Cambridge AS & A Level · Mathematics (9709) · Notes

Functions AS

Read online at pastopic.com/cambridge/mathematics-9709/notes/functions

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Read first: Quadratics

What you need to know

By the end of this topic you should be able to:

1. **Use the language of functions correctly.** Know what a function is, what its domain and range are, what makes a function one-one, and what an inverse function and a composite function are. Read and write the notation $f(x)$, $f : x \mapsto \ldots$, $fg(x)$ and $f^{-1}(x)$.
2. **Work out the range** of a function on the domain it is given, in straightforward cases such as straight lines, quadratics, square roots and simple fractions.
3. **Combine two functions** into a composite, write it as one expression, evaluate it and solve equations with it. Know that gf only exists if every output of f is an input that g accepts.
4. **Decide whether a function is one-one**, and if it is, find its inverse, together with the inverse's domain and range.
5. **Sketch a one-one function and its inverse on the same axes**, as reflections of each other in the line $y = x$, with that line drawn.
6. **Transform graphs.** Know what adding or multiplying by a constant, outside or inside the bracket, does to the graph of $y = f(x)$, including two or three changes in a row. Describe them as *translations*, *reflections* and *stretches*, and go both ways: description to equation, and equations or graphs to description. The graphs may be polynomials, trigonometric graphs or sketches with marked points.

Understand it

What a function is

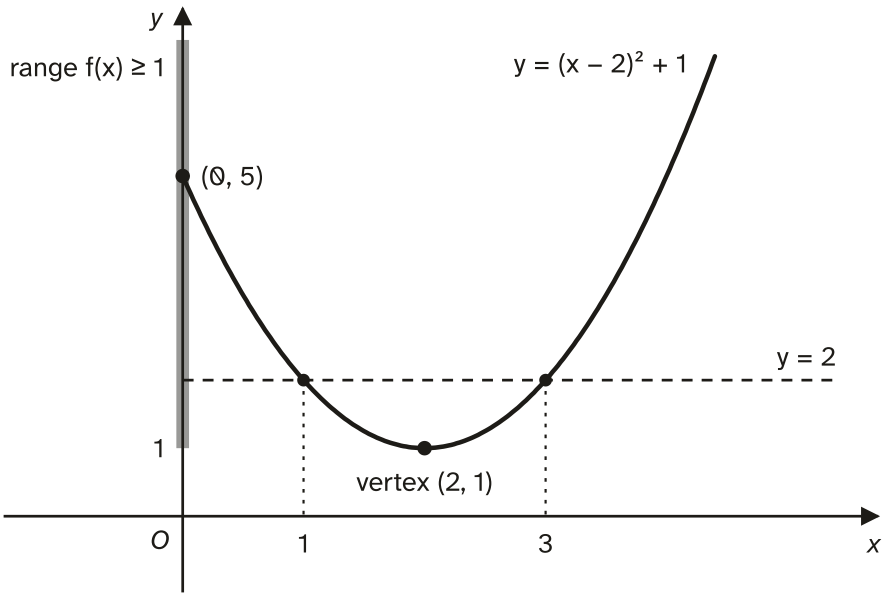
A function is a rule that takes an input and gives back **exactly one** output. We write $f(x) = 2x + 1$, or $f : x \mapsto 2x + 1$, and say "f of x".

- The **domain** is the set of inputs the function is allowed to take. It is part of the definition: "$f(x) = 2x + 1$ for $x \geqslant 0$" is a different function from "$f(x) = 2x + 1$ for $x \in \mathbb{R}$".
- The **range** is the set of outputs that actually come out when every input in the domain goes in. On a graph, the domain is spread along the x-axis and the range along the y-axis.

Because the range is a set of outputs, you write it with $f(x)$ (or y), never with x: "$f(x) \geqslant 1$", not "$x \geqslant 1$".

Finding a range: look at the graph

Picture the graph over the domain only, and read off the lowest and highest y-values. For a quadratic, complete the square first (Quadratics) to see the vertex.



For $f(x) = (x-2)^2 + 1$ with $x \geqslant 0$, the vertex $(2, 1)$ is inside the domain, so the range is $f(x) \geqslant 1$ (the starting value $f(0) = 5$ is not the lowest point). With domain $x \geqslant 3$ the vertex would be cut off: the graph would start at $f(3) = 2$ and only go up, so the range would be $f(x) \geqslant 2$. Always ask: **is the turning point inside the domain?**

One-one functions

A function is **one-one** if no two different inputs give the same output. On a graph, any horizontal line meets it at most once.

