Cambridge AS & A Level · Mathematics (9709) · Notes

Integration A2

Read online at pastopic.com/cambridge/mathematics-9709/notes/integration-p3

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Read first: Integration, Differentiation, Algebra, Trigonometry

What you need to know

By the end of this topic you should be able to:

1. **Integrate the standard functions by reversing differentiation**: e^{ax+b}, $\dfrac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$ and $\sec^2(ax+b)$ (all from Paper 2), and now also $\dfrac{1}{x^2+a^2}$, which integrates to an inverse tangent.
2. **Use trigonometric identities to make an integrand integrable**, for example double-angle formulae to remove a square such as $\sin^2 3x$, or an identity proved in an earlier part of the question.
3. **Integrate a rational function by splitting it into partial fractions**, for the denominators in the Algebra topic: three linear factors, a linear factor times a repeated linear factor, and a linear factor times a quadratic cx^2+d.
4. **Recognise an integrand of the form** $\dfrac{k\,f'(x)}{f(x)}$ (the top is a multiple of the derivative of the bottom) and integrate it to $k\ln|f(x)|$.
5. **Recognise when an integrand is a product and use integration by parts**, including the cases where one factor is $\ln$ or $\tan^{-1}$, and cases that need parts twice.
6. **Use a substitution you are given** to change an integral, definite or indefinite, into one you can do, and evaluate it.

Paper 1 and Paper 2 skills (areas, volumes of revolution, exact values with $\ln$ and e) are used again throughout. You are always **told** which substitution to use; choosing your own is not needed.

Understand it

A reminder from Paper 2

The Paper 2 Integration note ("Read first") explains these slowly. In short, integrate as if the bracket were just x, then divide by the number in front of x:

Integrand	Integral $(+c)$
e^{ax+b}	$\dfrac{1}{a}e^{ax+b}$

Integrand	Integral $(+c)$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\ln\lvert ax+b\rvert$
$\sin(ax+b)$	$-\dfrac{1}{a}\cos(ax+b)$
$\cos(ax+b)$	$\dfrac{1}{a}\sin(ax+b)$
$\sec^2(ax+b)$	$\dfrac{1}{a}\tan(ax+b)$

Squares of sines and cosines go through $\cos^2 A = \frac{1}{2}(1+\cos 2A)$ and $\sin^2 A = \frac{1}{2}(1-\cos 2A)$, and $\tan^2 A = \sec^2 A - 1$. All angles are in **radians**.

The inverse tangent integral

From Paper 3 differentiation, $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$. With $\frac{x}{a}$ in place of x, the chain rule gives

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{1/a}{1+x^2/a^2} = \frac{a}{a^2+x^2}.$$

Divide by a and read it backwards:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c.$$

This is in the formula list. Use it when the bottom is "x^2 plus a positive number" and the top is a constant.

- $\displaystyle\int \frac{5}{x^2+16}\,dx = \frac{5}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$, since $a=4$.
- $\displaystyle\int_0^2 \frac{1}{x^2+4}\,dx = \left[\tfrac{1}{2}\tan^{-1}\tfrac{x}{2}\right]_0^2 = \tfrac{1}{2}\tan^{-1}1 = \frac{\pi}{8}$.
- If x^2 has a coefficient, take it out first so the bottom starts with x^2: $\displaystyle\int \frac{1}{25+4x^2}\,dx = \frac{1}{4}\int \frac{1}{x^2+\frac{25}{4}}\,dx = \frac{1}{4}\times\frac{2}{5}\tan^{-1}\left(\frac{2x}{5}\right) = \frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right) + c$.

The formula list also gives two logarithm results, for "x^2 **minus** a number" and "a number minus x^2". For example, with $a=2$,

$$\int_0^1 \frac{1}{4-x^2}\,dx = \left[\frac{1}{4}\ln\frac{2+x}{2-x}\right]_0^1 = \frac{1}{4}\ln 3.$$

You could also get these by partial fractions; quoting the formula is quicker.

Recognising $\frac{f'(x)}{f(x)}$

By the chain rule, $\dfrac{d}{dx}\ln\lvert f(x)\rvert = \dfrac{f'(x)}{f(x)}$. So whenever the top is a **multiple of the derivative of the bottom**,

$$\int \frac{k\,f'(x)}{f(x)}\,dx = k\ln\lvert f(x)\rvert + c.$$

Differentiate the bottom and compare it with the top:

- $\displaystyle\int \frac{x^2}{x^3+4}\,dx$: the bottom differentiates to $3x^2$, and the top is $\frac{1}{3}$ of that, so the answer is $\frac{1}{3}\ln|x^3+4|+c$.
- $\displaystyle\int \frac{e^{2x}}{e^{2x}+5}\,dx = \frac{1}{2}\ln\left(e^{2x}+5\right)+c$ (no modulus needed: $e^{2x}+5>0$).
- $\displaystyle\int \tan 2x\,dx = \int \frac{\sin 2x}{\cos 2x}\,dx = -\frac{1}{2}\ln|\cos 2x|+c = \frac{1}{2}\ln|\sec 2x|+c$, because $\cos 2x$ differentiates to $-2\sin 2x$.

If the top is **not** a multiple, split the fraction. Here the x part is a log and the constant part is an inverse tangent:

$$\int \frac{3x+2}{x^2+4}\,dx = \int \frac{3x}{x^2+4}\,dx + \int \frac{2}{x^2+4}\,dx = \frac{3}{2}\ln\left(x^2+4\right)+\tan^{-1}\left(\frac{x}{2}\right)+c.$$

A close relative: when a function is multiplied by its own derivative, reverse the chain rule. $\dfrac{d}{dx}\cos^5 x = -5\cos^4 x\sin x$, so $\displaystyle\int \sin x\cos^4 x\,dx = -\frac{1}{5}\cos^5 x+c$; similarly $\displaystyle\int \sec^2 x\tan^3 x\,dx = \frac{1}{4}\tan^4 x+c$ and $\displaystyle\int \cos x\, e^{\sin x}\,dx = e^{\sin x}+c$. (The substitutions $u=\cos x$, $u=\tan x$ and $u=\sin x$ give the same answers.)

