{du}{dx}\,dx$

Substitution: replace dx using $\frac{du}{dx}$, replace every x, change the limits

$\cos^2 A = \frac{1}{2}(1+\cos 2A)$, $\sin^2 A = \frac{1}{2}(1-\cos 2A)$, $\tan^2 A = \sec^2 A - 1$

Partial fraction forms for $(ax+b)(cx+d)(ex+f)$, $(ax+b)(cx+d)^2$ and $(ax+b)(cx^2+d)$

$\ln a + \ln b = \ln ab$, $\ln a - \ln b = \ln \frac{a}{b}$, $k\ln a = \ln a^k$

3.6 Numerical solution of equations

If f is continuous and $f(a)$, $f(b)$ have opposite signs, then $f(x) = 0$ has a root between a and b

Iterative formula: $x_{n+1} = F(x_n)$, starting from x_1

If $x_n \to \alpha$, then $\alpha = F(\alpha)$: the limit is a root of $x = F(x)$

Checking an answer c: a sign change of $x - F(x)$ between $c-h$ and $c+h$, where h is half a unit in the last place ($h = 0.005$ for 2 decimal places, 0.0005 for 3)

$\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$, $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

$\ln e^k = k$, $e^{\ln k} = k$, $\ln a + \ln b = \ln ab$, $k \ln a = \ln a^k$

Triangle area $\frac{1}{2}ab\sin C$, so segment area $= \frac{1}{2}r^2(\theta - \sin\theta)$

3.7 Vectors

$|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}$

Unit vector in the direction of $\mathbf{v}$: $\dfrac{\mathbf{v}}{|\mathbf{v}|}$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ ("end minus start")

Midpoint of AB: $\frac{1}{2}(\mathbf{a} + \mathbf{b})$

Line: $\mathbf{r} = \mathbf{a} + t\mathbf{b}$; through A and B: $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$

Parallel: one direction vector is a multiple of the other

Perpendicular: $\mathbf{a} \cdot \mathbf{b} = 0$

Acute angle between lines: $\cos\theta = \dfrac{|\mathbf{b}_1 \cdot \mathbf{b}_2|}{|\mathbf{b}_1|\,|\mathbf{b}_2|}$

Foot of perpendicular from A: $\overrightarrow{AP} \cdot \mathbf{b} = 0$ for the general point P

Reflection of A in a line with foot F: $\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA}$

Area of triangle ABC: $\frac{1}{2}\,AB \times AC \times \sin A$; parallelogram: twice that

$\sin\theta = \sqrt{1 - \cos^2\theta}$ for $0^\circ < \theta < 180^\circ$

3.8 Differential equations

Separating: $\dfrac{dy}{dx} = f(x)g(y) \Rightarrow \displaystyle\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$

One constant of integration, found from the initial condition

$\ln|y| = f(x) + c \Rightarrow y = Ae^{f(x)}$

"Rate of change of x" $= \dfrac{dx}{dt}$; "proportional to" $= k\times$; "decreasing" means a negative rate

Connected rates: $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$

$\displaystyle\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|$, $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}$

$\int \tan x\,dx = -\ln|\cos x| = \ln|\sec x|$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ (rearranged from the given $\cos 2A$)

Vertical-sided tank with base area A: $V = Ah$; cylinder $V = \pi r^2 h$

3.9 Complex numbers

$\mathrm{i}^2 = -1$; $z = x + \mathrm{i}y$, $\mathrm{Re}\, z = x$, $\mathrm{Im}\, z = y$

$a + \mathrm{i}b = c + \mathrm{i}d \iff a = c$ and $b = d$

$z^* = x - \mathrm{i}y$; $zz^* = x^2 + y^2 = |z|^2$

Divide: multiply top and bottom by the conjugate of the bottom

$(z + w)^* = z^* + w^*$, $(zw)^* = z^* w^*$

$|z| = \sqrt{x^2 + y^2}$; $\tan(\arg z) = \dfrac{y}{x}$, quadrant from a sketch

$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta}$, $x = r\cos\theta$, $y = r\sin\theta$

$|z_1 z_2| = |z_1||z_2|$, $\arg(z_1 z_2) = \arg z_1 + \arg z_2\ (\pm 2\pi)$

$\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $\arg\dfrac{z_1}{z_2} = \arg z_1 - \arg z_2\ (\pm 2\pi)$

Real coefficients: roots $a \pm b\mathrm{i}$ in pairs, factor $z^2 - 2az + a^2 + b^2$