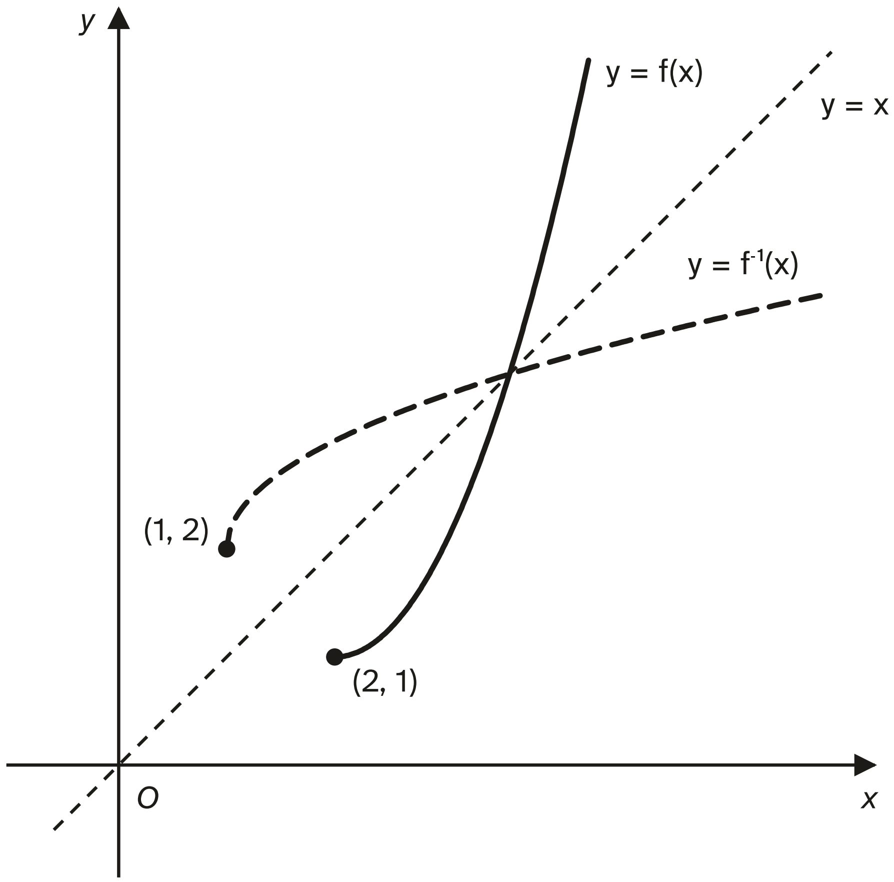
In the diagram, $y = 2$ meets the curve twice ($f(1) = f(3) = 2$), so f with domain $x \geqslant 0$ is **not** one-one: it has a turning point inside the domain. Cut the domain at the vertex, $x \geqslant 2$, and it becomes one-one.

Inverse functions

An inverse function f^{-1} undoes f: if f sends 3 to 2, then f^{-1} sends 2 back to 3. So $f^{-1}f(x) = ff^{-1}(x) = x$. Only one-one functions have inverses: if two inputs gave the same output, the inverse could not know which one to send it back to. Since inputs and outputs swap places:

- the **domain of** f^{-1} is the range of f;
- the **range of** f^{-1} is the domain of f.

Swapping inputs and outputs means swapping x and y on the graph. A point (a, b) on $y = f(x)$ becomes (b, a) on $y = f^{-1}(x)$, and that is a **reflection in the line** $y = x$.



Here $f(x) = (x-2)^2 + 1$ for $x \geqslant 2$ has range $f(x) \geqslant 1$, so $f^{-1}(x) = 2 + \sqrt{x-1}$ has domain $x \geqslant 1$ and range $f^{-1}(x) \geqslant 2$. The range of f^{-1} also tells you which square root to take: it must give values of at least 2, so it is $2 + \sqrt{\dots}$, not $2 - \sqrt{\dots}$.

Composite functions

$gf(x)$ means **do f first, then g**: $gf(x) = g\big(f(x)\big)$. For $f(x) = 2x+1$ and $g(x) = x^2$, $gf(x) = (2x+1)^2$ but $fg(x) = 2x^2 + 1$, so order matters. $ff(x)$ means apply f twice.

gf can only be formed if every output of f is something g is allowed to take: **the range of f must lie inside the domain of g.**

