Cambridge AS & A Level · Mathematics (9709) · Notes

Integration AS

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Read first: Integration, Differentiation, Trigonometry, Logarithmic and exponential functions, Algebra

What you need to know

By the end of this topic you should be able to:

1. **Integrate the new standard functions by reversing differentiation.** Integrate e^{ax+b}, $\dfrac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$ and $\sec^2(ax+b)$, together with constant multiples, sums and differences of them. You do this by "undoing" the derivatives you learnt in Paper 2 differentiation; the general method of integration by substitution is **not** needed.
2. **Use trigonometric identities to rewrite an integrand before integrating.** Squares such as $\cos^2 x$ or $\tan^2 3x$, and products such as $\sin x \cos x$, cannot be integrated as they stand. Identities, especially the double-angle formulae, turn them into terms you can integrate.
3. **Estimate a definite integral with the trapezium rule**, and use a sketch of the curve to decide, in simple cases, whether the estimate is too big (an over-estimate) or too small (an under-estimate).

The Paper 1 skills (the constant of integration, definite integrals, areas and volumes of revolution) are all used again here with these new functions. Integration by substitution, by parts and with partial fractions are Paper 3 topics.

Understand it

Reversing the new derivatives

In Paper 1 you integrated powers of x by asking "what did I differentiate to get this?". The same question works for every derivative from Paper 2 differentiation. Each line below is a derivative read backwards:

- $\dfrac{d}{dx}\,e^{ax+b} = a\,e^{ax+b}$, so $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$.
- $\dfrac{d}{dx}\ln(ax+b) = \dfrac{a}{ax+b}$, so $\displaystyle\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$.
- $\dfrac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$, so $\displaystyle\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$.
- $\dfrac{d}{dx}\sin(ax+b) = a\cos(ax+b)$, so $\displaystyle\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$.

- $\dfrac{d}{dx}\tan(ax+b) = a\sec^2(ax+b)$, so $\displaystyle\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+c$.

The pattern is always the same: integrate as if the bracket were just x, then **divide by a, the number in front of x**. Differentiating multiplies by a (the chain rule), so integrating must divide by it.

- $\displaystyle\int e^{4x-3}\,dx = \frac{1}{4}e^{4x-3}+c$ and $\displaystyle\int 6e^{-2x}\,dx = \frac{6}{-2}e^{-2x} = -3e^{-2x}+c$.
- $\displaystyle\int \frac{5}{2x+7}\,dx = \frac{5}{2}\ln|2x+7|+c$.
- $\displaystyle\int 6\sin 3x\,dx = -2\cos 3x + c$, $\displaystyle\int \cos\tfrac{1}{2}x\,dx = 2\sin\tfrac{1}{2}x + c$ and $\displaystyle\int \sec^2(4x-1)\,dx = \frac{1}{4}\tan(4x-1)+c$.

Three things to notice:

- **$\frac{1}{x}$ is the gap in the power rule.** $\int x^{-1}\,dx$ would need dividing by 0, which is why Paper 1 left it out. Its integral is $\ln|x|$. The modulus is there because $\ln$ only accepts positive numbers; when $ax+b$ is positive over the whole interval you are working on, you can write $\ln(ax+b)$.
- **The sign flips for sine.** The derivative of $\cos$ is $-\sin$, so the integral of $\sin$ is $-\cos$. Check a doubtful answer by differentiating it.
- **Angles are in radians.** All these results come from derivatives that are only true in radians.

This "divide by a" rule only works when the inside is **linear** ($ax+b$). Something like e^{x^2} is not in this course.

Rewriting with identities first

$\cos^2 x$, $\sin^2 5x$ and $\tan^2 x$ are not derivatives of anything on the list above, so you change them into things that are. Rearrange the double-angle formulae from MF19 ($\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$) and $1+\tan^2 A = \sec^2 A$:

$$\cos^2 A = \tfrac{1}{2}(1+\cos 2A), \qquad \sin^2 A = \tfrac{1}{2}(1-\cos 2A), \qquad \tan^2 A = \sec^2 A - 1.$$

The angle **doubles**, and the square disappears. For example:

- $\displaystyle\int \cos^2 x\,dx = \int \left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2x\right)dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c$.
- $\displaystyle\int \sin^2 5x\,dx = \int \left(\tfrac{1}{2}-\tfrac{1}{2}\cos 10x\right)dx = \tfrac{1}{2}x - \tfrac{1}{20}\sin 10x + c$. Here $A = 5x$, so $2A = 10x$.
- $\displaystyle\int 3\tan^2 x\,dx = \int (3\sec^2 x - 3)\,dx = 3\tan x - 3x + c$.

A product of sine and cosine of the same angle halves into a double angle: $\sin x\cos x = \tfrac{1}{2}\sin 2x$, so $\int \sin x\cos x\,dx = -\tfrac{1}{4}\cos 2x + c$.

