Cambridge AS & A Level · Mathematics (9709) · Notes

Integration AS

Read online at pastopic.com/cambridge/mathematics-9709/notes/integration-p1

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Read first: Differentiation

What you need to know

By the end of this topic you should be able to:

1. **Integrate as the reverse of differentiating.** Integrate any power of x, and any power of a linear bracket $(ax+b)^n$, for every rational n except $n=-1$. This includes constant multiples, sums and differences of such terms.
2. **Find the constant of integration** from extra information, for example finding the equation of a curve when you know its gradient function and one point it passes through.
3. **Evaluate definite integrals**, including simple *improper* ones: where one limit is infinite, such as $\int_1^\infty x^{-2}\,dx$, or where the function becomes infinite at a limit, such as $\int_0^1 x^{-1/2}\,dx$.
4. **Find areas** of regions bounded by a curve and lines parallel to the axes, between a curve and a straight line, or between two curves.
5. **Find volumes of revolution** when a region is turned a full circle about the x-axis or the y-axis, including regions that do not touch the axis they are turned about.

Integration of $\frac{1}{x}$, exponentials, trigonometric functions and the trapezium rule are **not** in this topic (they are in Paper 2 and Paper 3).

Understand it

Undoing differentiation

Differentiating turns a function into its gradient function. Integrating goes back the other way: given $\frac{dy}{dx}$, it finds y.

When you differentiate x^3 you multiply by the power and then take one off the power, giving $3x^2$. To reverse this, do the opposite steps in the opposite order: **add one to the power, then divide by the new power**. So

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1).$$

The rule breaks for $n=-1$ because you would divide by zero; that case needs logarithms, which come in Paper 2 and 3.

Why "+ c"?

Differentiating x^2, x^2+5 and x^2-7 all give $2x$, because a constant disappears. So when you integrate $2x$ you cannot know which constant was there: the answer is x^2+c. Graphically, $y=x^2+c$ is a whole family of identical curves, each shifted up or down. One known point on the curve picks out exactly one member of the family, and that is how you find c.

Brackets: the reverse chain rule

Differentiating $(3x+1)^5$ with the chain rule gives $5(3x+1)^4\times 3$: the extra $\times 3$ is the derivative of the inside. Reversing it, you must also **divide by the coefficient of x inside the bracket**:

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \qquad (n \neq -1).$$

This only works because the inside is *linear*. You cannot integrate $(x^2+1)^3$ this way; expand it first.

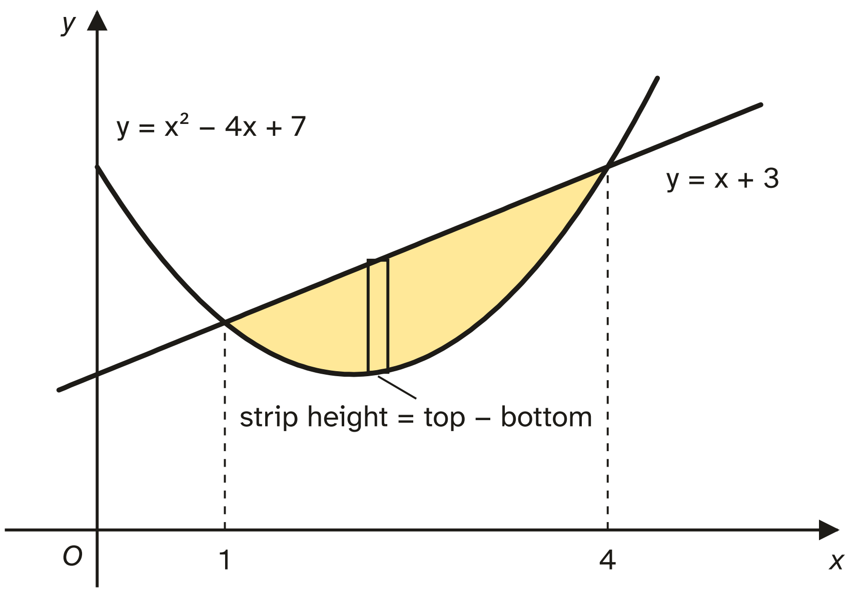
You can always check an integral by differentiating your answer: you should get back exactly what you started with.

