Cambridge AS & A Level · Mathematics (9709) · Notes

Logarithmic and exponential functions

A2

Read online at pastopic.com/cambridge/mathematics-9709/notes/logarithmic-and-exponential-functions-p3

Last checked 28 September 2026

Read first: Logarithmic and exponential functions, Algebra

What you need to know

This topic has the same four learning outcomes as the Paper 2 topic, but Paper 3 questions are harder: several steps, less help, and stricter answer forms. Read the Paper 2 note first. By the end of this topic you should be able to:

1. **Link logarithms and indices, and use the laws of logarithms.** $\log_a b = c$ means $a^c = b$. Use the product, quotient and power laws with $\ln$ and with other bases such as $\log_3$ and $\log_4$, including writing a plain number as a logarithm ($2 = \log_4 16$). Changing the base of a logarithm is **not** required.
2. **Know e^x and $\ln x$.** Know what they are, that they are inverse functions whose graphs are reflections in $y = x$, and the shape of $y = e^{kx}$ when k is positive (growth) and when k is negative (decay).
3. **Solve equations and inequalities with the unknown in a power**, by taking logarithms, including equations with two different bases and equations that turn into a quadratic in e^x or a^x.
4. **Turn a relationship into a straight line.** Take logarithms of a relationship such as $y = Ax^m$ or $y = Ab^x$ (or a less tidy one) to get a straight line, then find the unknown constants from its gradient and intercept.

Understand it

A quick reminder

A logarithm is a power: $\log_a b$ is the power you raise a to in order to get b. So $\log_2 32 = 5$, $\log_a a = 1$ and $\log_a 1 = 0$, and you can only take the log of a **positive** number. The three laws, for any base:

$$\log_a(pq) = \log_a p + \log_a q, \qquad \log_a\left(\frac{p}{q}\right) = \log_a p - \log_a q, \qquad \log_a(p^n) = n\log_a p.$$

$\ln x$ is $\log_e x$, where $e = 2.718\ldots$. The functions e^x and $\ln x$ undo each other: $\ln(e^x) = x$ and $e^{\ln x} = x$ for $x > 0$. So $y = e^x$ (always positive, through $(0, 1)$, the x-axis an asymptote) and $y = \ln x$ (only for $x > 0$, through $(1, 0)$, the y-axis an asymptote) are reflections of each other in $y = x$. The Paper 2 note draws these graphs and explains each fact.

What the laws can and cannot do

The laws only split **products, quotients and powers**. Nothing splits a sum or a difference:

- $\ln(x+4)$ is **not** $\ln x + \ln 4$, and $\ln(1-e^{-x})$ is **not** $\ln 1 - \ln e^{-x}$;
- $\ln(e^{2x}+5)$ is **not** $2x + \ln 5$.

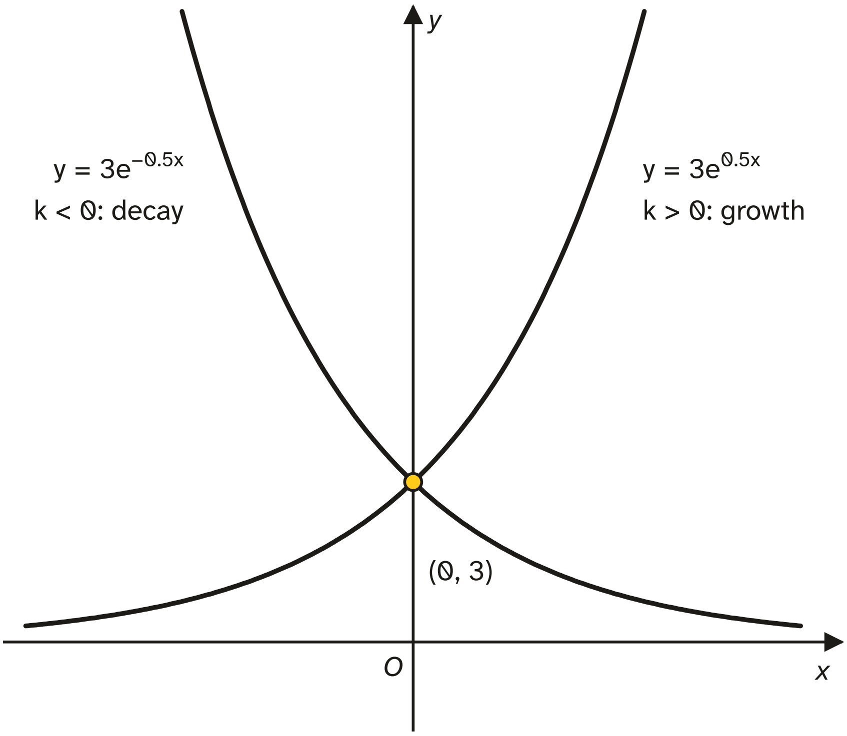
Numbers in front also need care. 7×2^{-x} is **not** 14^{-x}: only the 2 is raised to the power. Two factors can only be combined when they have the **same** power, as in $2^x \times 5^x = 10^x$. So $\ln(7 \times 2^{-x}) = \ln 7 - x \ln 2$.