Trigonometric integrals

Paper 3 questions often rewrite the integrand first, with an identity you are asked to prove in part (a), or with one of these:

- **Squares of multiple angles**: $\sin^2 3x = \frac{1}{2}(1-\cos 6x)$, so $\displaystyle\int \sin^2 3x\,dx = \frac{1}{2}x - \frac{1}{12}\sin 6x + c$. The angle doubles, and you divide by the new number.
- **Products of different angles**: adding the expansions of $\sin(4x+x)$ and $\sin(4x-x)$ gives $2\sin 4x\cos x = \sin 5x + \sin 3x$. So $\displaystyle\int \sin 4x\cos x\,dx = -\frac{1}{10}\cos 5x - \frac{1}{6}\cos 3x + c$.
- **After an $R\cos(x-\alpha)$ form**: $\cos x + \sqrt{3}\sin x = 2\cos\left(x-\frac{1}{3}\pi\right)$, so $\dfrac{1}{\left(\cos x+\sqrt{3}\sin x\right)^2} = \dfrac{1}{4}\sec^2\left(x-\frac{1}{3}\pi\right)$, which integrates to $\dfrac{1}{4}\tan\left(x-\frac{1}{3}\pi\right)$.
- **A power of $\cos x$ times $\sin x$**, as in the last section. $\sin 2x\cos x = 2\sin x\cos^2 x$, for instance, integrates to $-\frac{2}{3}\cos^3 x$.

Partial fractions

The Algebra note shows how to split a fraction. Each piece is then a standard integral:

Denominator	Partial fractions	Each piece integrates to
$(ax+b)(cx+d)(ex+f)$	$\dfrac{A}{ax+b}+\dfrac{B}{cx+d}+\dfrac{C}{ex+f}$	$\dfrac{A}{a}\ln\lvert ax+b\rvert$, and so on
$(ax+b)(cx+d)^2$	$\dfrac{A}{ax+b}+\dfrac{B}{cx+d}+\dfrac{C}{(cx+d)^2}$	the last one is **not** a log: $-\dfrac{C}{c(cx+d)}$

Denominator	Partial fractions	Each piece integrates to
$(ax+b)(cx^2+d)$	$\dfrac{A}{ax+b}+\dfrac{Bx+C}{cx^2+d}$	split the second: Bx part gives a log, C part an inverse tangent

- **The repeated factor.** $\dfrac{C}{(cx+d)^2}=C(cx+d)^{-2}$ is a power, so the Paper 1 rule applies: $\displaystyle\int_0^1 \frac{4}{(2x+1)^2}\,dx=\left[-\frac{2}{2x+1}\right]_0^1=-\frac{2}{3}+2=\frac{4}{3}$.
- **The quadratic factor.** $\dfrac{Bx+C}{cx^2+d}=\dfrac{Bx}{cx^2+d}+\dfrac{C}{cx^2+d}$, the log-plus-inverse-tangent pair from the last section. Treating the whole thing as one log, or as one inverse tangent, is wrong.
- **Top and bottom of the same degree**: the partial fractions start with a constant, $A+\dfrac{B}{\cdots}+\dfrac{C}{\cdots}$, which integrates to Ax.
- **Top of higher degree**: divide first, then deal with the remainder over the divisor.

Integration by parts

The product rule says $\dfrac{d}{dx}(uv)=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$. Integrate both sides and rearrange:

$$\int u\,\frac{dv}{dx}\,dx=uv-\int v\,\frac{du}{dx}\,dx.$$

This is in the formula list. It swaps one integral for another, and it helps when the new integral is **easier**. Choose:

- u = the factor that gets simpler when you differentiate it (a power of x), and
- $\dfrac{dv}{dx}$ = the factor you can integrate (e^{kx}, $\sin kx$, $\cos kx$, $\sec^2 x$, a power of x).
- **Exception:** $\ln$ and $\tan^{-1}$ are hard to integrate but easy to differentiate, so they are always u.

For $\displaystyle\int xe^{3x}\,dx$: $u=x$, $\dfrac{dv}{dx}=e^{3x}$, so $\dfrac{du}{dx}=1$ and $v=\frac{1}{3}e^{3x}$:

$$\int xe^{3x}\,dx=\frac{1}{3}xe^{3x}-\int\frac{1}{3}e^{3x}\,dx=\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}+c.$$

For $\displaystyle\int x\ln x\,dx$: $u=\ln x$, $\dfrac{dv}{dx}=x$, so $\dfrac{du}{dx}=\dfrac{1}{x}$ and $v=\frac{1}{2}x^2$:

$$\int x\ln x\,dx=\frac{1}{2}x^2\ln x-\int\frac{1}{2}x^2\cdot\frac{1}{x}\,dx=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+c.$$

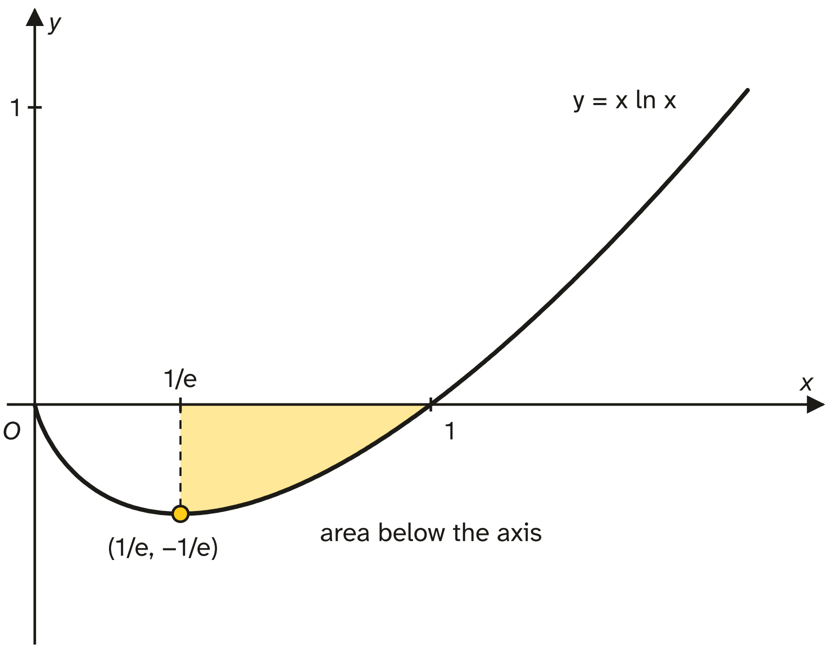
The hidden 1. A lone $\ln$ or $\tan^{-1}$ is a product with 1: take $u=\ln 2x$ and $\dfrac{dv}{dx}=1$, so $v=x$. Then $\displaystyle\int\ln 2x\,dx=x\ln 2x-\int x\cdot\frac{1}{x}\,dx=x\ln 2x-x+c$. In the same way, $\displaystyle\int\tan^{-1}3x\,dx=x\tan^{-1}3x-\int\frac{3x}{1+9x^2}\,dx=x\tan^{-1}3x-\frac{1}{6}\ln\left(1+9x^2\right)+c$, where the second integral is an $\frac{f'}{f}$.