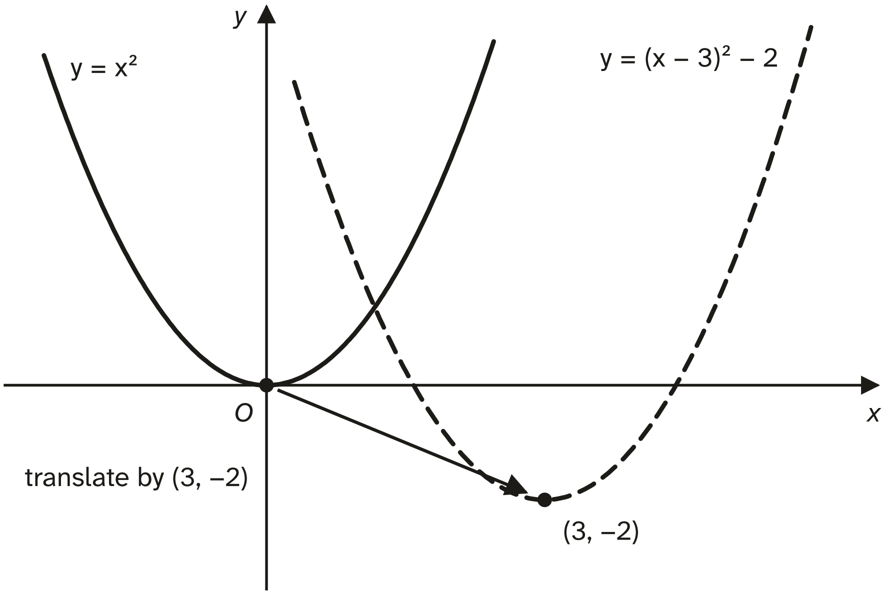
Transformations of graphs

Changes outside the bracket act on y, and do what they say.

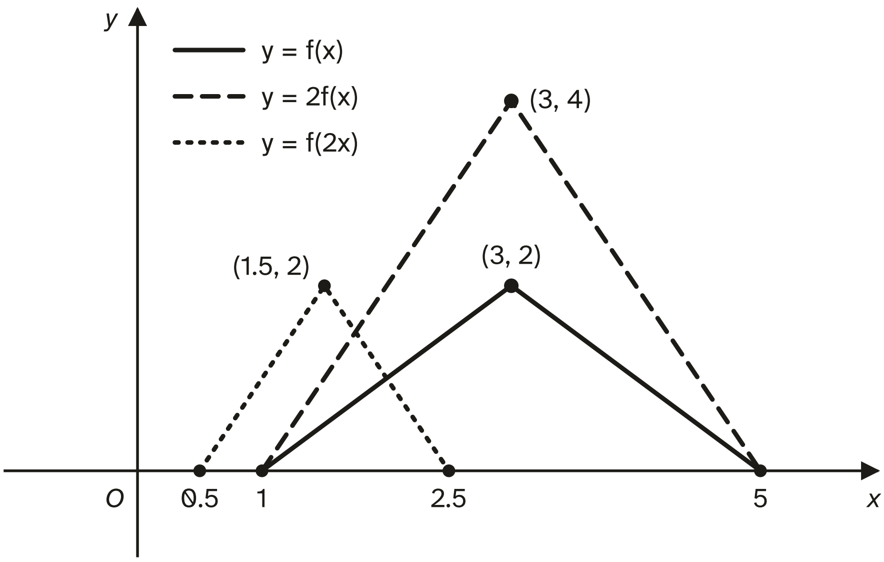
- $y = f(x) + a$: every y-value goes up by a. This is a **translation** by $\begin{pmatrix} 0 \\ a \end{pmatrix}$.
- $y = af(x)$: every y-value is multiplied by a. This is a **stretch parallel to the y-axis, scale factor a.** Points on the x-axis stay put.
- $y = -f(x)$: every y-value changes sign. This is a **reflection in the x-axis**.

Changes inside the bracket act on x, and do the opposite of what they seem to say.

- $y = f(x+a)$: the graph moves **left** by a. This is a translation by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$. Every value f had at $x = 0$ now happens at $x = -a$.
- $y = f(ax)$: the graph is squashed towards the y-axis. This is a **stretch parallel to the x-axis, scale factor $\frac{1}{a}$**. For example, $f(2x)$ at $x = 2$ gives what f gave at $x = 4$. Points on the y-axis stay put.
- $y = f(-x)$: left and right swap. This is a **reflection in the y-axis**.



So $y = (x-3)^2 - 2$ is $y = x^2$ translated by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$: the -3 inside moves it **right** 3, the -2 outside moves it **down** 2.



$y = 2f(x)$ doubles every height (the peak $(3, 2)$ goes to $(3, 4)$); $y = f(2x)$ halves every distance from the y-axis (the peak goes to $(1.5, 2)$).

Doing two or more transformations

Build the equation one step at a time, and change the **current** equation each time:

To do this to the graph	Do this to the equation
Translate by $\begin{pmatrix} p \\ q \end{pmatrix}$	Replace x by $(x-p)$, then add q
Stretch parallel to the y-axis, factor k	Multiply the whole right-hand side by k
Stretch parallel to the x-axis, factor k	Replace x by $\frac{x}{k}$
Reflect in the x-axis	Multiply the whole right-hand side by -1
Reflect in the y-axis	Replace x by $-x$

Changes to x and to y don't affect each other, but **two changes in the same direction depend on order**:

- Outside, follow the arithmetic: $y = 3f(x) + 1$ is "stretch by 3, then translate up 1"; the other way round gives $3f(x) + 3$.
- Inside, the order is reversed: $y = f(2x+6)$ is "translate by $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$, then stretch parallel to the x-axis, factor $\frac{1}{2}$" ($x \to x+6$ gives $f(x+6)$, then $x \to 2x$ gives $f(2x+6)$). If you stretch first, the translation is only $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$, because $f\big(2(x+3)\big) = f(2x+6)$.