$|z_1 - z_2|$ = distance between the points z_1 and z_2

$|z - a| = k$: circle; $|z - a| = |z - b|$: perpendicular bisector; $\arg(z - a) = \alpha$: half-line from a

Circle centre c, radius r, $d = |c|$: $|z|$ from $d - r$ to $d + r$; extreme $\arg z = \arg c \pm \sin^{-1}\frac{r}{d}$

4 Mechanics

4.1 Forces and equilibrium

Weight $W = mg$, with $g = 10\ \mathrm{m\ s^{-2}}$

Components of a force P at angle θ to a direction: $P\cos\theta$ along it, $P\sin\theta$ at right angles

On a slope at angle α: weight gives $mg\sin\alpha$ down the slope and $mg\cos\alpha$ into the slope

Resultant: $\sqrt{X^2 + Y^2}$ at angle $\tan^{-1}\dfrac{|Y|}{|X|}$ to the x-direction

Equilibrium: $\sum(\text{components}) = 0$ in each of two directions

Friction: $F \leqslant \mu R$; limiting friction $F = \mu R$

Limiting equilibrium on a slope with no other force along it: $\mu = \tan\alpha$

Newton's third law: action and reaction are equal in size and opposite in direction

$\tan\alpha = \frac{a}{b}$ gives $\sin\alpha = \dfrac{a}{\sqrt{a^2 + b^2}}$, $\cos\alpha = \dfrac{b}{\sqrt{a^2 + b^2}}$

4.2 Kinematics of motion in a straight line

$s = vt - \frac{1}{2}at^2$

The suvat formulae need **constant** acceleration

$v = \dfrac{ds}{dt}, \quad a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$

$v = \displaystyle\int a\,dt, \quad s = \int v\,dt$ (find the constant each time)

Gradient of a displacement–time graph = velocity

Gradient of a velocity–time graph = acceleration; area under it = displacement

Greatest or least velocity: $a = 0$; instantaneous rest: $v = 0$

Average speed $= \dfrac{\text{total distance}}{\text{total time}}$

Free motion under gravity: $a = -g = -10$ m s^{-2} (upwards positive)

4.3 Momentum

Momentum $= mv$, in N s (or kg m s^{-1}); a vector, so its sign shows its direction

Conservation of momentum: $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

Coalescing: $m_1u_1 + m_2u_2 = (m_1 + m_2)v$

Kinetic energy $= \frac{1}{2}mv^2$, in J

KE lost in a collision $=$ total KE before $-$ total KE after

From momentum p and kinetic energy E: $u = \dfrac{2E}{p}, m = \dfrac{p}{u}$

Weight $= mg$, with $g = 10$ m s^{-2}

Newton's second law: resultant force $= ma$

Friction when sliding: $F = \mu R$ (on a horizontal floor $R = mg$, so the deceleration is μg)

Sliding freely on a smooth slope at angle α: acceleration $g\sin\alpha$ down the slope

Rough slope: $R = mg\cos\alpha$, $F = \mu R$; acceleration $g\sin\alpha - \mu g\cos\alpha$ sliding down, deceleration $g\sin\alpha + \mu g\cos\alpha$ sliding up

Total energy lost over a motion $=$ (PE $+$ KE at start) $-$ (PE $+$ KE at end) $=$ work against friction $+$ KE lost at impacts

Potential energy gained when rising h: mgh; falling h from rest on a smooth track gives $v^2 = 2gh$

Constant speed (no acceleration): distance $=$ speed $\times$ time

4.4 Newton’s laws of motion

Newton's second law: resultant force $= ma$

Weight $W = mg$, with $g = 10\ \text{m s}^{-2}$

On a slope at angle α: weight gives $mg\sin\alpha$ down the slope and $mg\cos\alpha$ into it

Sliding friction: $F = \mu R$, against the direction of motion

Rough slope: $a = g\sin\alpha - \mu g\cos\alpha$ sliding down; $a = -(g\sin\alpha + \mu g\cos\alpha)$ sliding up, with no other force along it

Starts to slide or slides back only if $\mu < \tan\alpha$

Driving force $= \dfrac{P}{v}$, from power $P = Fv$

Constant speed (or greatest speed): resultant force $= 0$

Connected by a light inextensible string: same tension throughout, same size of acceleration