Exact answers

"Exact" answers in this topic keep e, $\ln$, π and surds. You will need the logarithm laws to tidy them:

- $2\ln 7 - 2\ln 3 = \ln 49 - \ln 9 = \ln \frac{49}{9}$ (a number in front goes up as a power; a difference of logs is the log of a quotient);
- $e^{\ln k} = k$, so $e^{-\ln 2} = \frac{1}{2}$ and $e^{2\ln 3} = e^{\ln 9} = 9$;
- $\ln 1 = 0$ and $e^0 = 1$.

The trapezium rule

Some integrals cannot be done exactly with the methods in this course, for example $\int \sqrt{1+e^x}\,dx$. The trapezium rule estimates the area under the curve instead. Split the interval from a to b into n strips of equal width

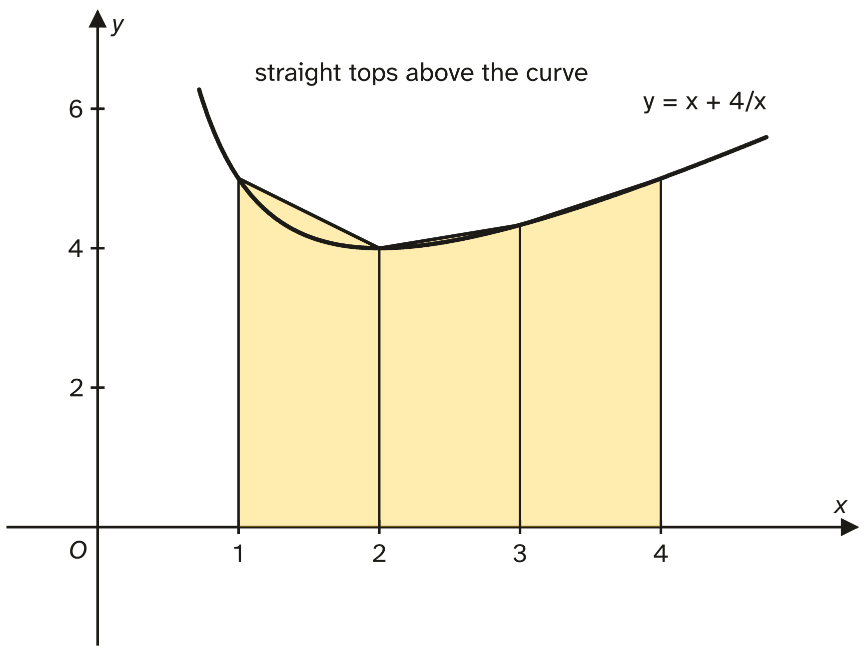
$$h = \frac{b-a}{n},$$

work out the heights $y_0, y_1, \ldots, y_n$ at the $n+1$ edges of the strips, and replace each strip by a trapezium. Adding the trapezia gives

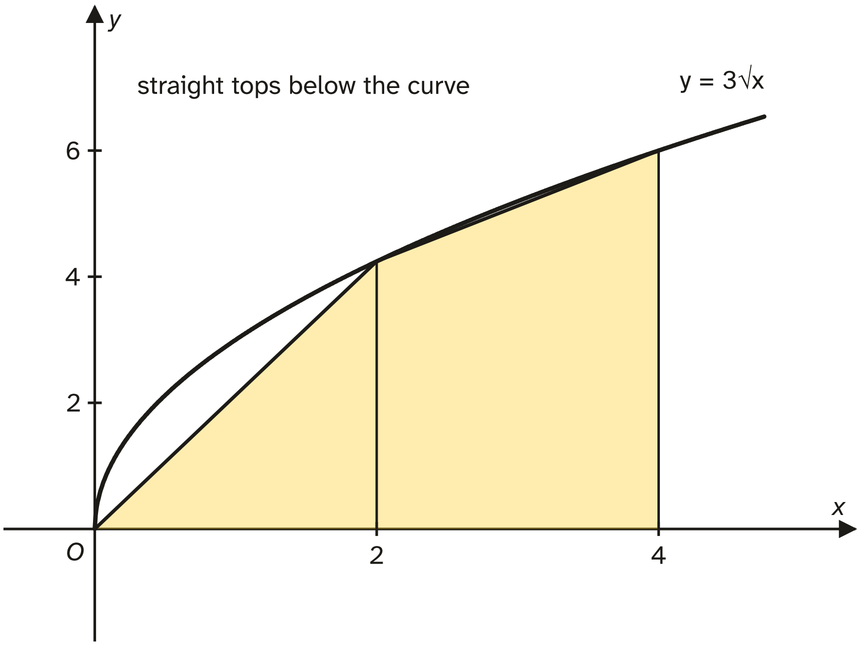
$$\int_a^b y\,dx \approx \tfrac{1}{2}h\left[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right].$$

The two end heights are counted once, every inside height twice (each one is shared by two trapezia).

Over or under? The trapezia have straight tops, and the curve does not. Where the curve **bends upwards** (like a cup, as $y = x + \frac{4}{x}$ does), each straight top lies **above** the curve, so the trapezia cover too much: the rule gives an **over-estimate**.