Key facts and formulas

None of these are in the MF19 list of formulae, so you must learn them all.

Fact	Status
$gf(x) = g\big(f(x)\big)$: f acts first	Learn it
gf exists only if the range of f lies inside the domain of g	Learn it
Only a one-one function has an inverse; $f^{-1}f(x) = ff^{-1}(x) = x$	Learn it
Domain of f^{-1} = range of f; range of f^{-1} = domain of f	Learn it
The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections in $y = x$	Learn it
$y = f(x) + a$: translation $\begin{pmatrix} 0 \\ a \end{pmatrix}$	Learn it
$y = f(x+a)$: translation $\begin{pmatrix} -a \\ 0 \end{pmatrix}$	Learn it
$y = af(x)$: stretch parallel to the y-axis, scale factor a	Learn it

Fact	Status
$y = f(ax)$: stretch parallel to the x-axis, scale factor $\frac{1}{a}$	Learn it
$y = -f(x)$: reflection in the x-axis; $y = f(-x)$: reflection in the y-axis	Learn it
$a(x+p)^2 + q$ has its vertex at $(-p, q)$: the least value is q if $a > 0$, the greatest value is q if $a < 0$	Learn it

How to do it

In the Paper 1 questions we have tagged for this topic (66 questions, 2021–2025), the types came up this often. Most questions mix two or three types.

Question type	Questions
Describe or carry out transformations of a graph	33
Find an inverse function (and sketch it)	30
Form or use a composite function	30
Find the range of a function	21
Explain whether an inverse or a composite exists	13

1. Describe or carry out transformations of a graph (33 questions)

From an equation to a description (for example, describe how $y = f(x)$ becomes $y = af(bx+c)+d$):

1. Split the changes into **inside** (act on x) and **outside** (act on y).
2. Name each one fully: a **translation** with its vector, a **stretch** with its scale factor *and* direction, a **reflection** with its mirror line.
3. Put them in a correct order (see "Doing two or more transformations") and say it: "first ... then ...".

Give exactly as many transformations as the question asks for. If a quadratic is involved, complete the square first, so you can see the translation: $x^2 - 6x + 11 = (x-3)^2 + 2$ is $y = x^2$ translated by $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

From a description to an equation:

Apply each transformation **in the order given**, using the table in "Understand it", and write the new equation after each step. Expand only at the end.

From two graphs to a description:

1. Find a feature that is easy to match on both graphs: a turning point, a flat point, an intercept or a corner.

2. Compare the size: how far is a second point from that feature, across and up, on each graph? If the heights double but the widths don't change, there is a stretch parallel to the y-axis, factor 2. If the widths halve, there is a stretch parallel to the x-axis, factor $\frac{1}{2}$. If the graph is upside down, there is a reflection in the x-axis.
3. Do the stretch or reflection first, then find the translation that takes the feature to its new position. Check with another point.

Variations you will meet:

- **Where does a point go?** Apply each step to the point's coordinates in order: $y = af(x) + b$ sends (p, q) to $(p,\ aq + b)$; $y = f(x + c)$ sends it to $(p - c,\ q)$; $y = f(cx)$ to $\left(\frac{p}{c},\ q\right)$. If you are given a point on the *new* graph and asked for the original, run the steps **backwards**, undoing each one.
- **A restricted domain moves too.** If f is defined for $x \leqslant 4$, then $y = f\left(\frac{x}{2}\right)$ needs $\frac{x}{2} \leqslant 4$, so its domain is $x \leqslant 8$. Changes outside the bracket leave the domain alone.
- **Turning points** move like any other point; a reflection in the x-axis turns a maximum into a minimum.
- **Trigonometric graphs** follow the same rules: $y = \sin(x - 30^\circ)$ is a translation by $\begin{pmatrix} 30 \\ 0 \end{pmatrix}$.

2. Find an inverse function (30 questions)

1. Write $y = f(x)$. If f is a quadratic, use its completed square form.
2. Rearrange to make x the subject.
   - With a square: isolate the squared bracket, then square root. **Choose the sign** using the domain of f (which is the range of f^{-1}). If the domain is on the right of the vertex, the bracket is positive, so take the positive root; if it is on the left, the bracket is negative, so take the negative root.
   - With a square root such as $y = a\sqrt{x + b} + c$: isolate the root, $\sqrt{x + b} = \frac{y - c}{a}$, then square both sides and rearrange. No sign choice is needed.
   - With a fraction such as $y = \frac{ax + b}{cx + d}$: multiply up, collect the x terms on one side, factorise out x and divide.
3. Swap to x: write $f^{-1}(x) = \ldots$ in terms of x, not y.
4. If asked, give the **domain of** f^{-1}: it is the range of f, so find that first. Write it with x: "$x \geqslant -3$".
5. Check with one value: $f^{-1}\big(f(a)\big)$ should give a.