A plain number can be written as a log in any base, which is how you get every term into one log: $1 = \ln e$, $3 = \ln e^3$, $1 = \log_3 3$, $2 = \log_3 9 = \log_4 16$.

Growth and decay

$y = Ae^{kx}$ (with $A > 0$) passes through $(0, A)$ and stays above the x-axis, which is an asymptote on one side.

- $k > 0$: the graph rises, faster and faster. This is **exponential growth**.
- $k < 0$: the graph falls and levels off towards the x-axis. This is **exponential decay**.



Models use this with time t. If a population is $P = 50e^{0.04t}$ after t hours, it starts at 50. It **doubles** when $e^{0.04t} = 2$, so $0.04t = \ln 2$ and $t = \frac{\ln 2}{0.04} = 17.3$ hours. It reaches 400 when $e^{0.04t} = 8$, so $t = \frac{\ln 8}{0.04} = 52.0$ hours. The doubling time does not depend on the starting value.

Getting the unknown out of a power

Take logs of both sides, then use the power law to bring each power down as a **bracket**. With two different bases:

$$7^{x-1} = 4(3^x) \;\Rightarrow\; (x-1)\ln 7 = \ln 4 + x\ln 3.$$

That is a linear equation. Collect the x terms: $x(\ln 7 - \ln 3) = \ln 4 + \ln 7$, so

$$x = \frac{\ln 28}{\ln \frac{7}{3}} = 3.93\ (3\text{ s.f.}).$$

The same answer comes from the laws of **indices** first: $\frac{7^x}{7} = 4 \times 3^x$, so $\left(\frac{7}{3}\right)^x = 28$, and one log finishes it. Either route is fine, and logs to any base give the same answer.

Inequalities work the same way, but remember that $\ln$ of a number between 0 and 1 is negative. For $0.3^x > 0.05$: $x\ln 0.3 > \ln 0.05$, and dividing by $\ln 0.3 < 0$ turns the sign round, so $x < \frac{\ln 0.05}{\ln 0.3} = 2.49$ (3 s.f.).

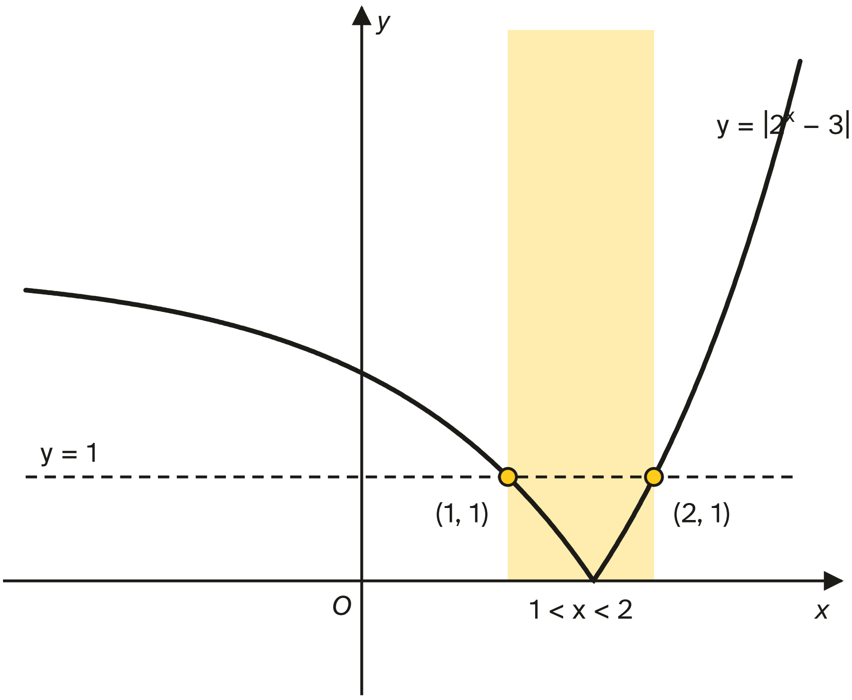
Hidden quadratics

$e^{2x} = (e^x)^2$ and $e^{-x} = \frac{1}{e^x}$, and the same holds for any base: $4^{-x} = \frac{1}{4^x}$. So an equation with e^x, e^{2x} and e^{-x} is really an equation in one letter $u = e^x$. Multiplying through by u clears e^{-x} and usually leaves a quadratic. Because e^x and a^x are **always positive**, a negative root for u gives no value of x, and you must say so.