Definite integrals and area

A definite integral $\int_a^b f(x)\,dx$ means: find any integral $F(x)$ of $f(x)$, then work out $F(b)-F(a)$. The $+c$ cancels, so you leave it out.

The reason this gives an area: split the region under a curve into very thin vertical strips of width δx. Each strip is almost a rectangle of height y, with area about $y\,\delta x$. Adding all the strips and letting their width shrink to zero gives exactly $\int_a^b y\,dx$.

The same strip idea explains the area between two graphs. Each strip now runs from the lower graph up to the upper one, so its height is **top minus bottom**:

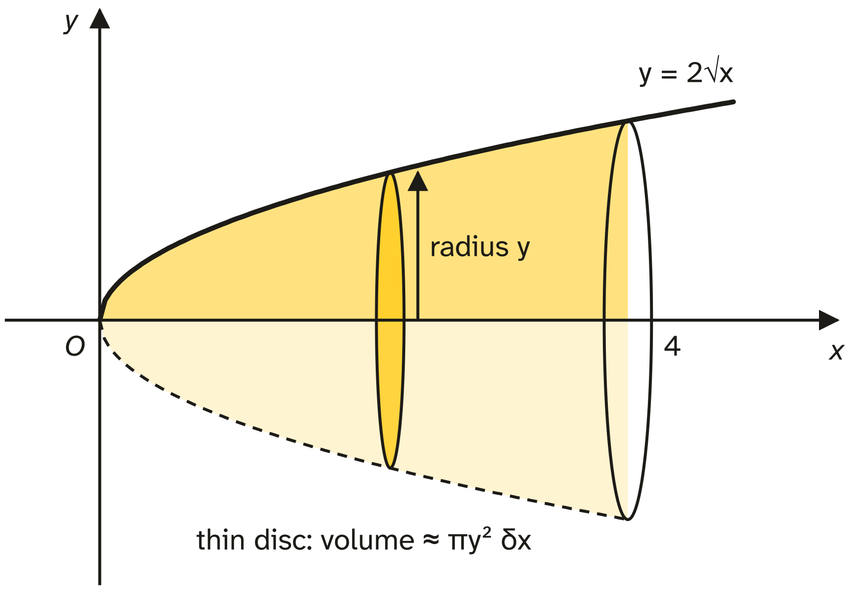


$$\text{Area} = \int_a^b (y_{\text{top}} - y_{\text{bottom}})\, dx.$$

Integrals count area **below** the x-axis as negative. If a region is below the axis, the integral comes out negative and the area is its size (drop the minus sign). If a region crosses the axis, split it at the crossing point and add the sizes of the parts.

Volumes of revolution

Turn the region under a curve a full circle (360°) about the x-axis and it sweeps out a solid. Slice that solid into thin discs: each disc has radius y (the height of the curve) and thickness δx, so its volume is about $\pi y^2\, \delta x$.