Sketch the inverse. Draw $y = x$ through the origin at 45° and reflect the curve in it: (a, b) goes to (b, a), end points go to end points, and a curve levelling off towards a horizontal line now levels off towards a vertical one.

Variations you will meet:

- **Solve $f^{-1}(x) = k$.** You don't need f^{-1}: the answer is $x = f(k)$.
- **Solve $f(x) = f^{-1}(x)$.** For an increasing function, the graphs can only meet on $y = x$, so solve $f(x) = x$ instead (and check the answer is in both domains).
- **The inverse of a composite**, $(fg)^{-1}$: form $fg(x)$ first as one expression, then find its inverse in the usual way.

- **A self-inverse function**, $f^{-1} = f$. This means f undoes itself, so $ff(x) = x$, and its graph is its own mirror image in $y = x$.
  - *Show that f is its own inverse* (or verify it for a given constant): find $f^{-1}(x)$ in the usual way, put in any given value of the constant, and simplify until it is **exactly** the same expression as $f(x)$. Say so in words: "$f^{-1}(x) = f(x)$, so f is self-inverse."
  - *Find the constant that makes f self-inverse*: find $f^{-1}(x)$ with the constant still in it, then make it match $f(x)$ term by term. For example, $f(x) = \dfrac{2x+5}{x+k}$: $y(x+k) = 2x+5 \Rightarrow x(y-2) = 5-ky$, so $f^{-1}(x) = \dfrac{-kx+5}{x-2}$. This is the same as $\dfrac{2x+5}{x+k}$ when $-k = 2$ and $k = -2$, so $k = -2$.

3. Form or use a composite function (30 questions)

1. **Get the order right.** $gf(x)$: write $g(\ldots)$ and put the whole of $f(x)$, in brackets, wherever x appears in g.
2. **Simplify.** Expand brackets. If there is a fraction inside a fraction, multiply its top and bottom by the small denominator. For example, with $f(x) = \dfrac{1}{x+1}$:
$$ff(x) = \frac{1}{\frac{1}{x+1}+1} = \frac{x+1}{1+(x+1)} = \frac{x+1}{x+2}.$$
3. **For a number**, such as $gf(3)$, find $f(3)$ first, then put that into g.
4. **For an equation**, such as $gf(x) = 20$ (or just $f(a) = 4$): form the expression, set it equal to the value and solve, showing your method. Or undo g first: $f(x) = g^{-1}(20)$. Then **reject any solution outside the domain of f**.

Variations you will meet:

- **Unknown constants.** A composite value (such as $gf(2) = 11$) or a given range gives an equation in the constant. Form the composite, then use the fact.
- **An identity.** "$fg(x) \equiv \ldots$" or "find h such that $gh(x) = \ldots$": either compare coefficients, or apply g^{-1} to both sides, so $h(x) = g^{-1}(\ldots)$.
- **The largest domain for a composite.** Find the range of g in terms of the unknown and make it fit inside the domain of f.

4. Find the range of a function (21 questions)

1. Picture (or sketch) the graph over the given domain only.
2. **Straight line:** work out the outputs at the two ends of the domain.
3. **Quadratic:** complete the square, $a(x+p)^2 + q$. If the vertex $x = -p$ is inside the domain, the range starts at q. If it isn't, the range starts at the value of f at the end of the domain.
4. **A fraction like $c + \dfrac{k}{x+d}$ or a square root:** look at both ends of the domain.
   - At an end that is **included** (like $x \geqslant 0$), work out the value there.

- As x gets very large, a fraction term tends to 0, so the function gets close to c but never reaches it.
- At an end that is **excluded because the denominator is zero there** (like $x > 1$ for $\frac{5}{x-1}$), the fraction term grows without limit, so that side of the range is unbounded.

For example, $f(x) = 2 + \dfrac{5}{x-1}$ for $x > 1$: near $x = 1$ the values are huge, and for large x they fall towards 2 without reaching it. The range is $f(x) > 2$, a strict inequality.

5. Write the answer with $f(x)$ or y, and choose $\geqslant$ or $>$ carefully: use $\geqslant$ only if that value is actually reached for some x in the domain.

Variations you will meet:

- **The range of a composite**, gf: find the range of f first, then feed that set of values into g.
- **The range is given; find a constant.** Write the least (or greatest) value in terms of the constant, and set it equal to the given value.
- **The range of an inverse** is the domain of the original.