Some equations only need the laws of indices: $3^{x+1} = 3^x + 12$ means $3 \times 3^x - 3^x = 12$, so $2 \times 3^x = 12$, $3^x = 6$ and $x = \frac{\ln 6}{\ln 3} = 1.631$ (3 d.p.).

A modulus around a power

$|2^x - 3| < 1$ means $2^x - 3$ lies between -1 and 1, so $2 < 2^x < 4$. Powers of 2 increase, so $1 < x < 2$.

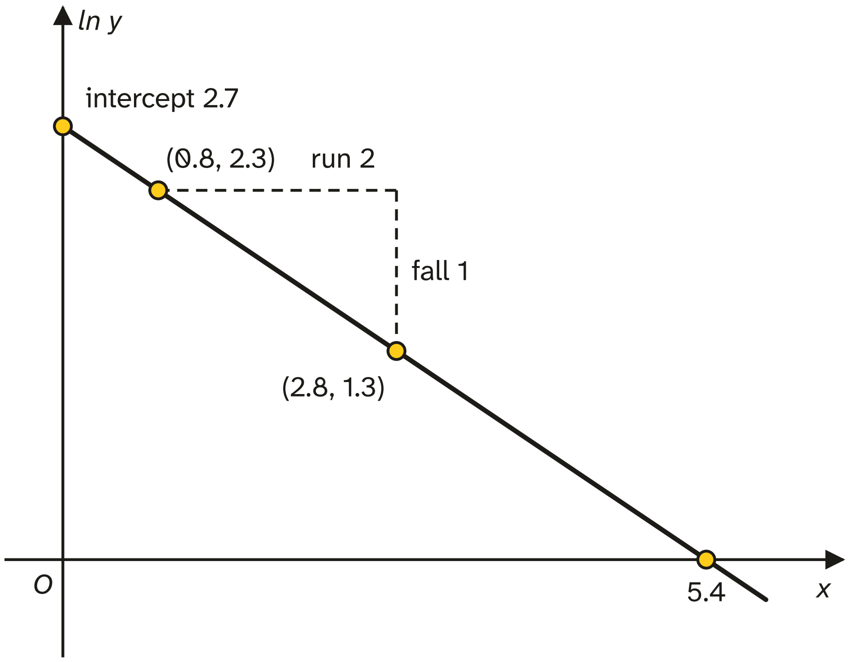


Turning a relationship into a straight line

Take logs of the relationship, then rearrange it to $Y = mX + c$, where Y and X are what the graph's axes show. For example:

Relationship	Log form	Plot	Gradient	Intercept
$y = Ax^m$	$\ln y = \ln A + m \ln x$	$\ln y$ against $\ln x$	m	$\ln A$
$y = Ab^x$	$\ln y = \ln A + x \ln b$	$\ln y$ against x	$\ln b$	$\ln A$
$y^2 = Ae^{bx}$	$\ln y = \frac{b}{2}x + \frac{1}{2}\ln A$	$\ln y$ against x	$\frac{b}{2}$	$\frac{1}{2}\ln A$
$a^y = bx^2$	$y = \frac{2}{\ln a}\ln x + \frac{\ln b}{\ln a}$	y against $\ln x$	$\frac{2}{\ln a}$	$\frac{\ln b}{\ln a}$

The vertical-axis variable must be **on its own** before you read off the gradient and intercept. The intercept is where the horizontal variable is 0: for a graph against $\ln x$ that is $\ln x = 0$, not $x = 0$.



Coordinates read from such a graph are already logarithms: the point $(0.8, 2.3)$ means $x = 0.8$ and $\ln y = 2.3$. Use them as they are.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. Nothing in this topic is in MF19, so you must learn all of it.

Formula	Status
$\log_a b = c \iff a^c = b$	Learn it
$\log_a(pq) = \log_a p + \log_a q$	Learn it
$\log_a\left(\dfrac{p}{q}\right) = \log_a p - \log_a q$	Learn it
$\log_a(p^n) = n\log_a p$, so $\frac{1}{2}\ln p = \ln\sqrt{p}$ and $-\ln p = \ln\frac{1}{p}$	Learn it
$\log_a a = 1$, $\log_a 1 = 0$; $\ln e = 1$, $\ln 1 = 0$; $n = \log_a a^n$	Learn it