Parts twice. With x^2 as u, one round of parts leaves x times something, so do parts again:

$$\int x^2 \cos 3x \, dx = \frac{1}{3}x^2 \sin 3x - \int \frac{2}{3}x \sin 3x \, dx = \frac{1}{3}x^2 \sin 3x + \frac{2}{9}x \cos 3x - \frac{2}{27}\sin 3x + c.$$

Definite integrals. $\int_a^b u \frac{dv}{dx} \, dx = \left[uv\right]_a^b - \int_a^b v \frac{du}{dx} \, dx$. The limits go into **both** parts.

Areas below the axis. A definite integral is negative where the curve is below the x-axis. The curve $y = x \ln x$ has its minimum at $\left(\frac{1}{e}, -\frac{1}{e}\right)$ and crosses the x-axis at $x = 1$:



Using the integral above, $\int_{1/e}^{1} x \ln x \, dx = \left(0 - \frac{1}{4}\right) - \left(-\frac{1}{2e^2} - \frac{1}{4e^2}\right) = \frac{3 - e^2}{4e^2} \approx -0.148$. The shaded **area** is the size of this: $\frac{e^2 - 3}{4e^2}$.

Integration by substitution

A substitution is the chain rule backwards: you replace x by a new variable u so that the integral becomes a standard one. You are always told what u is. Every piece of the integral must change:

1. **Differentiate** the substitution and write dx in terms of du (for example $u = 2x + 1$ gives $du = 2\,dx$, so $dx = \frac{1}{2}\,du$).
2. **Replace every** x, using the substitution rearranged if needed ($x = \frac{1}{2}(u - 1)$).
3. **Change the limits**: put each x-limit into the substitution to get the u-limit.
4. **Simplify and integrate** in u. For a definite integral there is no need to go back to x.

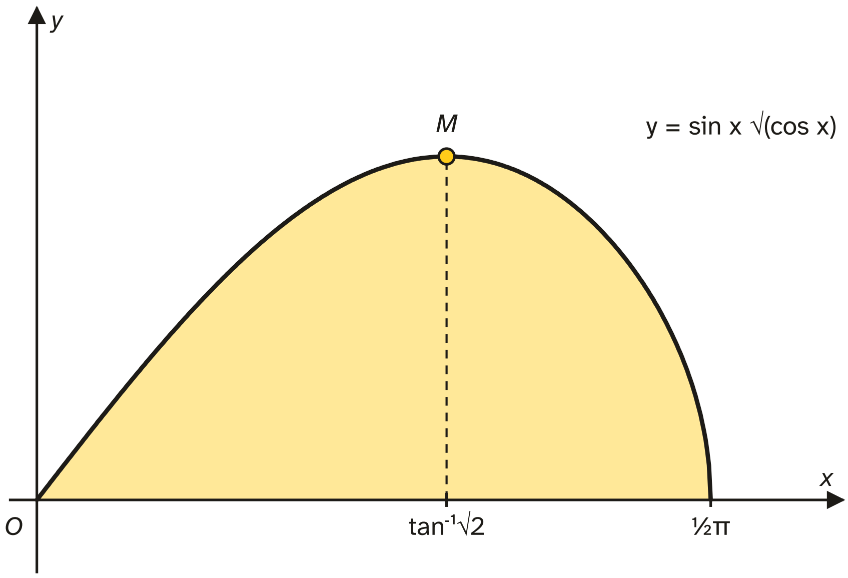
For $\int_0^4 \frac{x}{\sqrt{2x+1}} \, dx$ with $u = 2x + 1$: $x = 0$ gives $u = 1$ and $x = 4$ gives $u = 9$, so

$$\int_1^9 \frac{\frac{1}{2}(u-1)}{\sqrt{u}} \cdot \frac{1}{2} \, du = \frac{1}{4}\int_1^9 \left(u^{1/2} - u^{-1/2}\right) du = \frac{1}{4}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9 = \frac{1}{4}\left(12 + \frac{4}{3}\right) = \frac{10}{3}.$$

Trigonometric substitutions turn a sum of squares into one term. For $\int_0^2 \frac{1}{(x^2+4)^{3/2}}\,dx$ with $x = 2\tan\theta$: $dx = 2\sec^2\theta\,d\theta$, and $x^2+4 = 4\tan^2\theta + 4 = 4\sec^2\theta$, so $\left(x^2+4\right)^{3/2} = 8\sec^3\theta$. The limits become $\theta = 0$ and $\theta = \frac{1}{4}\pi$:

$$\int_0^{\pi/4} \frac{2\sec^2\theta}{8\sec^3\theta}\,d\theta = \frac{1}{4}\int_0^{\pi/4} \cos\theta\,d\theta = \frac{1}{4}\sin\frac{\pi}{4} = \frac{\sqrt{2}}{8}.$$

A minus sign can swap the limits. With $u = \cos x$, $du = -\sin x\,dx$, and $x = 0$ to $x = \frac{1}{2}\pi$ becomes $u = 1$ to $u = 0$. Since $-\int_1^0 = \int_0^1$, the minus sign and the upside-down limits cancel. Write this step out; don't make both changes silently. The curve in Example 4 is worked this way:



Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\int x^n\,dx = \frac{x^{n+1}}{n+1}\ (n \neq -1)$, $\int \frac{1}{x}\,dx = \ln\lvert x\rvert$, $\int e^x\,dx = e^x$	Given
$\int \sin x\,dx = -\cos x$, $\int \cos x\,dx = \sin x$, $\int \sec^2 x\,dx = \tan x$	Given
$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	Given
$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\frac{x-a}{x+a}\ (x > a)$, $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\frac{a+x}{a-x}\ (\lvert x\rvert < a)$	Given
$\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert$	Given
$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$	Given

Formula	Status
$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}$, $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\lvert ax+b\rvert$	Learn it
$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)$, $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)$, $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)$	Learn it
$\int \frac{1}{(ax+b)^2}\,dx = -\frac{1}{a(ax+b)}$	Learn it
$\int \tan x\,dx = \ln\lvert\sec x\rvert = -\ln\lvert\cos x\rvert$	Learn it
$\int f'(x)\big(f(x)\big)^n\,dx = \frac{\big(f(x)\big)^{n+1}}{n+1}\ (n \neq -1)$	Learn it
$\int_a^b u\frac{dv}{dx}\,dx = \Big[uv\Big]_a^b - \int_a^b v\frac{du}{dx}\,dx$	Learn it
Substitution: replace dx using $\frac{du}{dx}$, replace every x, change the limits	Learn it
$\sin^2 A + \cos^2 A = 1,\ 1 + \tan^2 A = \sec^2 A,\ \sin 2A = 2\sin A\cos A,$ $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$, and the $\sin(A \pm B)$, $\cos(A \pm B)$ expansions	Given
$\cos^2 A = \frac{1}{2}(1+\cos 2A),\ \sin^2 A = \frac{1}{2}(1-\cos 2A),\ \tan^2 A = \sec^2 A - 1$	Learn it
Partial fraction forms for $(ax+b)(cx+d)(ex+f)$, $(ax+b)(cx+d)^2$ and $(ax+b)(cx^2+d)$	Learn it
$\ln a + \ln b = \ln ab,\ \ln a - \ln b = \ln\frac{a}{b},\ k\ln a = \ln a^k$	Learn it

Arbitrary constants are left out of the table; write $+\,c$ with every indefinite integral.

How to do it

We have tagged 62 Paper 3 questions for this topic (2021–2025), all different. 10 of them are counted under two types, so the counts add up to more than 62. 45 of the 62 also belong to another topic: a stationary point first (Differentiation, 20), partial fractions or division first (Algebra, 14), an identity to prove first (Trigonometry, 6), or an iteration afterwards (Numerical solution of equations, 5).

Question type	Questions
Integration by parts	24
Integration by a given substitution	16
Partial fractions and other algebraic fractions	15

Question type	Questions
Trigonometric integrals	12
An integral equal to a value, then iteration	5

1. Integration by parts (24 questions)

1. **Choose u and $\frac{dv}{dx}$.** $\ln$ or $\tan^{-1}$ is always u; otherwise u is the power of x. Write down u, $\frac{du}{dx}$, $\frac{dv}{dx}$ and v before you start.
2. **Apply the formula**: $uv - \int v\frac{du}{dx}\,dx$. Tidy the new integrand (for example $\frac{1}{2}x^2 \cdot \frac{1}{x} = \frac{1}{2}x$).
3. **Do the second integral.** If it still has a power of x times e^{kx}, $\sin$ or $\cos$, use parts again, with the same choice of roles.
4. **Substitute the limits** into the whole expression, in radians, and simplify to the form asked.

For example, $\displaystyle\int_0^1 xe^{-2x}\,dx = \left[-\frac{1}{2}xe^{-2x}\right]_0^1 + \int_0^1 \frac{1}{2}e^{-2x}\,dx = \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^1 = \frac{1}{4} - \frac{3}{4}e^{-2}$.

Variations you will meet:

- **A lone logarithm**, such as $\ln kx$: use the hidden 1. $\int \ln kx\,dx = x\ln kx - x$, whatever the constant k.
- **A power of x times** $\ln x$, including negative and fractional powers such as $\dfrac{\ln x}{x^4}$, $\dfrac{\ln x}{\sqrt{x}}$ or $x^{-2/3}\ln x$. Write the power as x^n, and $u = \ln x$. The second integral is always a single power of x.
- **A power of x times an exponential**, such as $(k-x)e^{-x/3}$ or xe^{-ax} with a constant a. With a constant, keep a as a letter all the way through.
- **Parts twice**: $x^2\sin 2x$ or $x^2\cos\frac{1}{3}x$. Keep the signs in brackets, since the second round is subtracted. Check each $\frac{1}{k}$ carefully: $\int\cos\frac{1}{3}x\,dx = 3\sin\frac{1}{3}x$.
- **An inverse tangent**, alone, such as $\tan^{-1}(\frac{1}{2}x)$, or times x, such as $x\tan^{-1}x$ or $x\tan^{-1}(2x)$. With $x\tan^{-1}(kx)$, $v = \frac{1}{2}x^2$ and the second integral is $\int \dfrac{kx^2}{2(1+k^2x^2)}\,dx$; divide first: $\dfrac{x^2}{1+4x^2} = \dfrac{1}{4} - \dfrac{1}{4(1+4x^2)}$. Part (a) may ask for this division, or for $\frac{dy}{dx}$ from $\tan y = kx$.
- $x\sec^2 x$: $v = \tan x$, so the second integral is $\int\tan x\,dx = \ln|\sec x|$.
- **An area from a diagram**, usually after finding a stationary point. If the region is below the x-axis, the integral is negative and the area is its size.
- **After a substitution**: the new integral in u is a product, such as $2ue^{2u}$ or $2u\sin u$.
- **"Show that the integral is less than k for all $a > 1$"**: find the integral in terms of a, then explain why the terms subtracted from k are positive. For example, $\displaystyle\int_1^a \frac{\ln x}{x^3}\,dx = \frac{1}{4} - \frac{\ln a}{2a^2} - \frac{1}{4a^2} < \frac{1}{4}$, because $\ln a > 0$ and $a^2 > 0$ when $a > 1$.