5. Explain whether an inverse or a composite exists (13 questions)

- **Does f^{-1} exist?** Only if f is one-one. If the graph has a turning point inside the domain (or a periodic graph repeats), say "f is not one-one", with a reason or an example such as $f(1) = f(3)$. If f is increasing (or decreasing) throughout its domain, it is one-one.
- **The largest domain for which f is one-one**, "for $x \leqslant k$" or "for $x \geqslant k$": the boundary k is the x-coordinate of the vertex.
- **Can gf be formed?** Compare the range of f with the domain of g. "gf cannot be formed because the range of f is not contained in the domain of g", then point to a value that breaks it, for example "$f(0) = -5$, but g is only defined for $x \geqslant 0$".
- **Can ff be formed?** The same test: the range of f must fit inside the domain of f.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

The curve $y = f(x)$ has a minimum point at $(-1, 4)$ and crosses the y-axis at $(0, 5)$.

(a) Describe fully a sequence of transformations that maps the graph of $y = f(x)$ onto the graph of $y = 3 - 2f(x)$, making clear the order. **[4]**

(b) Find the points that $(-1, 4)$ and $(0, 5)$ are mapped to, and say whether the image of the minimum point is a maximum or a minimum. **[3]**

(a) All three changes are outside the bracket, so they act on y and follow the arithmetic: multiply first, then add. Rewrite as $y = -2f(x) + 3$.

First, a **stretch parallel to the y-axis, scale factor** 2. **B1**

Then a **reflection in the x-axis**. **B1** (These two can be swapped: both only multiply.)

Last, a **translation by** $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$. **B1**

B1 for the translation coming after the stretch and reflection. Translating first would give $-2\big(f(x) + 3\big) = -2f(x) - 6$.

(b) Each y-coordinate q becomes $3 - 2q$; x-coordinates stay the same. **M1**

$$(-1, 4) \to (-1, 3 - 8) = (-1, -5), \qquad (0, 5) \to (0, 3 - 10) = (0, -7)$$

A1

The reflection turns the graph upside down, so $(-1, -5)$ is a **maximum** point. **B1**

Check with $f(x) = (x + 1)^2 + 4$, which has this minimum and y-intercept: $3 - 2f(x) = -2(x + 1)^2 - 5$, with a maximum at $(-1, -5)$ and value -7 at $x = 0$. ✓

Example 2

The function f is defined by $f(x) = x^2 - 4x + 6$ for $x \geqslant 2$.

(a) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. **[4]**

(b) The graphs of $y = f(x)$ and $y = f^{-1}(x)$ meet at two points. Find their coordinates. **[3]**

(c) The domain of f is changed to $x \geqslant 0$. Explain why f no longer has an inverse. **[1]**

(a) $x^2 - 4x + 6 = (x - 2)^2 + 2$. **B1**

Let $y = (x - 2)^2 + 2$. Then $(x - 2)^2 = y - 2$, so $x - 2 = \pm\sqrt{y - 2}$. **M1**

For $x \geqslant 2$, $x - 2$ is not negative, so take the **positive** root:

$$f^{-1}(x) = 2 + \sqrt{x - 2}$$

A1

The vertex $(2, 2)$ is in the domain, so the range of f is $f(x) \geqslant 2$, and the domain of f^{-1} is $x \geqslant 2$. **B1**

(b) f is increasing on its domain, so the graphs of f and f^{-1} can only meet on the line $y = x$. Solve $f(x) = x$. **M1**

$$x^2 - 4x + 6 = x \;\Rightarrow\; x^2 - 5x + 6 = 0 \;\Rightarrow\; (x-2)(x-3) = 0$$

M1 for solving the quadratic, showing the method.

Both $x = 2$ and $x = 3$ are in both domains, so the points are $(2, 2)$ and $(3, 3)$. **A1**

Check: $f(3) = 9 - 12 + 6 = 3$ and $f^{-1}(3) = 2 + 1 = 3$. ✓

(c) The vertex $x = 2$ is now inside the domain, so f is not one-one: for example $f(1) = f(3) = 3$. **B1**

Example 3

Functions f, g and h are defined by

$$f(x) = 4 - \frac{6}{x+1} \;\text{ for } x \geqslant 0, \qquad g(x) = x^2 + 1 \;\text{ for } x \in \mathbb{R}, \qquad h(x) = x - 5 \;\text{ for } x \in \mathbb{R}.$$

(a) Find the range of f. **[2]**

(b) Find an expression for $f^{-1}(x)$. **[3]**

(c) Solve the equation $fg(x) = 2$. **[3]**

(d) Explain why the composite function fh cannot be formed. **[1]**

(a) $f(0) = 4 - 6 = -2$. As x increases, $\frac{6}{x+1}$ tends to 0, so $f(x)$ rises towards 4 but never reaches it.

$$-2 \leqslant f(x) < 4$$

B1 for each end, with the correct inequality signs.