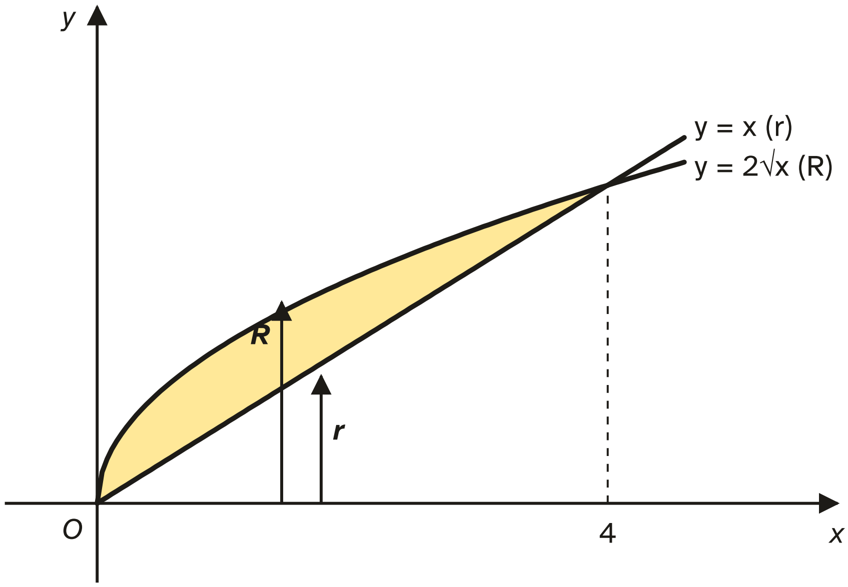


Adding the discs gives

$$V = \pi \int_a^b y^2\, dx.$$

Turning about the y-axis works the same way with the roles of x and y swapped: the slices are horizontal, their radius is x, and you integrate with respect to y between y-limits: $V = \pi \int_c^d x^2\, dy$.

If the region does **not** touch the axis, each slice is a *washer* (a disc with a hole). Its volume is the big disc minus the hole: $\pi R^2 - \pi r^2$, where R is the outer radius and r the inner radius.



So $V = \pi \int_a^b (R^2 - r^2)\, dx$. Note that this is **not** $\pi \int (R - r)^2\, dx$: squaring the difference gives the wrong solid.

Improper integrals

Sometimes the region stretches off to infinity, or the curve shoots up to infinity at one end. The area can still be finite. To find it, integrate as usual, then look at what your answer does at the awkward limit:

- If a limit is ∞, a term like $\frac{2}{x}$ or $\frac{1}{(3x-2)}$ tends to 0 as $x \to \infty$.
- If the function becomes infinite at $x = 0$ (like $x^{-1/2}$), its integral ($2x^{1/2}$) may still have a sensible value there, here 0.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$	Given
$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad (n \neq -1)$	Learn it
$\int kf(x)\, dx = k \int f(x)\, dx$, and integrate sums term by term	Learn it
$\int_a^b f(x)\, dx = F(b) - F(a)$, where F is an integral of f	Learn it
Area under a curve: $\int_a^b y\, dx$; area next to the y-axis: $\int_c^d x\, dy$	Learn it

Formula	Status
Area between two graphs: $\int_a^b (y_{\text{top}} - y_{\text{bottom}})\,dx$	Learn it
Volume about the x-axis: $V = \pi \int_a^b y^2\,dx$	Learn it
Volume about the y-axis: $V = \pi \int_c^d x^2\,dy$	Learn it
Washer: $V = \pi \int_a^b (R^2 - r^2)\,dx$	Learn it
Volume of a cone $= \frac{1}{3}\pi r^2 h$	Given
Volume of a cylinder $= \pi r^2 h$; area of a triangle $= \frac{1}{2}bh$; trapezium $= \frac{1}{2}(a+b)h$	Learn it

Useful index facts: $\sqrt{x} = x^{1/2}$, $\frac{1}{x^n} = x^{-n}$, $\frac{1}{\sqrt{x}} = x^{-1/2}$, and $\frac{k}{(ax+b)^n} = k(ax+b)^{-n}$. Always rewrite in these forms before integrating.

How to do it

In the Paper 1 questions we have tagged for this topic (72 questions, 2021–2025), the types came up this often. Some questions mix two types.

Question type	Questions
Find the equation of a curve from its gradient	32
Find an area	26
Find a volume of revolution	12
Evaluate an improper integral	3
Find an unknown constant from a definite integral	1

1. Find the equation of a curve from its gradient (32 questions)

You are given $\frac{dy}{dx}$ (or $f'(x)$) and a point on the curve.

1. Rewrite every term as a power: $\frac{8}{x^2} = 8x^{-2}$, $\frac{10}{(2x-3)^2} = 10(2x-3)^{-2}$.
2. Integrate each term: add one to the power, divide by the new power, and for a bracket also divide by the coefficient of x. Write $+c$.
3. Substitute the point's x and y values and solve for c.
4. Write the final answer as a full equation, $y = \ldots$ (or $f(x) = \ldots$), with c replaced by its value and no fractions inside fractions.