Formula	Status
$\ln(e^x) = x$ and $e^{\ln x} = x$ (for $x > 0$)	Learn it
$e^x > 0$ and $a^x > 0$ for every x; $\ln x$ exists only for $x > 0$	Learn it
$e^{2x} = (e^x)^2$, $e^{-x} = \frac{1}{e^x}$, $a^{x+n} = a^n \times a^x$	Learn it
$y = Ae^{kx}$: growth if $k > 0$, decay if $k < 0$; doubling (or halving) time $\frac{\ln 2}{\lvert k\rvert}$	Learn it
$y = Ax^m \Rightarrow \ln y = \ln A + m \ln x$	Learn it
$y = Ab^x \Rightarrow \ln y = \ln A + x \ln b$	Learn it
Straight line: gradient $= \frac{Y_2 - Y_1}{X_2 - X_1}$, and $Y = mX + c$	Learn it

How to do it

In the Paper 3 questions we have tagged for this topic (38 questions, 2021–2025, all different), the types came up this often. Each question has one main type here, so the counts add up to 38. They are short questions, usually among the first three on the paper.

Question type	Questions
Logarithmic equations	10
Straight-line graphs from logarithms	9
Exponential equations with different bases	8
Equations in a^x or e^x solved with a substitution	6
Rewriting a log relationship	3
A modulus with a power inside	2

1. Logarithmic equations (10 questions)

1. **Tidy each log term.** Move any number in front inside as a power: $2\ln(x+1) = \ln(x+1)^2$.
2. **Write any plain number as a log** in the same base: $5 = \ln e^5$, $1 = \log_3 3$, $2 = \log_4 16$.
3. **Combine into one log on each side** with the product and quotient laws.
4. **Remove the logs**: $\log_a P = \log_a Q$ gives $P = Q$.
5. **Solve** the equation that is left (often a quadratic), then **check** every answer in the original: each bracket inside a log must be positive. Reject the others and say why.

For example, $\log_3(x+6) = 2 - \log_3 x$ becomes $\log_3\big(x(x+6)\big) = \log_3 9$, so $x^2 + 6x - 9 = 0$ and $x = -3 \pm 3\sqrt{2}$. Only $x = -3 + 3\sqrt{2} = 1.24$ (3 s.f.) keeps $x > 0$.

Variations you will meet:

- **A number on its own**, such as $\ln(x+a) = 5 + \ln x$ or $\ln(x-a) = 7 - \ln x$: gather the logs, then $\ln(\ldots) = 5$ gives $(\ldots) = e^5$. Give the answer to the accuracy asked.
- **Cubes that cancel**: $\ln(x^3 - c) = 3\ln x - \ln 3$ gives $x^3 - c = \frac{x^3}{3}$, which is linear in x^3. Then take the cube root.
- **Other bases**, $\log_3$ or $\log_4$. Work in that base throughout; don't switch to $\ln$ (changing base is not in the syllabus, and you don't need it).
- **"Show that the equation can be written as a quadratic"** then **"hence solve"** the same equation, or one with x replaced by something like $2y$. Solve the quadratic, reject any root that makes a log bracket negative, then find y from $x = 2y$ (so y is half of x).
- **A log of an exponential expression**, such as $\ln(e^{2x} + 5) = 2x + \ln 2$: write the right side as one log, $\ln(2e^{2x})$, so $e^{2x} + 5 = 2e^{2x}$ and $e^{2x} = 5$. Or $\ln(2 + e^{-x}) = 1$: remove the log first, $2 + e^{-x} = e$, so $e^{-x} = e - 2$ and $x = -\ln(e-2) = 0.3309$ (4 d.p.).
- **Answer forms**: "exact" (a surd or an expression in e), or a stated number of decimal places or significant figures.

2. Straight-line graphs from logarithms (9 questions)

1. **Take logs of the relationship** and use the laws, keeping brackets: a^{2y-1} gives $(2y-1)\ln a$.
2. **Rearrange** so the vertical-axis variable is on its own, in the form $Y = mX + c$.
3. **Match**: the gradient m and the intercept c each equal an expression in the constants.
4. **Use the numbers**: gradient $= \frac{Y_2 - Y_1}{X_2 - X_1}$ from two points, then put one point into $Y = mX + c$ for c. Or put both points into the log equation and solve the two equations together.
5. **Undo the logs** to find the constants ($\ln A = 1.65 \Rightarrow A = e^{1.65}$), to the accuracy asked.