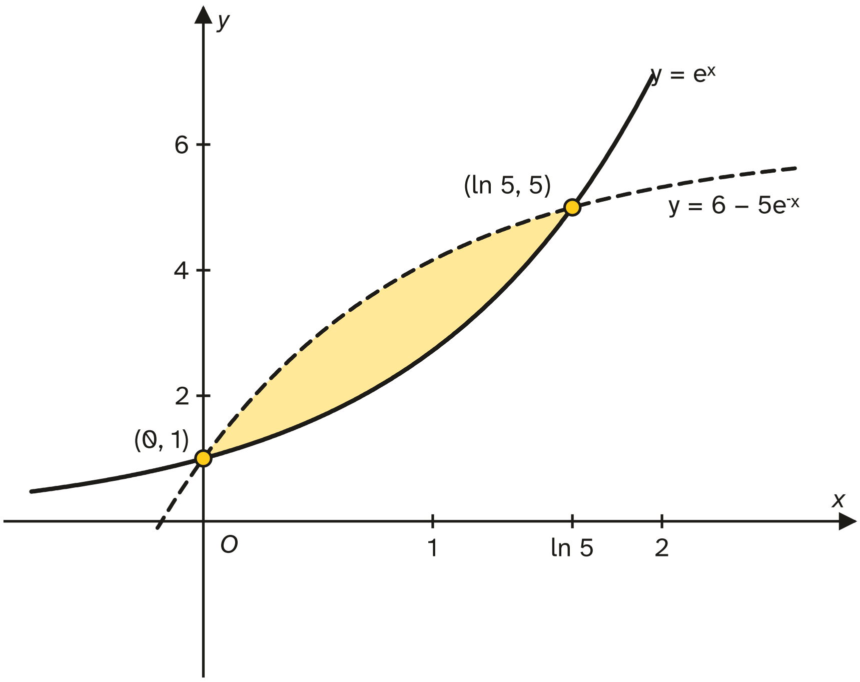
Where the curve **bends downwards** (like a cap), each straight top lies **below** the curve and misses a thin sliver: the rule gives an **under-estimate**. For $y = 3\sqrt{x}$ from 0 to 4 with two strips, $h = 2$ and the heights are 0, $3\sqrt{2}$ and 6, so the estimate is $\frac{1}{2} \times 2\left[0 + 2(3\sqrt{2}) + 6\right] = 6\sqrt{2} + 6 \approx 14.49$. The exact value is $\left[2x^{3/2}\right]_0^4 = 16$, which is bigger, as the picture predicts.



More strips make the trapezia hug the curve more closely, so the estimate improves. If the curve bends up in one part of the interval and down in another, a sketch alone cannot tell you which way the error goes.

Areas and volumes with the new functions

Areas and volumes work exactly as in Paper 1: area $= \int_a^b y\,dx$ for a curve above the x-axis, area between two graphs $= \int_a^b (y_{\text{top}} - y_{\text{bottom}})\,dx$, and volume about the x-axis $= \pi \int_a^b y^2\,dx$. What is new is that the limits often come from solving an exponential or trigonometric equation, and the answers contain $\ln$ and e.



For the two curves in the diagram, the region between $x = 0$ and $x = \ln 5$ has $y = 6 - 5e^{-x}$ on top, so its area is $\int_0^{\ln 5} (6 - 5e^{-x} - e^x)\,dx$. Example 3 works it out.

A region **below** the x-axis gives a negative integral. For example $\int_0^{\ln 2}(e^x - 2)\,dx = [e^x - 2x]_0^{\ln 2} = 1 - 2\ln 2 \approx -0.386$, so the area is $2\ln 2 - 1$. Work out parts above and below the axis separately and add their sizes.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$	Given
$\displaystyle\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c, \quad \int e^x\,dx = e^x + c$	Given
$\displaystyle\int \sin x\,dx = -\cos x + c, \quad \int \cos x\,dx = \sin x + c, \quad \int \sec^2 x\,dx = \tan x + c$	Given
$\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$	Learn it
$\displaystyle\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\lvert ax+b\rvert + c$	Learn it
$\displaystyle\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c, \quad \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c,$ $\displaystyle\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$	Learn it
$\sin^2 A + \cos^2 A = 1,\ 1 + \tan^2 A = \sec^2 A,\ \sin 2A = 2\sin A\cos A,$ $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$	Given
$\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$ expansions	Given
$\cos^2 A = \frac{1}{2}(1 + \cos 2A),\ \sin^2 A = \frac{1}{2}(1 - \cos 2A)$	Learn it
$\tan^2 A = \sec^2 A - 1,\ \sin A\cos A = \frac{1}{2}\sin 2A$	Learn it
Trapezium rule: $\displaystyle\int_a^b y\,dx \approx \tfrac{1}{2}h\left[y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n\right],\ h = \frac{b-a}{n}$	Learn it
Curve bends upwards: over-estimate; bends downwards: under-estimate	Learn it
Area $= \displaystyle\int_a^b (y_{\text{top}} - y_{\text{bottom}})\,dx$; volume about the x-axis $= \pi\displaystyle\int_a^b y^2\,dx$	Learn it
$\ln a + \ln b = \ln ab,\ \ln a - \ln b = \ln\frac{a}{b},\ k\ln a = \ln a^k,\ e^{\ln k} = k$	Learn it

Rewrite before integrating: $\frac{1}{e^x} = e^{-x}$, $\frac{1}{\cos^2 x} = \sec^2 x$, $\frac{k}{ax+b} = k \times \frac{1}{ax+b}$, and $\frac{1}{3x} = \frac{1}{3} \times \frac{1}{x}$.