Variations you will meet:

- **An unknown constant k in $\frac{dy}{dx}$.** Find k first from the extra fact you are given, for example "the gradient at $(2, 5)$ is 3" (substitute $x = 2$ into $\frac{dy}{dx}$ and set it equal to 3), or "there is a stationary point at $x = 2$" (set $\frac{dy}{dx} = 0$ at $x = 2$). Then integrate.
- **You are given $\frac{d^2y}{dx^2}$.** Integrate twice, with a new constant each time. Use "stationary point at $x = p$" to find the first constant, because $\frac{dy}{dx} = 0$ there. The y-value of the point is used only in the second step, for the second constant.
- **The curve passes through a maximum or minimum with a known y-coordinate.** Find its x-value by solving $\frac{dy}{dx} = 0$ first, then use that point to find c.

2. Find an area (26 questions)

1. **Find the limits.** Read them from the question or the diagram, or find where the graphs meet by solving the equations simultaneously. Show how you solved it; don't just quote calculator roots.
2. **Decide what to integrate.**
   - Region between a curve, the x-axis and vertical lines: $\int_a^b y\,dx$.
   - Region between two graphs: $\int_a^b (y_{\text{top}} - y_{\text{bottom}})\,dx$. Use the diagram to see which is on top.
   - A straight line forms one edge: you can integrate it, but it's often quicker to use the area of a triangle or trapezium.
   - Region bounded by a curve, the y-axis and horizontal lines: either rearrange to $x = \ldots$ and find $\int_c^d x\,dy$, or take a rectangle and subtract the area under the curve.
3. **Integrate**, showing each integrated term.
4. **Substitute both limits** and show it: $F(b) - F(a)$ written out. Don't assume the lower limit gives 0; for example $(3x+4)^{3/2}$ at $x = 0$ is 8, not 0.
5. **Build the final area** from the pieces: add or subtract triangles, rectangles or other integrals as the diagram shows. If a region is below the x-axis, use the size of the integral.

Tangents and normals. Some areas are bounded by a tangent or normal to the curve. Find its equation first with differentiation, then find where it meets the axis. Usually the region is a triangle under the line minus (or plus) an integral under the curve.

3. Find a volume of revolution (12 questions)

1. **Which axis?** About the x-axis: $\pi \int y^2\,dx$ with x-limits. About the y-axis: rearrange to x^2 in terms of y, then $\pi \int x^2\,dy$ with y-limits.
2. **Square before integrating.** Write out y^2 (or x^2) in full and simplify, for example $\left(\frac{1}{2}x + \frac{4}{x}\right)^2 = \frac{1}{4}x^2 + 4 + \frac{16}{x^2}$. For $y = \sqrt{6x+5}$, $y^2 = 6x + 5$.
3. **Does the region touch the axis?** If not, subtract the hole. Either work out $\pi \int (R^2 - r^2)\,dx$ in one go, or find the two volumes separately and subtract. When the inner boundary is a horizontal line $y = k$, the hole is a cylinder, $\pi k^2 \times \text{length}$; when it is a line through the origin, the hole is a cone.
4. **Integrate, substitute both limits, keep π.** Give an exact answer (a multiple of π) unless a decimal is asked for.

Unknown limits. If a limit is a letter (for example the region from $x = a$ to $x = 2a$), integrate as normal, substitute the letters, simplify, and then use the given condition (for example $V \geqslant 46\pi$) to get an equation or inequality in that letter. Solve it, and reject values the question rules out, such as negative a.

4. Evaluate an improper integral (3 questions)

1. Rewrite the function as a power and integrate as usual.
2. At an infinite limit, write what each term tends to, for example $\frac{-2}{3x-2} \to 0$ as $x \to \infty$. At a limit where the function is infinite (like $x^{-1/2}$ at 0), substitute into your *integrated* expression, which is usually finite.
3. Subtract in the usual order: (value at the top limit) $-$ (value at the bottom limit).