2. Integration by a given substitution (16 questions)

1. **Differentiate the substitution** and write dx (or $d\theta$) in terms of du.

2. **Replace every** x, including the x's inside the dx. Rearrange the substitution if needed, such as $x^2 = u + 3$ or $\cos x = u - 2$.
3. **Change the limits**, and write both conversions down.
4. **Simplify**, then integrate in u. Deal with any minus sign by swapping the limits, shown as a separate step.

Variations you will meet:

- **"Show that $\int \dots\, dx = \int_a^b \dots\, du$, where a and b are to be found"**, then "hence" evaluate. Split odd powers: with $u = x^2 - k$, $x^3\, dx = x^2 \cdot x\, dx = (u + k) \cdot \frac{1}{2}\, du$. Then divide each term by the root, $\dfrac{u+3}{\sqrt{u}} = u^{1/2} + 3u^{-1/2}$, and integrate term by term.
- **u = the expression inside a root or a power**, such as $u = 3 - 2x$, $u = 2 + \cos x$ or $u = 3 + 2\tan x$. The rest of the integrand usually becomes du (for example $\sec^2 x\, dx = \frac{1}{2}\, du$ when $u = 3 + 2\tan x$).
- **$u = \sin x$ or $u = \cos x$ with double angles**: first write $\sin 2x = 2 \sin x \cos x$ and $\cos 2x = 2\cos^2 x - 1$ (or $1 - 2\sin^2 x$), so everything left is in one trig function times du.
- $u = \sqrt{x}$: $x = u^2$, so $dx = 2u\, du$. An infinite limit stays infinite; as $u \to \infty$, $\tan^{-1} u \to \frac{1}{2}\pi$. For example, $\displaystyle\int_4^\infty \frac{1}{\sqrt{x}\,(x+4)}\, dx = \int_2^\infty \frac{2}{u^2+4}\, du = \left[\tan^{-1}\frac{u}{2}\right]_2^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.
- $x = a\tan\theta$, for $x^2 + a^2$ in the bottom: use $1 + \tan^2\theta = \sec^2\theta$. The result is often $\cos^2\theta$, which needs the double-angle identity (type 4).
- $u = e^x$: $du = e^x\, dx$, so $dx = \dfrac{du}{u}$. The result is often a fraction in u for partial fractions. For example, $\displaystyle\int_0^{\ln 3} \frac{e^{2x}}{e^x + 1}\, dx = \int_1^3 \frac{u}{u+1}\, du = \Big[u - \ln(u+1)\Big]_1^3 = 2 - \ln 2$.
- **An integrand of the form** $\dfrac{k}{a^2 - u^2}$ after the substitution: use the formula-list result, with the u-limits.
- **"Using the substitution ..., or otherwise"**: you may spot the answer directly (for example as an $e^{f(x)}$ reversed chain rule), but show a method.
- **An area or a volume**, with the curve drawn after a stationary-point part. Use the x-limits from the diagram, then convert them.

3. Partial fractions and other algebraic fractions (15 questions)

1. **Split** the fraction into partial fractions (see the Algebra note): choose the form from the denominator, and find the constants by substituting values of x and comparing coefficients.
2. **Integrate term by term**: $\dfrac{A}{ax+b} \to \dfrac{A}{a}\ln|ax+b|$; $\dfrac{C}{(cx+d)^2} \to -\dfrac{C}{c(cx+d)}$; $\dfrac{Bx+C}{x^2+k}$ splits into a log and an inverse tangent.
3. **Substitute the limits**, then combine the logs into the form asked.

For example, with a constant $a > 0$: $\dfrac{3a}{(x+a)(x+4a)} = \dfrac{1}{x+a} - \dfrac{1}{x+4a}$, so

$$\int_0^{2a} \frac{3a}{(x+a)(x+4a)}\, dx = \left[\ln\frac{x+a}{x+4a}\right]_0^{2a} = \ln\frac{3a}{6a} - \ln\frac{a}{4a} = \ln\frac{1}{2} - \ln\frac{1}{4} = \ln 2.$$

Variations you will meet:

- **A constant a in the fraction**: treat it as a number. Inside each log the a's cancel when you combine the limits, as above, but the partial fraction constants themselves may contain a, so a can multiply the log terms as well as come from a constant term integrated to Ax. Take a out as a common factor to reach a form such as $a(p+\ln q)$. For example, $\dfrac{x^2+5ax+5a^2}{(x+a)(x+2a)} = 1 + \dfrac{a}{x+a} + \dfrac{a}{x+2a}$, so its integral from 0 to a is $a + a\ln\dfrac{2a}{a} + a\ln\dfrac{3a}{2a} = a(1+\ln 3)$.
- **Numerator and denominator of the same degree**: the partial fractions start with a constant.
- **A repeated factor $(cx+d)^2$**: its term integrates to a fraction, not a third log.
- **A quadratic factor x^2+k** or $1+kx^2$: the constant B may come out as 0; that is allowed. Split $\dfrac{Bx+C}{x^2+k}$ in two. For $\dfrac{1}{1+3x^2} = \dfrac{1}{3} \times \dfrac{1}{x^2+\frac{1}{3}}$, the result is $\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\,x\right)$: the $\sqrt{3}$ is **inside** the inverse tangent.
- **Divide first**, when part (a) asks for a quotient and remainder. For example $\dfrac{x^3+2x^2+3x+5}{x^2+1} = x + 2 + \dfrac{2x+3}{x^2+1}$, which integrates to $\frac{1}{2}x^2 + 2x + \ln\left(x^2+1\right) + 3\tan^{-1}x + c$.
- **No partial fractions needed**: a fraction such as $\dfrac{x(x+1)}{x^2+4}$ is $1 + \dfrac{x-4}{x^2+4}$, then a log and an inverse tangent.
- **After the substitution $u = e^x$**, the fraction in u is split and integrated in the same way, with the limits in u.
- **Answer forms**: "a single logarithm", "$\ln\frac{a}{b}$", "$a + b\ln c$", "$p\ln q + r\ln s$ with q and s prime", "$a\pi - \ln b$". Use the log laws, and keep a number in front of a log unless the form says otherwise.