How to do it

We have tagged 79 Paper 2 questions for this topic (2021–2025). Many appear word for word in two or three exam variants, which leaves 53 different questions. 6 of the 53 mix types, so the counts below add up to more than 53.

Question type	Questions
Trigonometric integrals using identities	20
Area of a region	13
Trapezium rule, over- or under-estimate	7
Divide first, then integrate	7
An integral equal to a value: find or show the constant	6
Exact integrals of e^{ax+b} and $\frac{1}{ax+b}$	5
Volume of revolution	2
Equation of a curve from its gradient	1

1. Trigonometric integrals using identities (20 questions)

1. **Look at each term.** $\sin$, $\cos$ and $\sec^2$ of a linear angle can be integrated straight away. Squares, cubes and products cannot.
2. **Rewrite** each awkward term with an identity: $\cos^2$ and $\sin^2$ become $\frac{1}{2}(1 \pm \cos 2A)$; $\tan^2$ becomes $\sec^2 - 1$; $\sin A \cos A$ becomes $\frac{1}{2}\sin 2A$; $\frac{1}{\cos^2 A}$ is $\sec^2 A$. Double the angle **inside**.
3. **Integrate term by term**, dividing by the number in front of x: $\cos 6x \to \frac{1}{6}\sin 6x$.
4. **For a definite integral**, substitute the limits in radians with exact values ($\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\tan\frac{\pi}{4} = 1$), and simplify.

For example, $(2 - \sin x)^2 = 4 - 4\sin x + \sin^2 x = \frac{9}{2} - 4\sin x - \frac{1}{2}\cos 2x$, so

$$\int (2 - \sin x)^2 \, dx = \tfrac{9}{2}x + 4\cos x - \tfrac{1}{4}\sin 2x + c.$$

Variations you will meet:

- **A single square**, such as $k\cos^2 x$ or $k\sin^2 4x$: one identity, then integrate. With $\sin^2 4x$ the new angle is $8x$, and you divide by 8.
- **A squared bracket**, such as $(\sec x + k\cos x)^2$: expand first. The cross term $\sec x \cos x = 1$, $\sec^2 x$ integrates directly and $\cos^2 x$ needs the identity.
- **Two different kinds of term**, such as $\tan^2 x + k\sin^2 2x$ or $\tan^2 \frac{1}{2}x + \sin x$: treat each one on its own. $\tan^2 \frac{1}{2}x = \sec^2 \frac{1}{2}x - 1$ integrates to $2\tan\frac{1}{2}x - x$.
- **A fraction that turns into** $\sec^2$: for example $\frac{2\sin^2 x}{\sin^2 2x} = \frac{2\sin^2 x}{4\sin^2 x\cos^2 x} = \frac{1}{2}\sec^2 x$. Write everything in sin and cos, then cancel.

- **"Hence" after an identity in part (a).** The earlier part turns something like $\sin 2\theta\,(a\cot\theta + b\tan\theta)$ into $p + q\cos 2\theta$; integrate that, not the original. If the integral uses $2x$ where the identity used θ, replace θ by $2x$ **everywhere**: $3 + 2\sin 2\theta$ becomes $3 + 2\sin 4x$.
- **Cubes** from a triple-angle identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ gives $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$, so $\int \cos^3 x\,dx = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x + c$.
- **After $R\cos(\theta - \alpha)$ form.** $\int \big(R\cos(2\theta - \alpha) + k\big)\,d\theta = \frac{R}{2}\sin(2\theta - \alpha) + k\theta + c$. And $\dfrac{1}{\big(R\cos(3\theta - \alpha)\big)^2} = \dfrac{1}{R^2}\sec^2(3\theta - \alpha)$, which integrates to $\dfrac{1}{3R^2}\tan(3\theta - \alpha)$.
- **A derivative from part (a) used backwards.** If you have shown $\frac{d}{dx}(\tan^2 x) = 2\tan x + 2\tan^3 x$, then $\int(\tan x + \tan^3 x)\,dx = \frac{1}{2}\tan^2 x + c$.
- **Limits from another part**, for example the x-coordinates of the points where a gradient has a given value, or of two maximum points.

2. Area of a region (13 questions)

1. **Find the boundaries.** Where the curve meets an axis, put $x = 0$ or $y = 0$. Where two curves meet, set them equal. Exponential equations often become a quadratic in e^x: multiply through by e^x, then let $u = e^x$.
2. **Decide what is on top.** Area $= \int (y_{\text{top}} - y_{\text{bottom}})\,dx$. If a straight edge is part of the boundary, it is often easier to find a triangle or rectangle and subtract (or add) the area under the curve.
3. **Integrate, substitute the limits, and simplify** with $e^{\ln k} = k$ and the log laws.
4. Give the answer in the form asked for, such as $p\ln 2 - q$ or $\ln a + b + ce^d$.