5. Find an unknown constant from a definite integral (1 question)

The integrand or a limit contains a letter and you are told the value of the integral. Integrate, substitute the limits exactly as for a number, set the result equal to the given value, and solve. Areas "in terms of k" work the same way: treat k as a number throughout and simplify powers carefully, for example $(4k)^{3/2} = 8k^{3/2}$.

Worked examples

Example 1

A curve is such that $\dfrac{dy}{dx} = \dfrac{12}{(3x-2)^3}$. The curve passes through the point $(1, 5)$. Find the equation of the curve. **[4]**

Rewrite as a power: $\dfrac{dy}{dx} = 12(3x-2)^{-3}$.

Add one to the power ($-3 \to -2$), divide by the new power and by the 3 inside the bracket:

$$y = \frac{12(3x-2)^{-2}}{(-2)\times 3} + c = -2(3x-2)^{-2} + c$$

B1 for $(3x-2)^{-2}$ (the power), **B1** for the coefficient -2 (dividing by -2 and by 3).

Substitute $x = 1$, $y = 5$: $5 = -2(1)^{-2} + c$, so $c = 7$. **M1** for substituting the point into an integrated expression.

$$y = 7 - \frac{2}{(3x-2)^2}$$

A1, written as a full equation.

Check: differentiating $-2(3x-2)^{-2}$ gives $-2 \times (-2)(3x-2)^{-3} \times 3 = 12(3x-2)^{-3}$. ✓

Example 2

The curve $y = x^2 - 4x + 7$ and the line $y = x + 3$ meet at the points A and B.

(a) Find the coordinates of A and B. **[3]**

(b) Find the area of the region enclosed by the curve and the line. **[4]**

(a) Set the equations equal: $x^2 - 4x + 7 = x + 3$, so $x^2 - 5x + 4 = 0$. **M1** for forming a three-term quadratic and solving it.

$(x-1)(x-4) = 0$, so $x = 1$ or $x = 4$. **A1**

$A(1, 4)$ and $B(4, 7)$. **A1**

(b) From the diagram in "Understand it", the line is on top between $x = 1$ and $x = 4$.

$$\text{Area} = \int_1^4 \left[(x+3) - (x^2 - 4x + 7)\right] dx = \int_1^4 (-x^2 + 5x - 4)\, dx$$

M1 for top minus bottom (or integrating each graph separately).

$$= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x\right]_1^4$$

A1 for correct integration.

$$= \left(-\frac{64}{3} + 40 - 16\right) - \left(-\frac{1}{3} + \frac{5}{2} - 4\right) = \frac{8}{3} - \left(-\frac{11}{6}\right)$$

M1 for substituting both limits correctly and subtracting.

$$= \frac{9}{2}$$

A1

Another way: the area under the line is a trapezium, $\frac{1}{2}(4+7) \times 3 = \frac{33}{2}$. The area under the curve is $\left[\frac{x^3}{3} - 2x^2 + 7x\right]_1^4 = 12$. The area between them is $\frac{33}{2} - 12 = \frac{9}{2}$.

Example 3

The region between the curve $y = 2\sqrt{x}$ and the line $y = x$ is rotated through 360° about the x-axis. Find the exact volume of the solid formed. **[5]**

Limits: $2\sqrt{x} = x \Rightarrow 4x = x^2 \Rightarrow x(x-4) = 0$, so $x = 0$ and $x = 4$. **B1**

The region does not touch the x-axis (except at O), so each slice is a washer. The outer radius is the curve, $R = 2\sqrt{x}$, and the inner radius is the line, $r = x$ (see the third diagram in "Understand it").

$$V = \pi \int_0^4 \left[(2\sqrt{x})^2 - x^2\right] dx = \pi \int_0^4 (4x - x^2)\, dx$$

M1 for $\pi \int (y_1^2 - y_2^2)\, dx$. (Using $(2\sqrt{x} - x)^2$ would score nothing.)

$$= \pi \left[2x^2 - \frac{x^3}{3}\right]_0^4$$

A1 for correct integration.

$$= \pi \left(32 - \frac{64}{3}\right) - 0$$

M1 for substituting both limits.

$$= \frac{32\pi}{3}$$

A1, exact, with π.