4. Trigonometric integrals (12 questions)

1. **Rewrite** the integrand so that every term is a standard integral: use the identity proved in part (a), or a double-angle or compound-angle formula.
2. **Integrate** term by term, dividing by the number in front of x.
3. **Substitute the limits** in radians, using exact values, and simplify.

For example, $\displaystyle\int_0^{\pi/4} \sin 2x(1-\sin 2x)\,dx = \int_0^{\pi/4}\left(\sin 2x - \frac{1}{2} + \frac{1}{2}\cos 4x\right)dx = \left[-\frac{1}{2}\cos 2x - \frac{1}{2}x + \frac{1}{8}\sin 4x\right]_0^{\pi/4} = \frac{1}{2} - \frac{\pi}{8}$.

Variations you will meet:

- **A single square**, such as $k\cos^2 5x$: $\frac{1}{2}k(1+\cos 10x)$, so the new angle is $10x$.
- **"Hence" after an identity**: $\frac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$ (then $\sec^2\theta - 1$); $\operatorname{cosec} 2\theta - \cot 2\theta = \tan\theta$ (then $\ln|\sec\theta|$); $\cos^4\theta - \sin^4\theta = \cos 2\theta$; a product such as $\sin 3x\cos 2x$ as half a sum of sines.
- **A power of $\cos x$ times $\sin x$**, perhaps after an identity: $\cos x\sin 4x$ written as $8\sin x\cos^4 x - 4\sin x\cos^2 x$, or $\sin x\cos 2x = 2\sin x\cos^2 x - \sin x$. Each $\sin x\cos^n x$ integrates to $-\frac{1}{n+1}\cos^{n+1} x$.
- **After $R\cos(2x+\alpha)$**: $\dfrac{k}{R^2\cos^2(2x+\alpha)} = \dfrac{k}{R^2}\sec^2(2x+\alpha)$, which integrates to $\dfrac{k}{2R^2}\tan(2x+\alpha)$.
- **After a substitution $x = a\tan\theta$**, which leaves $\cos^2\theta$.

- **A volume of revolution**, $\pi \int y^2\,dx$: squaring $\cos x\sqrt{\sin 2x}$ gives $\cos^2 x \sin 2x = 2\sin x \cos^3 x$.
- **An area from a diagram**: take the size of any part below the x-axis, and use the limits where the curve meets the axis.

5. An integral equal to a value, then iteration (5 questions)

1. **Integrate**, almost always by parts, and substitute the limits, one of which is the unknown a.
2. **Set equal** to the given number.
3. **Rearrange** to the printed form, one step per line. Work out what to isolate by looking at that form (for example get $\ln a$ alone, then take $\exp$ of both sides).

For example, $\displaystyle\int_0^a xe^x\,dx = 3$ gives $\left[xe^x - e^x\right]_0^a = (a-1)e^a + 1 = 3$, so $(a-1)e^a = 2$ and $a = 1 + 2e^{-a}$.

Variations you will meet: the integrand is xe^{kx}, $x^n \ln x$ or $\frac{\ln x}{\sqrt{x}}$; the printed form may use $\exp(\ldots)$ for $e^{\cdots}$, or a cube root. Parts (b) and (c), a sign-change check and an iteration to 2 decimal places with each step to 4 decimal places, use the method in the Numerical solution of equations topic, in radians where trigonometry appears.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **DM1** method mark that needs the M mark before it.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

Find the exact value of $\displaystyle\int_1^4 \sqrt{x}\,\ln x\,dx$, giving your answer in the form $a\ln 2 + b$, where a and b are rational. **[5]**

$\ln x$ must be u: $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$, $\dfrac{dv}{dx} = x^{1/2}$, $v = \dfrac{2}{3}x^{3/2}$.

$$\int \sqrt{x}\,\ln x\,dx = \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{3/2}\cdot\frac{1}{x}\,dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}\,dx.$$

M1 for parts reaching the form $px^{3/2}\ln x - q\int x^{1/2}\,dx$, **A1** for the correct expression.

$$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2}.$$

A1

$$\left[\frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2}\right]_1^4 = \left(\frac{16}{3}\ln 4 - \frac{32}{9}\right) - \left(0 - \frac{4}{9}\right).$$

DM1 for substituting both limits (note $4^{3/2} = 8$ and $\ln 1 = 0$).

Since $\ln 4 = 2\ln 2$, the answer is $\frac{32}{3}\ln 2 - \frac{28}{9}$. **A1**

Example 2

(a) Use the substitution $u = \sqrt{x}$ to show that $\int_1^4 e^{\sqrt{x}}\,dx = \int_1^2 2ue^u\,du$. **[4]**

(b) Hence find the exact value of $\int_1^4 e^{\sqrt{x}}\,dx$. **[4]**

(a) $x = u^2$, so $\frac{dx}{du} = 2u$ and $dx = 2u\,du$. **B1**

Replace $\sqrt{x}$ by u and dx by $2u\,du$: $\int e^{\sqrt{x}}\,dx = \int e^u \cdot 2u\,du$. **M1** for substituting for both $\sqrt{x}$ and dx, **A1** for $2ue^u$.

Limits: $x = 1$ gives $u = 1$, and $x = 4$ gives $u = 2$. So the integral is $\int_1^2 2ue^u\,du$. **A1**, the given result with the limits shown.

(b) Integrate by parts, differentiating the factor $2u$ (to 2) and integrating e^u (to e^u).

$$\int 2ue^u\,du = 2ue^u - \int 2e^u\,du = 2ue^u - 2e^u.$$

M1 for the form $aue^u - b\int e^u\,du$, **A1** for $2ue^u - 2e^u$.

$$\left[2ue^u - 2e^u\right]_1^2 = \left(4e^2 - 2e^2\right) - (2e - 2e) = 2e^2.$$

DM1 for using the u-limits 1 and 2, **A1** for $2e^2$.