Variations you will meet:

- **Bounded by the curve and the line segment joining its intercepts A on the y-axis and B on the x-axis**: area $=$ triangle OAB $-$ area under the curve (for a curve that bends upwards).
- **Bounded by the curve, a horizontal line $y = k$ and a vertical line**: use a rectangle, then subtract the area under the curve.
- **Between two curves and the y-axis**: one limit is 0 and the other is where the curves meet.
- **Part of the region is below the x-axis**: integrate each curve separately, take the size of the negative one, and add. Or integrate $y_{\text{top}} - y_{\text{bottom}}$ in one go, which is always positive.
- **Between the line joining two maximum points and the curve**: the two maxima have the same height k, so the line is horizontal and the region lies below it. Area $=$ rectangle (height k, width the gap between the x-coordinates) $-$ the integral of the curve between them. Find the points first by differentiating.
- **A limit found by iteration** in an earlier part: it is a decimal, so give the area to the accuracy asked, such as 2 significant figures.
- **The limit needs finding first**, for example where a curve such as $y = 4e^x - e^{2x}$ meets the x-axis: $e^x(4 - e^x) = 0$ gives $e^x = 4$ and $x = \ln 4$.
- **A curve from Paper 1 alongside a new one**, such as $y = \sqrt{ax + b}$ with $y = \sin^2 x$: integrate the root with the Paper 1 rule for $(ax + b)^n$ and the square with the identity.

3. Trapezium rule, over- or under-estimate (7 questions)

1. **Width:** $h = \dfrac{b-a}{n}$, where n is the number of intervals (strips). There are $n+1$ y-values.
2. **Table** the x-values and y-values. Keep 4 or 5 significant figures, or exact values if the answer is in exact form.
3. **Apply** $\frac{1}{2}h\,[\text{ends} + 2 \times \text{middles}]$ and round to the accuracy asked.
4. **Over or under:** sketch the curve with the trapezia. Curve bending upwards means the tops are above it: over-estimate. Bending downwards: under-estimate. Say **why**, in terms of the trapezium tops and the curve.

Variations you will meet:

- **An answer in logarithms**, such as "show that the estimate is $\ln 12$": keep every y-value as a log, then combine them with the log laws. Decimals do not show a given log answer.
- **Deduce another area.** If region A was estimated and the exact total $A+B$ is known, then $B \approx$ total − estimate of A. The error flips: an over-estimate of A gives an **under-estimate** of B.
- **A region then rotated**, or **a stationary point first**: the trapezium rule is one part of a longer question.
- **Trigonometric y-values**: set the calculator to radians.
- **"Use a graph to show..."**: the sketch must be clearly curved, with the straight tops of the trapezia drawn.

4. Divide first, then integrate (7 questions)

1. **Divide** the polynomial by the linear bracket (long division; see the Algebra note): $\dfrac{p(x)}{ax+b} = q(x) + \dfrac{r}{ax+b}$, where $q(x)$ is the quotient and r the remainder.
2. **Integrate** $q(x)$ term by term, and the remainder term to $\dfrac{r}{a}\ln(ax+b)$.
3. **Substitute the limits** and combine the logarithms into the form asked, such as $a + \ln b$.

For example, $6x^2 + 7x + 5 = (2x+1)(3x+2) + 3$, so

$$\int \frac{6x^2+7x+5}{2x+1}\,dx = \int \left(3x + 2 + \frac{3}{2x+1}\right) dx = \tfrac{3}{2}x^2 + 2x + \tfrac{3}{2}\ln(2x+1) + c.$$

Variations you will meet:

- **An unknown constant in** $p(x)$, with the integral given as $a + \ln 64$, say: the remainder contains the constant, so match the log term to find it and the other terms to find a.
- **A negative remainder** gives $a - \ln b$; take care with the sign.
- **Indefinite** ("find $\int \ldots\, dx$"): include $+c$.
- **Synthetic division** by $4x-3$ gives the quotient for $x - \frac{3}{4}$; divide it by 4 afterwards.

5. An integral equal to a value: find or show the constant (6 questions)

1. **Integrate** and substitute the limits, which contain the unknown a.

2. **Set equal** to the given value.
3. **Rearrange** to what is asked. With exponentials, first get the e term alone on one side, then take logs. With logarithms, combine them into one log, then remove it.

For example, $\int_0^a 2e^{2x}\,dx = 15$ gives $e^{2a} - 1 = 15$, so $e^{2a} = 16$ and $a = \frac{1}{2}\ln 16 = 2\ln 2$.