(b) Let $y = 4 - \dfrac{6}{x+1}$. Then

$$\frac{6}{x+1} = 4 - y, \qquad x + 1 = \frac{6}{4-y}, \qquad x = \frac{6}{4-y} - 1 = \frac{2+y}{4-y}.$$

M1 for the first two steps of the rearrangement, **M1** for the last two.

$$f^{-1}(x) = \frac{x+2}{4-x}$$

A1, in terms of x. Check: $f(0) = -2$ and $f^{-1}(-2) = 0$. ✓

(c) The range of g is $g(x) \geqslant 1$, inside the domain of f, so fg exists.

$$fg(x) = 4 - \frac{6}{x^2+2}$$

B1 for putting $g(x)$ into f (note $x^2 + 1 + 1 = x^2 + 2$).

$$4 - \frac{6}{x^2+2} = 2 \;\Rightarrow\; \frac{6}{x^2+2} = 2 \;\Rightarrow\; x^2 + 2 = 3$$

M1 for solving.

$$x = 1 \text{ or } x = -1$$

A1. Both are allowed, since the domain of g is all real numbers.

(d) The range of h (all real numbers) is not contained in the domain of f ($x \geqslant 0$): for example $h(0) = -5$, and $f(-5)$ is not defined. **B1**

Example 4

The function f is defined by $f(x) = (x-4)^2 - 9$ for $x \leqslant 4$.

(a) State the range of f. **[1]**

(b) The graph of $y = f(x)$ is stretched parallel to the x-axis with scale factor 2, and then reflected in the x-axis, to give the graph of $y = g(x)$. Find $g(x)$ in the form $p - q(x-r)^2$, and state the domain of g. **[4]**

(c) Find an expression for $g^{-1}(x)$ and state the domain of g^{-1}. **[4]**

(d) Solve the equation $g^{-1}(x) = 2$. **[2]**

(a) The vertex $(4, -9)$ is at the end of the domain, so the least value is -9: $f(x) \geqslant -9$. **B1**

(b) Stretch parallel to the x-axis, factor 2: replace x by $\frac{x}{2}$. **M1**

$$y = \left(\tfrac{x}{2} - 4\right)^2 - 9 = \tfrac{1}{4}(x-8)^2 - 9$$

since $\frac{x}{2} - 4 = \frac{1}{2}(x-8)$, and squaring the $\frac{1}{2}$ gives $\frac{1}{4}$.

Reflect in the x-axis: multiply the whole right-hand side by -1. **M1**

$$g(x) = 9 - \tfrac{1}{4}(x-8)^2$$

A1

The domain moves with the graph: the old condition $x \leqslant 4$ now applies to $\frac{x}{2}$, so $\frac{x}{2} \leqslant 4$ and the domain of g is $x \leqslant 8$. **B1**

Check with the vertex: the stretch sends $(4, -9)$ to $(8, -9)$ and the reflection sends that to $(8, 9)$, which is the vertex of g and the end of its domain. ✓

(c) The reflection turns the range of f upside down, so the range of g is $g(x) \leqslant 9$, and the domain of g^{-1} is $x \leqslant 9$. **B1**

Let $y = 9 - \frac{1}{4}(x-8)^2$. Then $(x-8)^2 = 4(9-y)$, so $x - 8 = \pm 2\sqrt{9-y}$. **M1**

The domain of g is $x \leqslant 8$, so $x - 8$ is never positive: take the **negative** root. **M1**

$$g^{-1}(x) = 8 - 2\sqrt{9-x}$$

A1, in terms of x. Check: $g(0) = 9 - \frac{1}{4}(64) = -7$, and $g^{-1}(-7) = 8 - 2\sqrt{16} = 0$. ✓

(d) You don't need the expression from (c): $g^{-1}(x) = 2$ means $x = g(2)$. **M1**

$$x = 9 - \tfrac{1}{4}(2-8)^2 = 9 - 9 = 0$$

A1. Check: $g^{-1}(0) = 8 - 2\sqrt{9} = 2$. ✓

Common mistakes

- **Loose words for transformations.** "Move", "squash" or "compress" may lose marks, and "right 3, down 4" in place of a vector or x- and y-directions is penalised. Say *translation* with a vector, *stretch* with a scale factor and a direction, *reflection* with the mirror line.
- **Getting inside changes backwards.** $y = f(2x)$ is a stretch with factor $\frac{1}{2}$, not 2, and $f(x+1)$ moves the graph 1 to the **left**. To translate by $\begin{pmatrix}1\\0\end{pmatrix}$, replace x by $x-1$.
- **No order, the wrong order, or the wrong way round.** Write "first ... then ...", remember that two changes in the same direction depend on order, and check which graph is mapped onto which.
- **Using x for a range.** Write $f(x) \geqslant -3$, not $x \geqslant -3$ (which looks like a domain and scores nothing).
- **Using a vertex that is outside the domain.** Then the range starts at the end of the domain.
- **Keeping $\pm$ in an inverse, or leaving it in terms of y.** Choose one sign using the domain of f (the negative root, for a domain left of the vertex, is the one most often missed), and finish with $f^{-1}(x) = \dots$ in terms of x.
- **Confusing $f^{-1}(x)$ with $f'(x)$ or $\frac{1}{f(x)}$.** The inverse undoes f; it is not the derivative and not the reciprocal.
- **Composing in the wrong order, or multiplying.** $fg(x)$ means $f(g(x))$, with g first. It never means $f(x) \times g(x)$.
- **Not substituting the whole inner function.** If $f(x) = 2x^2 + 3$, then $ff(x) = 2(2x^2+3)^2 + 3$: the whole bracket is squared.
- **Keeping solutions that are outside the domain.** After solving $gf(x) = k$ or $f(x) = g(x)$, check each answer against the domain and reject any that don't fit, saying why.
- **Weak "explain" answers.** For "why is there no inverse?" (or "why does it have one?"), use the words "f is (not) one-one" and give a reason, such as a turning point in the domain or two inputs with the same output. For "why can't gf be formed?", compare the range of f with the domain of g; it is not about square roots of negatives.
- **Sketching the inverse badly.** Reflect in $y = x$ (drawn through the origin), not in an axis; the inverse must not bend back on itself.