Check with a cone: the volume under the line is a cone of radius 4 and height 4, $\frac{1}{3}\pi(4^2)(4) = \frac{64\pi}{3}$. The volume under the curve is $\pi \int_0^4 4x\,dx = 32\pi$. The difference is $32\pi - \frac{64\pi}{3} = \frac{32\pi}{3}$. ✓

Example 4

A curve has a stationary point at $(2, 3)$, and $\dfrac{d^2y}{dx^2} = 3 - \dfrac{16}{x^3}$.

(a) Find an expression for $\dfrac{dy}{dx}$. **[3]**

(b) Find the equation of the curve. **[3]**

(c) Determine the nature of the stationary point at $(2, 3)$. **[1]**

(a) $\dfrac{d^2y}{dx^2} = 3 - 16x^{-3}$. Integrate:

$$\frac{dy}{dx} = 3x - \frac{16x^{-2}}{-2} + c = 3x + 8x^{-2} + c$$

B1 for $3x$ and $8x^{-2}$.

At a stationary point $\frac{dy}{dx} = 0$, so at $x = 2$: $6 + \frac{8}{4} + c = 0$, giving $c = -8$. **M1** for using $\frac{dy}{dx} = 0$ at $x = 2$ (not $\frac{dy}{dx} = 3$).

$$\frac{dy}{dx} = 3x + \frac{8}{x^2} - 8$$

A1

(b) Integrate again, with a new constant:

$$y = \frac{3x^2}{2} - 8x^{-1} - 8x + d$$

B1 for all three terms.

At $(2, 3)$: $3 = 6 - 4 - 16 + d$, so $d = 17$. **M1**

$$y = \frac{3x^2}{2} - \frac{8}{x} - 8x + 17$$

A1

(c) At $x = 2$: $\dfrac{d^2y}{dx^2} = 3 - \dfrac{16}{8} = 1 > 0$, so the point is a **minimum**. **B1**, with the value and the conclusion both stated.

Common mistakes

- **Leaving out** $+c$, or finding c and then not writing the final equation. Without c you lose the last marks, and every later part that uses the equation goes wrong. Always finish with "$y = \ldots$".
- **Forgetting to divide by the coefficient of x in a bracket.** $\int (4x-7)^{1/2}\,dx$ is $\frac{(4x-7)^{3/2}}{\frac{3}{2}\times 4}$, not $\frac{(4x-7)^{3/2}}{\frac{3}{2}}$. Differentiate your answer to check.
- **Dividing by the wrong thing.** Divide only by the new power (and, for a bracket, by the coefficient of x inside it). The number in front of a term stays as a multiplier: $\int 3x^{1/2}\,dx = 3 \times \frac{x^{3/2}}{\frac{3}{2}} = 2x^{3/2} + c$, not $\frac{x^{3/2}}{\frac{3}{2}}$.
- **Differentiating instead of integrating**, or treating the curve as a straight line through the point. "The gradient of a curve is..., find the equation of the curve" always means integrate.
- **Working with powers carelessly.** Rewrite $\frac{1}{\sqrt{x}}$ as $x^{-1/2}$ before you start, and remember that $-3+1 = -2$, not -4.
- **Forgetting to square y for a volume.** It is $\pi \int y^2\,dx$, not $\pi \int y\,dx$. For a washer, square each radius separately, $R^2 - r^2$, never $(R-r)^2$.
- **Forgetting the hole.** If the region does not reach the axis, subtract the cylinder, cone or inner volume.
- **Mixing the axes.** For a turn about the y-axis, you need x^2 in terms of y and y-limits. Using $\pi \int y^2\,dx$ or x-limits scores nothing.
- **Squaring terms one at a time when solving for intersections.** From $\sqrt{3x-2} = \frac{1}{2}x + 1$, squaring gives $3x - 2 = \frac{1}{4}x^2 + x + 1$, not $3x - 2 = \frac{1}{4}x^2 + 1$.
- **Assuming the lower limit gives zero.** Always substitute it. $\left[(3x+4)^{3/2}\right]$ at $x = 0$ is 8.
- **Using the y-coordinate of a stationary point in the wrong step.** When integrating $\frac{d^2y}{dx^2}$, the first constant comes from $\frac{dy}{dx} = 0$; the y-value is only used for the second constant.
- **Rounding an exact answer.** If the question says "exact", giving 33.5 instead of $\frac{32\pi}{3}$ loses the final mark.