Variations you will meet:

- **"Show that $a = \ldots$"**, a rearrangement ready for iteration. Look at the target form to see what to isolate, show every step, then use the iteration (Numerical solution of equations) in the next part.
- **Limits in terms of** a, such as $-2a$ to a, or a to $a+14$. Substitute carefully; $\ln(a+14) - \ln a = \ln\frac{a+14}{a}$, **not** $\frac{\ln(a+14)}{\ln a}$.
- **The value is an area**, such as "the shaded area is 120 square units".
- **A log value**, such as $\ln 2$ or $\ln 10$: collect every log into one, then compare what is inside.

6. Exact integrals of e^{ax+b} and $\frac{1}{ax+b}$ (5 questions)

1. Integrate with the "divide by a" rule.
2. Substitute both limits; write $e^{\text{something}}$ in simplest form, such as $e^3 - e$.
3. Combine the logs into one, such as $\ln 25$.

For example, $\int_0^{\ln 2} e^{3x}\,dx = \frac{1}{3}\left(e^{3\ln 2} - 1\right) = \frac{1}{3}(8-1) = \frac{7}{3}$, and $\int_2^5 \frac{3}{x-1}\,dx = 3\ln 4 - 3\ln 1 = \ln 64$.

Variations you will meet: a sum such as $4e^{2x} - 2e^{-x}$ (integrate each term); a pair of parts, one with $\frac{k}{ax+b}$ giving $\ln a$ and one with e^{ax+b}; a coefficient to spot, such as $\int \frac{1}{3x}\,dx = \frac{1}{3}\ln x$.

7. Volume of revolution (2 questions)

1. $V = \pi\int_a^b y^2\,dx$. Square the **whole** of y first: $(1+e^{kx})^2 = 1 + 2e^{kx} + e^{2kx}$, and $\left(\sqrt{f(x)}\right)^2 = f(x)$.
2. Rewrite any trig squares with the identities, integrate, and keep π in an exact answer.

For example, the region under $y = 2\sec x$ from 0 to $\frac{\pi}{4}$ rotated about the x-axis gives $V = \pi\int_0^{\pi/4} 4\sec^2 x\,dx = \pi\left[4\tan x\right]_0^{\pi/4} = 4\pi$.

8. Equation of a curve from its gradient (1 question)

Integrate $\frac{dy}{dx}$, **keep** $+c$, then substitute the given point to find c. For example, $\frac{dy}{dx} = 2e^{2x} + 3$ through $(0, 5)$ gives $y = e^{2x} + 3x + c$, and $5 = 1 + c$, so $y = e^{2x} + 3x + 4$.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

(a) Find the exact value of $\int_1^4 \frac{6}{3x-2}\,dx$, giving your answer in the form $\ln k$. **[3]**

(b) Find the exact value of $\int_0^{1/2} 4e^{1-2x}\,dx$. **[3]**

(a) Divide by the 3 in front of x:

$$\int \frac{6}{3x-2}\,dx = \frac{6}{3}\ln(3x-2) = 2\ln(3x-2).$$

B1

$$\Big[2\ln(3x-2)\Big]_1^4 = 2\ln 10 - 2\ln 1 = 2\ln 10.$$

M1 for substituting both limits and using a log law. $2\ln 10 = \ln 10^2 = \ln 100$. **A1**

(b) Divide by -2:

$$\int 4e^{1-2x}\,dx = \frac{4}{-2}e^{1-2x} = -2e^{1-2x}.$$

M1 for the form ke^{1-2x}, **A1** for $k=-2$.

$$\Big[-2e^{1-2x}\Big]_0^{1/2} = -2e^0 - \left(-2e^1\right) = 2e-2.$$

A1

Example 2

(a) Show that $(\sin x + 2\cos x)^2$ can be written as $\frac{5}{2} + \frac{3}{2}\cos 2x + 2\sin 2x$. **[3]**

(b) Hence find the exact value of $\int_0^{\pi/4} (\sin x + 2\cos x)^2\,dx$. **[4]**

(a) Expand:

$$(\sin x + 2\cos x)^2 = \sin^2 x + 4\sin x\cos x + 4\cos^2 x.$$

Use $\sin^2 x + \cos^2 x = 1$ and $2\sin x\cos x = \sin 2x$: this is $1 + 3\cos^2 x + 2\sin 2x$. **M1**

Now $3\cos^2 x = \frac{3}{2}(1+\cos 2x)$. **M1** for the double-angle identity for $\cos^2 x$.

$$1 + \tfrac{3}{2} + \tfrac{3}{2}\cos 2x + 2\sin 2x = \tfrac{5}{2} + \tfrac{3}{2}\cos 2x + 2\sin 2x.$$

A1, the given answer with every step shown.

(b)

$$\int \left(\tfrac{5}{2} + \tfrac{3}{2}\cos 2x + 2\sin 2x\right) dx = \tfrac{5}{2}x + \tfrac{3}{4}\sin 2x - \cos 2x.$$

M1 for the form $k_1 x + k_2 \sin 2x + k_3 \cos 2x$, **A1** for all three terms correct.

$$\left(\frac{5\pi}{8} + \frac{3}{4}\sin\frac{\pi}{2} - \cos\frac{\pi}{2}\right) - \left(0 + 0 - \cos 0\right) = \frac{5\pi}{8} + \frac{3}{4} - 0 + 1.$$

M1 for substituting both limits in radians. The answer is $\dfrac{5\pi}{8} + \dfrac{7}{4}$. **A1**

Example 3

The curves $y = e^x$ and $y = 6 - 5e^{-x}$ are shown in the third diagram in "Understand it".