Newton's third law: equal and opposite forces on the two objects

4.5 Energy, work and power

Work done by a constant force: $W = Fd\cos\theta$, in J

Work done against a resistance F over a distance d: Fd

Kinetic energy: $\mathrm{KE} = \frac{1}{2}mv^2$

Gravitational potential energy: $\mathrm{PE} = mgh$ (h vertical); along a slope, $h = d\sin\alpha$

Work–energy: $\mathrm{KE}_1 + \mathrm{PE}_1 + \mathrm{WD\ by\ driving\ forces} = \mathrm{KE}_2 + \mathrm{PE}_2 + \mathrm{WD\ against\ resistances}$

Smooth, gravity only: $\mathrm{KE} + \mathrm{PE}$ is constant

Average power $= \dfrac{\text{work done}}{\text{time}}$; work done by an engine at constant power $= Pt$

$P = Fv$, so driving force $= \dfrac{P}{v}$

Vehicle uphill: $\dfrac{P}{v} - R - mg\sin\alpha = ma$; top speed when $a = 0$

Friction when sliding: $F = \mu R$

5 Probability & Statistics 1

5.1 Representation of data

Variance $=$ (standard deviation)^2

Coded data: $\bar{x} = a + \dfrac{\sum(x-a)}{n}$; sd $= \sqrt{\dfrac{\sum(x-a)^2}{n} - \left(\dfrac{\sum(x-a)}{n}\right)^2}$

$\sum(x-a) = \sum x - na$

Back to totals: $\sum x = n\bar{x}$, $\sum x^2 = n(\sigma^2 + \bar{x}^2)$

Median: the $\frac{n+1}{2}$th value; Q_1, Q_3: medians of the lower and upper halves

Range $=$ largest $-$ smallest; IQR $= Q_3 - Q_1$

Frequency density $= \dfrac{\text{frequency}}{\text{class width}}$
Midpoint $= \frac{1}{2}$(lower boundary + upper boundary)
From a cumulative frequency graph: median at $\frac{n}{2}$, quartiles at $\frac{n}{4}$ and $\frac{3n}{4}$, pth percentile at $\frac{p}{100}n$

5.2 Permutations and combinations

n different objects in a line: $n!$ ways; $0! = 1$
${}^nP_r = \dfrac{n!}{(n-r)!}$, arrangements of r from n different objects
n objects, p alike, q alike, ...: $\dfrac{n!}{p!\,q!\cdots}$ arrangements
${}^nC_r = {}^nC_{n-r}$, and ${}^nP_r = {}^nC_r \times r!$
Together: glue into a block, then $\times$ the block's inside orders
Not together: total $-$ together; or arrange the rest, then use the $m+1$ gaps around m objects
Exactly k between two items in n places: $n-k-1$ positions for the pair
Scenarios: multiply within a scenario, add across scenarios
Equal-sized unlabelled groups: divide by $r!$ for r groups of the same size

5.3 Probability

Equally likely outcomes: $\mathrm{P}(A) = \dfrac{\text{outcomes in } A}{\text{all outcomes}}$
$\mathrm{P}(A') = 1 - \mathrm{P}(A)$; $\mathrm{P}(\text{at least one}) = 1 - \mathrm{P}(\text{none})$
Exclusive events: $\mathrm{P}(A \text{ or } B) = \mathrm{P}(A) + \mathrm{P}(B)$
Independent events: $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$
Any events: $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B \mid A)$ (multiply along a tree)
Conditional probability: $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$
Exclusive with $\mathrm{P}(A), \mathrm{P}(B) > 0$: never independent

5.4 Discrete random variables

$\sum P(X = x) = 1$
Standard deviation $= \sqrt{\mathrm{Var}(X)}$
The four conditions for a binomial model (fixed n, two outcomes, constant p, independent trials)
$\mathrm{Geo}(p)$: $P(X > n) = (1-p)^n$ and $P(X \leqslant n) = 1 - (1-p)^n$

rth success on trial n: $\binom{n-1}{r-1}p^r(1-p)^{n-r}$

5.5 The normal distribution

$X \sim N(\mu, \sigma^2)$: mean μ, **variance** σ^2

Standardising: $Z = \dfrac{X-\mu}{\sigma}$

$P(Z > a) = 1 - \Phi(a)$; $P(-a < Z < a) = 2\Phi(a) - 1$

$P(X = a) = 0$ for a continuous variable, so $<$ and $\leqslant$ give the same probability

Normal approximation valid when $np > 5$ and $nq > 5$

$B(n, p) \approx N(np,\ npq)$ with a continuity correction (± 0.5)