How to get the marks

- **Show the integration.** Each correctly integrated term usually earns a mark, even if a later step goes wrong. Write the integrated expression before you substitute.
- **Show both limits substituted**, for example $\left(32 - \frac{64}{3}\right) - (0)$. An answer straight from a calculator's integration function, with no working, gets no credit.
- **Write the full equation.** "Find the equation of the curve" needs $y = \ldots$ at the end; "find $f(x)$" needs $f(x) = \ldots$. Tidy any fractions inside fractions.
- **"Exact"** means fractions, surds and π, no decimals. Otherwise give non-exact answers to 3 significant figures (angles to 1 decimal place in degrees), unless the question asks for something else. Don't round in the middle of your working.

- **"Show that"** means the answer is given, so every step must be visible and your working must reach it exactly. A missing step loses the mark even if the final line is right.
- **"Hence"** means use the previous part (for example, the points you just found become your limits).
- **"Determine the nature"** needs a value and a conclusion: work out $\frac{d^2y}{dx^2}$ at the point, say whether it is positive or negative, and state "minimum" or "maximum". If you test the gradient either side, give the x-values you used and the gradients you got.
- **Use the diagram.** It shows which graph is on top, where the region starts and stops, and whether the region touches the axis. Sketch your own if there isn't one, and mark the limits on it.
- **Method marks follow through.** If an earlier answer was wrong but your method is right, you can still earn the method marks. Keep going rather than leaving a blank.

Quick check

1. Find $\displaystyle\int \sqrt{4x+1}\,dx$.
2. Find the exact value of $\displaystyle\int_2^\infty \frac{6}{(3x-2)^2}\,dx$.
3. A curve $y = f(x)$ passes through $(4, 10)$, and $f'(x) = 3\sqrt{x} - \dfrac{8}{x^2}$. Find $f(x)$.
4. The region bounded by the curve $y = x^2 + 1$ for $x \geqslant 0$, the y-axis and the line $y = 5$ is rotated through 360° about the y-axis. Find the exact volume of the solid formed.
5. Given that $a > 1$ and $\displaystyle\int_1^a (4x-3)\,dx = 15$, find the value of a.

Answers

1. $\dfrac{(4x+1)^{3/2}}{\frac{3}{2}\times 4}+c=\dfrac{1}{6}(4x+1)^{3/2}+c.$
2. $\displaystyle\int_2^{\infty} 6(3x-2)^{-2}\,dx=\left[-\frac{2}{3x-2}\right]_2^{\infty}=0-\left(-\frac{2}{4}\right)=\frac{1}{2}.$
3. $f(x)=2x^{3/2}+\dfrac{8}{x}+c$. At $x=4$: $16+2+c=10$, so $c=-8$ and $f(x)=2x^{3/2}+\dfrac{8}{x}-8.$
4. $x^2=y-1$, with y from 1 to 5: $V=\pi\displaystyle\int_1^5 (y-1)\,dy=\pi\left[\frac{y^2}{2}-y\right]_1^5=\pi\left(\frac{15}{2}+\frac{1}{2}\right)=8\pi.$
5. $\left[2x^2-3x\right]_1^a=2a^2-3a+1=15$, so $2a^2-3a-14=0$, $(2a-7)(a+2)=0$. Since $a>1$, $a=\frac{7}{2}$.

Past questions to try

- **9709/12/M/J/25 Q6**: the area between a curve and a line.
- **9709/13/O/N/24 Q9**: the area between two curves.
- **9709/11/M/J/23 Q10**: an exact volume for a region that does not touch the axis.
- **9709/12/F/M/24 Q1**: an improper integral.