(a) Show that the curves meet at the points where $x = 0$ and $x = \ln 5$. **[3]**

(b) Find the exact area of the region enclosed by the two curves, giving your answer in the form $a\ln 5 - b$. **[5]**

(a) $e^x = 6 - 5e^{-x}$. Multiply through by e^x:

$$e^{2x} = 6e^x - 5 \;\Rightarrow\; (e^x)^2 - 6e^x + 5 = 0.$$

M1 for a quadratic in e^x.

$(e^x - 1)(e^x - 5) = 0$, so $e^x = 1$ or $e^x = 5$. **M1** for solving it. So $x = 0$ or $x = \ln 5$. **A1**

(b) Between the points, $6 - 5e^{-x}$ is on top (at $x = 1$ it is about 4.16, while $e^1 \approx 2.72$).

$$\text{Area} = \int_0^{\ln 5} \left(6 - 5e^{-x} - e^x\right) dx.$$

M1 for top minus bottom with the limits 0 and $\ln 5$.

$$= \Big[6x + 5e^{-x} - e^x\Big]_0^{\ln 5}.$$

M1 for the form $k_1 x + k_2 e^{-x} + k_3 e^x$, **A1** for the correct signs and coefficients.

At $x = \ln 5$: $e^{\ln 5} = 5$ and $e^{-\ln 5} = \frac{1}{5}$, so the value is $6\ln 5 + 1 - 5 = 6\ln 5 - 4$. At $x = 0$: $0 + 5 - 1 = 4$. **M1** for both limits with $e^{\pm \ln 5}$ simplified.

$$\text{Area} = 6\ln 5 - 4 - 4 = 6\ln 5 - 8.$$

A1 (about 1.66, which is sensible for the thin region in the diagram).

Example 4

The region R is bounded by the curve $y = x + \dfrac{4}{x}$, the x-axis and the lines $x = 1$ and $x = 4$.

(a) Use the trapezium rule with three intervals to find an approximation to the area of R. Give your answer correct to 3 significant figures. **[3]**

(b) Find the exact area of R, giving your answer in the form $a + b\ln 2$. **[3]**

(c) Explain, with reference to the shape of the curve, whether your answer to (a) is an over-estimate or an under-estimate. **[2]**

(a) $h = \frac{4-1}{3} = 1$. The y-values at $x = 1, 2, 3, 4$ are $5,\ 4,\ \frac{13}{3},\ 5$. **B1**

$$\text{Area} \approx \tfrac{1}{2} \times 1 \times \left[5 + 2\left(4 + \tfrac{13}{3}\right) + 5\right] = \tfrac{1}{2} \times \tfrac{80}{3} = 13.33\ldots$$

M1 for the correct formula with $h = 1$. 13.3 **A1**

(b)

$$\int_1^4 \left(x + \frac{4}{x}\right) dx = \left[\tfrac{1}{2}x^2 + 4\ln x\right]_1^4.$$

M1 for the form $k_1 x^2 + k_2 \ln x$, **A1** for the correct integral.

$$= (8 + 4\ln 4) - \left(\tfrac{1}{2} + 0\right) = \tfrac{15}{2} + 8\ln 2.$$

A1, using $4\ln 4 = 4\ln 2^2 = 8\ln 2$. (This is about 13.05.)

(c) Over-estimate. **B1** The curve bends upwards between $x = 1$ and $x = 4$, so the straight top of each trapezium lies above the curve (see the first diagram in "Understand it"). **B1** for the reason. It agrees with $13.33 > 13.05$.

Common mistakes

- **Forgetting to divide by** a, or multiplying by it instead. $\int e^{5x}\,dx = \frac{1}{5}e^{5x}$, not $5e^{5x}$; $\int \frac{1}{3x}\,dx = \frac{1}{3}\ln x$, not $3\ln x$ or $\ln 3x$ with no factor. Examiner reports often mention wrong coefficients on log terms. Differentiate your answer to check.
- **Getting the sign wrong for sine and cosine.** $\int \sin kx\,dx = -\frac{1}{k}\cos kx$; $\int \cos kx\,dx = +\frac{1}{k}\sin kx$.
- **Treating a power as a coefficient.** $\cos^2 x$ is not $\cos 2x$ or $2\cos x$. It needs the identity $\frac{1}{2}(1 + \cos 2x)$. Reports say this confusion is common when an expansion produces a $\cos^2$ term.
- **Not doubling the angle.** $\sin^2 2x = \frac{1}{2}(1 - \cos 4x)$, not $\frac{1}{2}(1 - \cos 2x)$. And when a part (a) identity in θ is used with $2x$, every 2θ becomes $4x$.
- **Breaking the log laws.** $\ln(a + 14) - \ln a$ is $\ln \frac{a+14}{a}$, not $\frac{\ln(a+14)}{\ln a}$. A "right" answer reached this way gets no accuracy mark.
- **Taking logs too early.** In $e^{2a} + 3 = 12$, you cannot take $\ln$ of each term. Get e^{2a} alone first ($e^{2a} = 9$), then take logs.