Variations you will meet:

- $\ln y$ **against** x, from $y = ab^x$, $ay = b^x$, $ky = e^{cx}$ or $Ay = b^x$. For $ay = b^x$: $\ln a + \ln y = x\ln b$, so $\ln y = x\ln b - \ln a$: gradient $\ln b$, intercept $-\ln a$.
- $\ln y$ **against** $\ln x$, for example from $x^n y^3 = C$: $n\ln x + 3\ln y = \ln C$, so $\ln y = -\frac{n}{3}\ln x + \frac{1}{3}\ln C$. The gradient is $-\frac{n}{3}$, not $-n$.
- y **against** $\ln x$, from $a^y = bx^2$ or $x = A(2^{-y})$. For the second, $\ln x = \ln A - y\ln 2$, so $y = -\frac{1}{\ln 2}\ln x + \frac{\ln A}{\ln 2}$. The **exact** gradient is $-\frac{1}{\ln 2}$. The line meets the y-axis where $\ln x = 0$.
- y **against** x, from something like $a^{y+1} = b^{2x-y}$: $(y+1)\ln a = (2x - y)\ln b$. Collect the y terms: $y(\ln a + \ln b) = 2x\ln b - \ln a$, which is a straight line. If you are then told $a = b^2$, write $\ln a = 2\ln b$ and the $\ln b$ cancels: $3y = 2x - 2$, so $y = \frac{2}{3}x - \frac{2}{3}$.
- **"Show that the graph is a straight line" or "explain why"**: write the log equation with the vertical variable on its own, say it has the form $Y = mX + c$ and name X and Y.
- **Two pairs of values instead of a graph**, such as "when $x = 1.5$, $\ln y = 0.9$": they are two points on the line; use them the same way.
- **A model with time**, such as $P = ae^{kt}$ with the gradient and intercept of $\ln P$ against t given: $\ln P = \ln a + kt$, so k is the gradient and $a = e^{\text{intercept}}$. Then "the time to double" is $\frac{\ln 2}{k}$, whatever a is.

- **Accuracy**: questions ask for 1 or 2 significant figures, 2 decimal places, or the nearest integer. Keep more figures until the end.

3. Exponential equations with different bases (8 questions)

1. **Take $\ln$ of both sides.** Split products and quotients first: $\ln\left(6 \times 5^{-3x}\right) = \ln 6 - 3x \ln 5$ and $\ln \frac{2}{7^x} = \ln 2 - x \ln 7$.
2. **Bring each power down in a bracket**: $\ln\left(9^{2-5x}\right) = (2-5x)\ln 9$.
3. **Collect the x terms** on one side, factorise out x, and divide.
4. **Give the form asked for.**

For example, $2^{x-3} = 3^{1-2x}$ gives $(x-3)\ln 2 = (1-2x)\ln 3$, so $x(\ln 2 + 2\ln 3) = \ln 3 + 3\ln 2$ and $x = \frac{\ln 24}{\ln 18}$.

Variations you will meet:

- **"Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers"**: combine the logs on the top and on the bottom, $\ln 3 + 3\ln 2 = \ln 24$. Don't cancel a $\ln$ from top and bottom: $\frac{\ln 24}{\ln 18}$ is **not** $\frac{24}{18}$, and it is not $\ln \frac{24}{18}$. Stop at the required form.
- **A decimal answer** (3 significant figures, or 2 or 3 decimal places): evaluate the same expression at the end.
- **Coefficients**, such as $5 \times 2^{x-1} = 3 \times 7^x$ or $3(2^{3x+1}) = 5^x$: each coefficient becomes a separate $\ln$ term. Never merge it with the base.
- **Fractions and negative powers**, such as $\frac{4}{7^x}$ or 5^{-3x}: it can help to multiply up first, so $3^{2x+1} = \frac{4}{7^x}$ becomes $3^{2x+1} \times 7^x = 4$.
- **The indices route**: get every x into powers with the same index, such as $\left(\frac{5}{2}\right)^x = 12$, then take logs once.

4. Equations in a^x or e^x solved with a substitution (6 questions)

1. **Write every term with u**, where $u = e^x$ (or 4^x, or e^{2x}): $e^{-x} = \frac{1}{u}$, $e^{2x} = u^2$, $2^{x-3} = \frac{u}{8}$, $3^{x+1} = 3u$.
2. **Clear fractions** (multiply by u, or by a denominator such as $e^x - 3$) and get everything on one side.
3. **Solve for u**: usually a three-term quadratic, sometimes linear.
4. **Reject any $u \leq 0$**, saying why ($e^x > 0$ for every x).
5. **Take logs**: $e^x = u$ gives $x = \ln u$; $4^x = u$ gives $x = \frac{\ln u}{\ln 4}$.

Variations you will meet:

- **A fraction**, such as $\frac{e^x + ke^{-x}}{e^x - c} = d$: multiply across, then multiply by e^x. See Example 3.
- **"Show that the equation has only one real root"**: you must show the other root of the quadratic is negative and say that e^x cannot be negative. "Math error" from the calculator is not a reason.
- **e^{2x} with e^{-2x}**: let $u = e^{2x}$. Then $x = \frac{1}{2}\ln u$.
- **Another base with a negative power**, such as $4^x = c + 4^{-x}$: multiply by 4^x to get $u^2 - cu - 1 = 0$.