How to get the marks

- **"Describe fully"**: transformations are marked element by element. A stretch needs the word, the scale factor and the direction; a translation needs the word and the vector. A final mark is often for the correct order with no extra transformations.

- **Inverse functions** earn method marks for making x the subject, even if the sign is wrong, but the final mark needs the right single sign and the answer in terms of x, written as $f^{-1}(x) = \ldots$.
- **"State the domain of f^{-1}"** or **"state the range"**: one mark, for a correct inequality in the right letter. The domain of f^{-1} uses x; a range uses $f(x)$ or y.
- **"Show that"** (for example, simplifying a composite to a given form): the answer is given, so show every line of simplification, including clearing fractions inside fractions. Testing a few values instead of doing the algebra scores nothing.
- **"Solve"**: show your method for any quadratic (factorise, formula or completed square), then keep only answers inside the domain.
- **"Hence"**: use the part before, such as the completed square form.
- **"Explain"** needs a reason in words that uses the key idea: *one-one*, or *range of* ... *not within the domain of* ....
- **Sketches**: the right shape in the right quadrants, end points or intercepts that match the original, and the line $y = x$ drawn when you sketch an inverse.

Quick check

1. The function f is defined by $f(x) = 5 - 2x$ for $-1 \leqslant x \leqslant 3$. Find the range of f.
2. Functions f and g are defined for all real x by $f(x) = 2x - 1$ and $g(x) = x^2$. Solve the equation $fg(x) = gf(x)$.
3. Functions f and g are defined by $f(x) = \sqrt{x-3}$ for $x \geqslant 3$ and $g(x) = x^2 + c$ for $x \in \mathbb{R}$, where c is a constant. Find the least value of c for which fg can be formed. For that value of c, solve $fg(x) = 4$.
4. The graph of $y = f(x)$ is reflected in the y-axis and then translated by $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$. Write down the equation of the new graph in terms of f. The point $(1, 2)$ lies on $y = f(x)$; find the corresponding point on the new graph.
5. The function f is defined by $f(x) = \dfrac{3x+4}{x-3}$ for $x \neq 3$. Find $f^{-1}(x)$. What do you notice, and what does it tell you about $ff(x)$?

Answers

1. f is decreasing. $f(-1) = 7$ and $f(3) = -1$, so $-1 \leqslant f(x) \leqslant 7$.
2. $fg(x) = 2x^2 - 1$ and $gf(x) = (2x-1)^2 = 4x^2 - 4x + 1$. Then $2x^2 - 1 = 4x^2 - 4x + 1$ gives $2x^2 - 4x + 2 = 0$, so $(x-1)^2 = 0$ and $x = 1$.
3. The range of g is $g(x) \geqslant c$. It must lie inside the domain of f, $x \geqslant 3$, so $c \geqslant 3$ and the least value is $c = 3$. Then $fg(x) = \sqrt{x^2 + 3 - 3} = 4$ gives $x^2 = 16$, so $x = 4$ or $x = -4$ (both allowed, as g is defined for all x).
4. Reflect: $y = f(-x)$. Translate: replace x by $x - 4$, giving $y = f\big(-(x-4)\big) = f(4-x)$. The point: $(1, 2) \to (-1, 2) \to (3, 2)$. Check: $f(4-3) = f(1) = 2$.
5. $y(x-3) = 3x + 4 \Rightarrow xy - 3x = 3y + 4 \Rightarrow x(y-3) = 3y + 4$, so $f^{-1}(x) = \dfrac{3x+4}{x-3}$. This is the same as $f(x)$: f is its own inverse, so $ff(x) = x$.

Past questions to try

- **9709/13/O/N/24 Q5**: three transformations read from two graphs.
- **9709/11/O/N/24 Q11**: range, a reflection and translation, and sketching an inverse.
- **9709/12/F/M/24 Q9**: a range of a composite that fixes a constant.
- **9709/11/M/J/23 Q8**: constants from composite values, then an inverse and its domain.