Example 3

Let $f(x) = \frac{3x^2 + 4x + 9}{(x+1)\left(x^2+3\right)}$.

(a) Express $f(x)$ in partial fractions. **[5]**

(b) Hence find the exact value of $\int_0^3 f(x)\,dx$, giving your answer in the form $p\ln 2 + q\pi\sqrt{3}$, where p and q are rational. **[5]**

(a) The form is $\frac{A}{x+1} + \frac{Bx + C}{x^2 + 3}$. **B1**

$$3x^2 + 4x + 9 \equiv A\left(x^2+3\right) + (Bx + C)(x+1).$$

M1 for a correct method. $x = -1$: $3 - 4 + 9 = 4A$, so $A = 2$. **A1**

x^2 terms: $3 = A + B$, so $B = 1$. **A1** Constant terms: $9 = 3A + C$, so $C = 3$. **A1**

$$f(x) = \frac{2}{x+1} + \frac{x+3}{x^2+3}.$$

(b) Split the second fraction: $\dfrac{x}{x^2+3} + \dfrac{3}{x^2+3}$.

- $\displaystyle\int \frac{2}{x+1}\,dx = 2\ln(x+1)$. **B1**
- $\displaystyle\int \frac{x}{x^2+3}\,dx = \frac{1}{2}\ln\left(x^2+3\right)$, an $\frac{f'}{f}$ with $k = \frac{1}{2}$. **B1**
- $\displaystyle\int \frac{3}{x^2+3}\,dx = \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} = \sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}$, from the formula with $a = \sqrt{3}$. **B1**

At $x = 3$: $2\ln 4 + \frac{1}{2}\ln 12 + \sqrt{3}\tan^{-1}\sqrt{3} = 2\ln 4 + \frac{1}{2}\ln 12 + \frac{\sqrt{3}}{3}\pi$. At $x = 0$: $0 + \frac{1}{2}\ln 3 + 0$. **M1** for substituting both limits and evaluating the inverse tangent in radians.

$$2\ln 4 + \frac{1}{2}\ln\frac{12}{3} + \frac{\sqrt{3}}{3}\pi = 2\ln 4 + \frac{1}{2}\ln 4 + \frac{1}{3}\pi\sqrt{3} = 5\ln 2 + \frac{1}{3}\pi\sqrt{3}.$$

A1, so $p = 5$ and $q = \frac{1}{3}$.

Example 4

The diagram in "Understand it" shows the curve $y = \sin x\sqrt{\cos x}$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Find the exact x-coordinate of M. **[5]**

(b) Using the substitution $u = \cos x$, find the exact area of the region bounded by the curve and the x-axis. **[5]**

(a) Product rule, with $\dfrac{d}{dx}(\cos x)^{1/2} = -\dfrac{\sin x}{2\sqrt{\cos x}}$:

$$\frac{dy}{dx} = \cos x\sqrt{\cos x} - \frac{\sin^2 x}{2\sqrt{\cos x}}.$$

M1 for the product rule with the chain rule, **A1** for a correct derivative.

Set it to 0 and multiply by $2\sqrt{\cos x}$: $2\cos^2 x - \sin^2 x = 0$. **M1**

Divide by $\cos^2 x$: $\tan^2 x = 2$. **A1**

M is in $0 < x < \frac{1}{2}\pi$, so $\tan x = \sqrt{2}$ and $x = \tan^{-1}\sqrt{2}$. **A1** (about 0.955).

(b) $u = \cos x$ gives $du = -\sin x\,dx$. **B1**

The curve meets the x-axis at $x = 0$ ($\sin x = 0$) and $x = \frac{1}{2}\pi$ ($\cos x = 0$), so the limits become $u = 1$ and $u = 0$.

$$\text{Area} = \int_0^{\pi/2} \sqrt{\cos x}\, \sin x\, dx = \int_1^0 \sqrt{u}\,(-du).$$

M1 for substituting for $\cos x$ and for $\sin x\, dx$.

$$= \int_0^1 u^{1/2}\, du.$$

A1 for the correct integral, with the minus sign used to swap the limits.

$$= \left[\frac{2}{3}u^{3/2}\right]_0^1 = \frac{2}{3}.$$

M1 for integrating to $ku^{3/2}$ and using the u-limits, **A1** for $\frac{2}{3}$.

Common mistakes

- **Stopping after one round of parts.** Examiners report candidates who applied parts once to something like $x^2\cos(\ldots)$ and did not see that the new integral needed parts again. If a power of x is still there, go round again.
- **Missing the hidden 1.** Examiners report candidates who did not see how to integrate a lone inverse tangent by parts, and brought in a factor that wasn't there. Take $u = \tan^{-1}(\ldots)$ or $u = \ln(\ldots)$, with $\frac{dv}{dx} = 1$.
- **Coefficient and sign slips** in $\int \sin \frac{1}{2}x\, dx = -2\cos \frac{1}{2}x$ (not $-\frac{1}{2}$), and in $\int \tan x\, dx$, which some confuse with the derivative of $\tan x$. Examiners also see the sign of $\ln|\cos x|$ lost. Differentiate your answer to check.
- **Wrong partial fraction form.** Reports mention a constant over a quadratic factor instead of $Bx + C$, $Bx + C$ over a repeated linear factor, and missing brackets when multiplying out. The wrong form can't be integrated in part (b).
- **Integrating $\frac{Bx+C}{x^2+k}$ in one go.** Many treated the whole fraction as one inverse tangent or one log, or kept an x in the coefficient. Split it first. Examiners also see the $\sqrt{3}$ put on the wrong side in $\tan^{-1}\left(\sqrt{3}\,x\right)$.
- **Turning a repeated factor into a log.** $\int (x+2)^{-2}\, dx = -(x+2)^{-1}$; a third log term is wrong.
- **Substitution slips**: not replacing dx (or replacing it with the wrong factor, such as $\frac{1}{u}$ instead of $2u$), putting x-limits into a u-integral, or a minus sign that disappears without the limits being swapped.
- **Degrees.** $\tan^{-1} 1 = \frac{1}{4}\pi$, not 45. Calculus with trigonometry is always in radians.
- **A negative area.** A region below the x-axis gives a negative integral; examiners note that only a few candidates realised the area is the size of it.
- **Ignoring "hence".** When part (a) gives an identity or a substituted integral, not using it leads to a much harder integral, and marks are lost.
- **Too little working for a given answer.** A step such as $6\ln 8 = 18\ln 2$, or combining logs, must be shown, not just the final printed line. Reports also warn against working backwards from a printed answer to hide an error.
- **Logs of negative numbers.** With a term such as $\frac{1}{3a-x}$, the integral is $-\ln|3a-x|$; writing $\ln(x-3a)$ without a modulus leads to logs of negative numbers at the limits.