- **Only indices needed**, such as $3^{x+1} = 3^x + 12$ or $2^{x-3} = 2^x - 14$: this is linear in u. Don't take logs term by term; collect the u terms first.

5. Rewriting a log relationship (3 questions)

1. **Combine the logs into one** with the laws (half a log becomes a square root).
2. **Remove the log**: $\ln P = x$ gives $P = e^x$; $\ln P = 3$ gives $P = e^3$.
3. **Rearrange** for the letter asked for.

For example, $x = \ln(3y+1) - \ln(y-2)$ gives $e^x = \frac{3y+1}{y-2}$, so $ye^x - 2e^x = 3y + 1$, $y(e^x - 3) = 1 + 2e^x$ and $y = \frac{1+2e^x}{e^x-3}$.

Variations you will meet:

- **Two letters with a constant**, such as $\ln(y+2) - 3\ln x = 2$: $\ln \frac{y+2}{x^3} = 2$, so $y = e^2x^3 - 2$. With $-\frac{1}{2}\ln(q+1)$, the term becomes $\sqrt{q+1}$ in the denominator, and you square at the end.
- **"Express $\ln(\dots)$ in terms of a and b"**: write each given fact as a linear equation in $\ln p$ and $\ln q$, solve them together, then substitute. For $\ln(p^2q) = a$ and $\ln \frac{p}{q} = b$: $2\ln p + \ln q = a$ and $\ln p - \ln q = b$, so $\ln p = \frac{a+b}{3}$ and $\ln q = \frac{a-2b}{3}$. Then $\ln(pq^3) = \ln p + 3\ln q = \frac{4a-5b}{3}$.

6. A modulus with a power inside (2 questions)

1. **Remove the modulus.** For an equation $|f| = g$ you can square both sides or split into $f = g$ and $f = -g$. For $|f| < c$ write $-c < f < c$.
2. **Isolate the power** in each part, such as 2^x or 5^x.
3. **Take logs** of each positive value (a power can't be negative or zero).
4. **Combine** the answers for an inequality into one range.

For example, $|2^x - 3| = 1$ gives $2^x = 4$ or $2^x = 2$, so $x = 2$ or $x = 1$; and $|2^x - 3| < 1$ gives $1 < x < 2$, as the second diagram shows.

Variations you will meet:

- **A multiple of the power on the other side**, such as $3|4^x - 2| = 4^x$: splitting gives $3(4^x - 2) = 4^x$ and $3(4^x - 2) = -4^x$, so $4^x = 3$ or $4^x = 1.5$, and two answers: $x = 0.792$ and $x = 0.292$ (3 s.f.).
- **A power like 3^{x+1} inside**: $|3^{x+1} - 4| < 2$ gives $2 < 3^{x+1} < 6$, so $\frac{\ln 2}{\ln 3} < x + 1 < \frac{\ln 6}{\ln 3}$. Subtract 1 and give the range as one statement with strict signs: $-0.369 < x < 0.631$ (3 s.f.).

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

Solve the equation $2^{2x-1} = 7(3^{-x})$. Give your answer in the form $\dfrac{\ln a}{\ln b}$, where a and b are integers. **[4]**

Take $\ln$ of both sides, splitting the product on the right:

$$(2x-1)\ln 2 = \ln 7 - x\ln 3.$$

M1 for using the power law (and the product law), **A1** for this correct linear equation.

Collect the x terms: $2x\ln 2 + x\ln 3 = \ln 7 + \ln 2$, so $x(\ln 4 + \ln 3) = \ln 14$. **M1** for solving for x.

$$x = \frac{\ln 14}{\ln 12}.$$

A1, only in this form ($x \approx 1.06$ alone would lose it).

Example 2

(a) Show that the equation $\log_2(3x+10) = 2 + 2\log_2(x-1)$ can be written as a quadratic equation in x. **[3]**

(b) Hence solve the equation $\log_2(3e^y+10) = 2 + 2\log_2(e^y-1)$, giving your answer correct to 3 decimal places. **[2]**

(a) $2 = \log_2 4$. **B1**

$2\log_2(x-1) = \log_2(x-1)^2$, so the right side is $\log_2 4 + \log_2(x-1)^2 = \log_2\left(4(x-1)^2\right)$. **M1** for the power and product laws.

$$3x + 10 = 4(x-1)^2 = 4x^2 - 8x + 4 \;\Rightarrow\; 4x^2 - 11x - 6 = 0.$$

A1

(b) The equation is part (a) with $x = e^y$. Solve the quadratic:

$$x = \frac{11 \pm \sqrt{121+96}}{8} = \frac{11 \pm \sqrt{217}}{8}, \quad \text{so } x = 3.216\ldots \text{ or } x = -0.466\ldots$$

$x = -0.466$ would make $x - 1$ negative (and e^y is never negative), so reject it. **M1** for solving the quadratic and using $e^y = 3.216\ldots$.