- **Giving a decimal when "exact" is asked.** $2e-2$ or $\ln 100$ is exact; 3.44 or 4.61 loses the final mark.
- **The wrong strip width or number of y-values.** Three intervals need four y-values; h is the interval length divided by the number of strips.
- **Explaining over- or under-estimates with rounding.** The reason must be about the **shape**: the trapezium tops are above (or below) the curve because it bends up (or down).
- **Adding an area below the axis as a negative.** A region below the x-axis has a negative integral; use its size when adding areas.
- **Dropping the remainder term** after dividing, or using a synthetic-division quotient without dividing it by the leading coefficient of the divisor.
- **Working in degrees.** Every angle in these integrals is in radians.

How to get the marks

- **Method marks are for the form.** Mark schemes typically give **M1** for an integral "of the form ke^{2x-1}" or "$k\ln(4x+1)$" even if k is wrong, and **A1** for the right k. So always write the integral down before substituting the limits.
- **Show the limits going in**, for example $(2\ln 10)-(2\ln 1)$. A mark for "applying the limits correctly" needs to see it.
- **"Show that"** means the answer is printed. Every step must be visible: the identity used, the rearrangement, the log law. Checking the printed answer by substituting it is not a proof.
- **"Hence"** means use the previous part, usually an identity that rewrites the integrand.
- **"Exact"** means keep $\ln$, e, π and surds. Simplify $e^{\ln k}$ and combine logs if a form such as $\ln a$ or $a+\ln b$ is asked for.
- **Indefinite integrals**: write $+c$. Some schemes condone its absence, but not all.
- **Trapezium rule**: show the y-values and the formula with your h, then round to the accuracy asked (often 3 significant figures; sometimes 2 decimal places).
- **"State, with a reason"**: one clear word (over or under), then the reason about the trapezium tops and the curve. A sketch helps.
- **Accuracy**: non-exact answers to 3 significant figures unless told otherwise; angles in radians.

Quick check

1. Find $\displaystyle\int\left(6e^{3x+1}+\frac{10}{5x-1}\right)dx$.
2. Find the exact value of $\displaystyle\int_0^{\pi/8}\tan^2 2x\,dx$.
3. Find the exact value of $\displaystyle\int_0^{\pi/6}4\cos^2 3x\,dx$.
4. Given that $\displaystyle\int_1^a\frac{4}{2x-1}\,dx=\ln 49$, where $a>1$, find the value of a.

5. Show that $\dfrac{4x^2+4x+5}{2x+1} = 2x+1+\dfrac{4}{2x+1}$, and hence find the exact value of $\displaystyle\int_0^2 \frac{4x^2+4x+5}{2x+1}\,dx$ in the form $a+\ln b$.

Answers

1. $\int 6e^{3x+1}\,dx = 2e^{3x+1}$ and $\int \frac{10}{5x-1}\,dx = 2\ln|5x-1|$, so the answer is $2e^{3x+1} + 2\ln|5x-1| + c$.
2. $\tan^2 2x = \sec^2 2x - 1$, so the integral is $\left[\frac{1}{2}\tan 2x - x\right]_0^{\pi/8} = \frac{1}{2}\tan\frac{\pi}{4} - \frac{\pi}{8} = \frac{1}{2} - \frac{\pi}{8}$.
3. $4\cos^2 3x = 2 + 2\cos 6x$, so the integral is $\left[2x + \frac{1}{3}\sin 6x\right]_0^{\pi/6} = \frac{\pi}{3} + \frac{1}{3}\sin\pi = \frac{\pi}{3}$.
4. $[2\ln(2x-1)]_1^a = 2\ln(2a-1) - 2\ln 1 = \ln(2a-1)^2$. So $(2a-1)^2 = 49$, $2a-1 = 7$ (it is positive as $a > 1$), and $a = 4$.
5. $(2x+1)^2 = 4x^2 + 4x + 1$, so $4x^2 + 4x + 5 = (2x+1)^2 + 4$; divide by $2x+1$. Then $\left[x^2 + x + 2\ln(2x+1)\right]_0^2 = 6 + 2\ln 5 - 0 = 6 + \ln 25$.

Past questions to try

- **9709/23/M/J/22 Q7**: the area between a root curve and a curve that needs the $\sin^2$ identity.
- **9709/22/O/N/22 Q6**: an exact area with one curve above the x-axis and one below, giving both a $\ln$ and an e term.
- **9709/21/M/J/21 Q6**: the trapezium rule and an exact integral combined, with an over- or under-estimate.
- **9709/22/F/M/23 Q5**: an integral equal to $\ln 10$, rearranged for iteration.