Expected number $= n\times$ probability

6 Probability & Statistics 2

6.1 The Poisson distribution

Conditions: events occur singly, independently, at random, at a constant mean rate

λ is proportional to the length of the interval (rate $\times$ time)

$P(X = 0) = e^{-\lambda}$, $P(X \geqslant 1) = 1 - e^{-\lambda}$

$X \sim Po(\lambda), Y \sim Po(\mu)$ independent $\Rightarrow X + Y \sim Po(\lambda + \mu)$

$B(n, p) \approx Po(np)$ when $n > 50$ and $np < 5$

$Po(\lambda) \approx N(\lambda, \lambda)$ when $\lambda > 15$, with a continuity correction

Standardising: $z = \dfrac{x-\lambda}{\sqrt{\lambda}}$

6.2 Linear combinations of random variables

$E(aX + b) = aE(X) + b$

$\operatorname{Var}(aX + b) = a^2 \operatorname{Var}(X)$

$E(aX + bY) = aE(X) + bE(Y)$

$\operatorname{Var}(aX + bY) = a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y)$, for independent X and Y

$\operatorname{Var}(X - Y) = \operatorname{Var}(X) + \operatorname{Var}(Y)$ for independent X and Y

Total of n independent values: mean $n\mu$, variance $n\sigma^2$; one value times n: mean $n\mu$, variance $n^2\sigma^2$

X normal $\Rightarrow aX + b$ normal; X, Y independent normal $\Rightarrow aX + bY$ normal

$X \sim Po(\lambda), Y \sim Po(\mu)$ independent $\Rightarrow X + Y \sim Po(\lambda + \mu)$

Standardising: $z = \dfrac{x - \mu}{\sigma}$

6.3 Continuous random variables

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

$f(x) \geqslant 0$ for all x, and $\displaystyle\int f(x)\,dx = 1$ over the whole interval

$P(X = a) = 0$, so $<$ and $\leqslant$ give the same probability

Median m: $\displaystyle\int_{\text{start}}^{m} f(x)\,dx = \tfrac{1}{2}$; lower quartile $\tfrac{1}{4}$, upper quartile $\tfrac{3}{4}$

Graph symmetrical about $x = c$: $E(X) = c$ and median $= c$

$P(X > a) = 1 - P(X < a)$

$\displaystyle\int \cos(ax)\,dx = \frac{\sin(ax)}{a}$, $\displaystyle\int \sin(ax)\,dx = -\frac{\cos(ax)}{a}$

Triangle $\tfrac{1}{2}bh$, trapezium $\tfrac{1}{2}(a+b)h$, circle πr^2

6.4 Sampling and estimation

$E(\bar{X}) = \mu$, $\operatorname{Var}(\bar{X}) = \dfrac{\sigma^2}{n}$; exactly normal if X is normal

Coded data: $\bar{x} = a + \dfrac{\Sigma(x-a)}{n}$, and $s^2 = \dfrac{1}{n-1}\left(\Sigma(x-a)^2 - \dfrac{(\Sigma(x-a))^2}{n}\right)$

Confidence interval for μ: $\bar{x} \pm z\dfrac{\sigma}{\sqrt{n}}$ (use s for σ if unknown and n large)

Confidence interval for p: $\hat{p} \pm z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$, with $\hat{p} = \dfrac{x}{n}$

z for an $\alpha\%$ interval: $\Phi(z) = \frac{1}{2}\left(1 + \frac{\alpha}{100}\right)$; level from z: $\alpha\% = 2\Phi(z) - 1$

Width $= 2 \times z \times$ standard error; centre $=$ the sample mean (or $\hat{p}$)

6.5 Hypothesis tests

Test statistic for a mean: $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$

Critical value of $\bar{x}$: $\mu_0 - z\dfrac{\sigma}{\sqrt{n}}$ (lower tail) or $\mu_0 + z\dfrac{\sigma}{\sqrt{n}}$ (upper tail)

Two-tailed test: compare each tail with $\frac{\alpha}{2}$ (e.g. ± 1.96 at 5%)

Normal approximations: $B(n,p) \approx N\big(np, np(1-p)\big)$; $\operatorname{Po}(\lambda) \approx N(\lambda,\lambda)$ for $\lambda > 15$; with a continuity correction

$P(\text{Type I}) = P(\text{reject } H_0 \mid H_0 \text{ true}) := \alpha$ for a normal test, the rejection region's probability for a discrete one

$$P(\text{Type II}) = P(\text{do not reject } H_0 \mid \text{the true value})$$