$y = \ln 3.2163\ldots = 1.168$ (3 d.p.). **A1**, with the other root rejected.

Example 3

Find the real root of the equation $\dfrac{2e^x + 3e^{-x}}{e^x - 1} = 5$, giving your answer correct to 3 decimal places. Your working should show clearly that the equation has only one real root. **[5]**

Multiply across: $2e^x + 3e^{-x} = 5e^x - 5$. Multiply every term by e^x and collect on one side:

$$3e^{2x} - 5e^x - 3 = 0.$$

B1 for this quadratic in e^x.

$$e^x = \frac{5 \pm \sqrt{25 + 36}}{6} = \frac{5 \pm \sqrt{61}}{6}.$$

M1 for solving the quadratic, **A1** for $e^x = \frac{5+\sqrt{61}}{6} = 2.1350\ldots$

$x = \ln 2.1350\ldots = 0.758$ (3 d.p.). **A1**

The other root, $\frac{5-\sqrt{61}}{6} = -0.468\ldots$, is negative, and $e^x > 0$ for every x, so it gives no value of x. The equation has only one real root. **B1**

Example 4

The variables x and y satisfy the equation $y^2 = Ae^{bx}$, where A and b are constants. The graph of $\ln y$ against x is a straight line passing through the points $(0.8, 2.3)$ and $(2.8, 1.3)$.

(a) Find the value of b, and the value of A correct to 3 significant figures. **[5]**

(b) Find the value of x for which $y = 1$. **[2]**

(a) Take ln: $2\ln y = \ln A + bx$, so

$$\ln y = \frac{b}{2}x + \frac{1}{2}\ln A.$$

B1 for a correct log equation.

Gradient $= \dfrac{1.3 - 2.3}{2.8 - 0.8} = -0.5$, so $\dfrac{b}{2} = -0.5$ and $b = -1$. **M1** for equating the gradient to $\frac{b}{2}$, **A1**

Put $(0.8, 2.3)$ into the line: $2.3 = -0.5 \times 0.8 + c$, so the intercept $c = 2.7$. Then $\frac{1}{2}\ln A = 2.7$, so $\ln A = 5.4$. **M1** for finding the intercept and linking it to $\ln A$.

$A = e^{5.4} = 221$ (3 s.f.). **A1**

(b) $y = 1$ means $\ln y = 0$: $0 = -0.5x + 2.7$, so $x = 5.4$. **M1 A1**

The third diagram in "Understand it" shows this line.

Common mistakes

- **Splitting the log of a sum.** $\ln(1 - e^{-x})$ is not $\ln 1 - \ln e^{-x}$, and $\ln(e^{3x} + 4)$ is not $3x + \ln 4$. Examiner reports point this out again and again, and it earns nothing. Remove the log instead (if $\ln(1 - e^{-x}) = -2$, then $1 - e^{-x} = e^{-2}$), or substitute $u = e^x$ first.
- **Merging a coefficient with a base.** 4×3^{-x} is not 12^{-x}, and $2(5^{x+1})$ is not 10^{x+1}. The coefficient is a separate $\ln$ term.
- **Dropping brackets** when bringing a power down: $\ln 2^{3x-1} = (3x - 1)\ln 2$, not $3x - 1 \ln 2$.
- **Keeping impossible roots.** A value that makes a log bracket negative, or e^x negative, must be rejected, with a reason. Many candidates lose the final mark this way. "Math error" is not a reason.
- **Solving for the wrong letter after "hence".** If the new equation has $2y$ where the old one had x, then y is **half** of the root, not the root itself.
- **Cancelling logs.** $\frac{\ln 6}{\ln 14}$ can't be simplified by "cancelling $\ln 2$". Combine terms into single logs instead.
- **Taking logs term by term** in an equation like $5^{x-1} = 5^x - 20$. Use the laws of indices ($5^{x-1} = \frac{5^x}{5}$) and collect the 5^x terms.
- **Straight-line mix-ups.** Don't put the graph's coordinates into the original equation as if they were x and y, and don't take logs of them again: $(0.42, 1.57)$ on a $\ln y$ against $\ln x$ graph already means $\ln x = 0.42$, $\ln y = 1.57$. For a graph against $\ln x$, the intercept is at $\ln x = 0$; "$\ln 0$" never appears in a correct method.
- **Index slips after squaring a modulus.** Squaring $|4^x - 2|$ gives $4^{2x} - 4(4^x) + 4$, and slips such as writing 3×4^x as 12^x are common. Splitting into two linear equations in 4^x avoids the squaring altogether.
- **Wrong accuracy.** 3 decimal places and 3 significant figures are different, and so are 3 and 4 decimal places. An exact answer when a decimal was asked for, or the other way round, also loses the last mark.