- **Log law errors**, such as $3\ln 3$ written as $\ln 9$.

How to get the marks

- **Method marks are for the form.** Mark schemes typically award **M1** for reaching the right shape, such as "$pxe^{x/2} - q\int e^{x/2}\,dx$" or "$a\ln(x+2a) + b\ln(x+3a)$", even with wrong numbers, then **A1** for the right numbers. Write the integral out before you substitute limits.
- **The limits mark usually depends on the method marks.** It is often a **DM1** "having integrated twice" or "having split correctly", so an earlier method error can cost it too. Show the substitution of **both** limits, including the lower one, even when a term is 0.
- **"Find the exact value"** means keep π, e, $\ln$ and surds. A decimal answer loses the accuracy mark, and some schemes give **no** marks for an answer with no working. Evaluate trig values exactly: $\sin\frac{1}{3}\pi = \frac{\sqrt{3}}{2}$.
- **"Show that"** or a printed answer: every step must be visible. For a substitution, find du, convert each limit ($x = \sqrt{7} \Rightarrow u = 4$, and so on) and swap limits in separate lines.
- **"Using integration by parts" or "using the substitution"**: the stated method must be used; mark schemes can refuse credit for another method.
- **"Hence"**: use the previous part. After a substitution part, the next integral must be done in u.
- **Give the form asked for**: "$a + \ln b$ where a and b are integers", "a single logarithm", "$p\pi + q\sqrt{3}$". An equivalent form may lose the last mark.
- **Accuracy**: exact answers where asked; otherwise non-exact answers to 3 significant figures, and angles in radians. In an iteration, each value to 4 decimal places and the final answer to the stated accuracy.
- **Follow-through**: a wrong partial fraction constant still allows method marks in the integration, so keep going.
- **Indefinite integrals**: include $+c$.

Quick check

1. Find the exact value of $\displaystyle\int_0^3 \frac{2x+6}{x^2+9}\,dx$.
2. Find the exact value of $\displaystyle\int_0^{\pi/6} x\sin 3x\,dx$.
3. By writing $\cos 4x\cos 2x$ as a sum of two cosines, find the exact value of $\displaystyle\int_0^{\pi/6} \cos 4x\cos 2x\,dx$.
4. Use the substitution $u = x+1$ to find the exact value of $\displaystyle\int_0^3 x\sqrt{x+1}\,dx$.
5. Express $\dfrac{x+5}{(x+1)(x+3)^2}$ in partial fractions, and hence find the exact value of $\displaystyle\int_0^1 \frac{x+5}{(x+1)(x+3)^2}\,dx$.

Answers

1. Split: $\dfrac{2x}{x^2+9}$ is an $\frac{f'}{f}$ and $\dfrac{6}{x^2+9}$ an inverse tangent with $a=3$. $\left[\ln\left(x^2+9\right)+2\tan^{-1}\frac{x}{3}\right]_0^3 = \ln 18 - \ln 9 + 2\times\frac{1}{4}\pi = \ln 2 + \frac{1}{2}\pi$.
2. Parts with $u=x$, $v=-\frac{1}{3}\cos 3x$: $\left[-\frac{1}{3}x\cos 3x+\frac{1}{9}\sin 3x\right]_0^{\pi/6} = -\frac{\pi}{18}\cos\frac{\pi}{2}+\frac{1}{9}\sin\frac{\pi}{2}-0=\frac{1}{9}$.
3. $\cos 4x\cos 2x=\frac{1}{2}(\cos 6x+\cos 2x)$, from adding the expansions of $\cos(4x+2x)$ and $\cos(4x-2x)$. So the integral is $\frac{1}{2}\left[\frac{1}{6}\sin 6x+\frac{1}{2}\sin 2x\right]_0^{\pi/6}=\frac{1}{2}\left(0+\frac{1}{2}\times\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{8}$.
4. $x=u-1$, $dx=du$, limits 1 and 4: $\displaystyle\int_1^4 (u-1)u^{1/2}\,du=\left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]_1^4=\left(\frac{64}{5}-\frac{16}{3}\right)-\left(\frac{2}{5}-\frac{2}{3}\right)=\frac{116}{15}$.
5. $\dfrac{A}{x+1}+\dfrac{B}{x+3}+\dfrac{C}{(x+3)^2}$ with $x+5\equiv A(x+3)^2+B(x+1)(x+3)+C(x+1)$. $x=-1$: $A=1$; $x=-3$: $C=-1$; x^2 terms: $B=-1$. So the fractions are $\dfrac{1}{x+1}-\dfrac{1}{x+3}-\dfrac{1}{(x+3)^2}$, and the integral is $\left[\ln(x+1)-\ln(x+3)+\dfrac{1}{x+3}\right]_0^1=\left(\ln 2-\ln 4+\frac{1}{4}\right)-\left(-\ln 3+\frac{1}{3}\right)=\ln\frac{3}{2}-\frac{1}{12}$.

Past questions to try

- **9709/33/M/J/23 Q7**: a substitution with $u=\cos x$ to show, then parts on the new integral.
- **9709/32/M/J/23 Q9**: partial fractions with a repeated factor, then a printed answer with $\ln$.
- **9709/31/O/N/21 Q4**: the substitution $u=\sqrt{x}$ with an infinite limit, giving an inverse tangent.
- **9709/31/M/J/21 Q4**: an identity for $\tan^2\theta$, then a definite integral.