How to get the marks

- **Show the log step.** The first method mark is for a correct law used on the actual terms, such as $(2x - 1)\ln 2$ and $\ln 7 - x\ln 3$ in Example 1. Answers with no working score nothing on most of these questions.
- **Then show the equation being solved** (a linear equation in x, or a quadratic in u). A dependent method mark needs it.
- **Any base is fine** for equations in powers: $\log_{10}$ gives the same answer as $\ln$, as long as you use one base throughout. But when the question is about $\ln x$ or $\ln y$ (straight-line graphs), work in $\ln$.
- **"In the form $\frac{\ln a}{\ln b}$" means exactly that**, with integers. An equivalent such as $\log_{40} 48$ or a decimal loses the final mark.
- **Accuracy is strict.** Mark schemes say "3 d.p. required" or "must be 3 s.f.": give exactly what is asked, and round only at the end.
- **"Exact"** means a surd, a fraction or an expression in e or $\ln$, such as $x = \sqrt{2}$ or $x = -3 + 3\sqrt{2}$.

- **Reject clearly.** Say which root you reject and why: "$e^x > 0$", or "$x - 1 < 0$, so $\ln(x-1)$ is undefined". Leaving a wrong extra answer loses the final mark.
- **"Show that the graph is a straight line"**: write the equation with the vertical variable on its own and **compare** it with $Y = mX + c$, naming what X and Y are. "It is linear" on its own does not score.
- **"Show that ... can be written as a quadratic"**: every law must be visible, and finish with the quadratic $=$ 0 with three terms.
- **"State the exact value of the gradient"**: an expression like $-\frac{1}{\ln 3}$, not a decimal.

Quick check

1. The positive numbers p and q satisfy $\ln(pq^2) = a$ and $\ln\left(\frac{p^3}{q}\right) = b$. Express $\ln(p^2 q)$ in terms of a and b.
2. Find the set of values of x for which $2^{x+3} < 7^x$, giving the boundary correct to 3 significant figures.
3. Solve the equation $2^x = 5 + 6(2^{-x})$, giving your answer correct to 3 decimal places.
4. Solve the equation $\ln x + \ln(x+2) = 1 + \ln 3$, giving your answer correct to 3 decimal places.
5. Find the set of values of x for which $|3^x - 5| < 2$, giving the non-integer boundary correct to 3 significant figures.

Answers

1. $\ln p + 2\ln q = a$ and $3\ln p - \ln q = b$. Solving: $\ln p = \dfrac{a+2b}{7}$ and $\ln q = \dfrac{3a-b}{7}$. So $\ln(p^2 q) = 2\ln p + \ln q = \dfrac{5a+3b}{7}$.
2. $(x+3)\ln 2 < x\ln 7$, so $3\ln 2 < x(\ln 7 - \ln 2)$. $\ln 7 - \ln 2 = \ln 3.5 > 0$, so the sign stays: $x > \dfrac{\ln 8}{\ln 3.5} = 1.66$ (3 s.f.).
3. Let $u = 2^x$: $u = 5 + \dfrac{6}{u}$, so $u^2 - 5u - 6 = 0$ and $(u-6)(u+1) = 0$. $2^x = -1$ is impossible, so $2^x = 6$ and $x = \dfrac{\ln 6}{\ln 2} = 2.585$ (3 d.p.).
4. $\ln\big(x(x+2)\big) = \ln 3e$, so $x^2 + 2x - 3e = 0$ and $x = -1 \pm \sqrt{1+3e}$. The negative root makes $\ln x$ undefined, so $x = -1 + \sqrt{1+3e} = 2.026$ (3 d.p.).
5. $-2 < 3^x - 5 < 2$, so $3 < 3^x < 7$. Then $1 < x < \dfrac{\ln 7}{\ln 3}$, that is $1 < x < 1.77$ (3 s.f.).

Past questions to try

- **9709/33/M/J/22 Q3**: a $\log_3$ equation that becomes a quadratic, then "hence" with $2y$ in place of x.
- **9709/31/M/J/21 Q2**: a hidden quadratic in e^x, showing there is only one real root.
- **9709/32/M/J/21 Q3**: a straight line of y against $\ln x$, with an exact gradient and the intercept.
- **9709/33/O/N/23 Q1**: a modulus inequality with a power inside.