Cambridge AS & A Level · Mathematics (9709) · Notes

Numerical solution of equations AS

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Read first: Logarithmic and exponential functions, Algebra

What you need to know

By the end of this topic you should be able to:

1. **Find roughly where a root is.** Sketch two graphs to see how many times they cross (each crossing is a root) and roughly where, and show that a root lies between two given values by checking for a **change of sign**.
2. **Use the notation for a sequence of approximations**, $x_1, x_2, x_3, \ldots$, and understand what it means for the sequence to **converge** to a root.
3. **Use an iterative formula** $x_{n+1} = F(x_n)$. Understand how it comes from the equation being solved (the root satisfies $x = F(x)$), rearrange an equation into a given form $x = F(x)$, and iterate to find a root to a stated accuracy. Know that an iteration **may fail to converge**.

You do **not** need to know the condition that decides whether an iteration converges. Newton–Raphson and other methods are not in the syllabus.

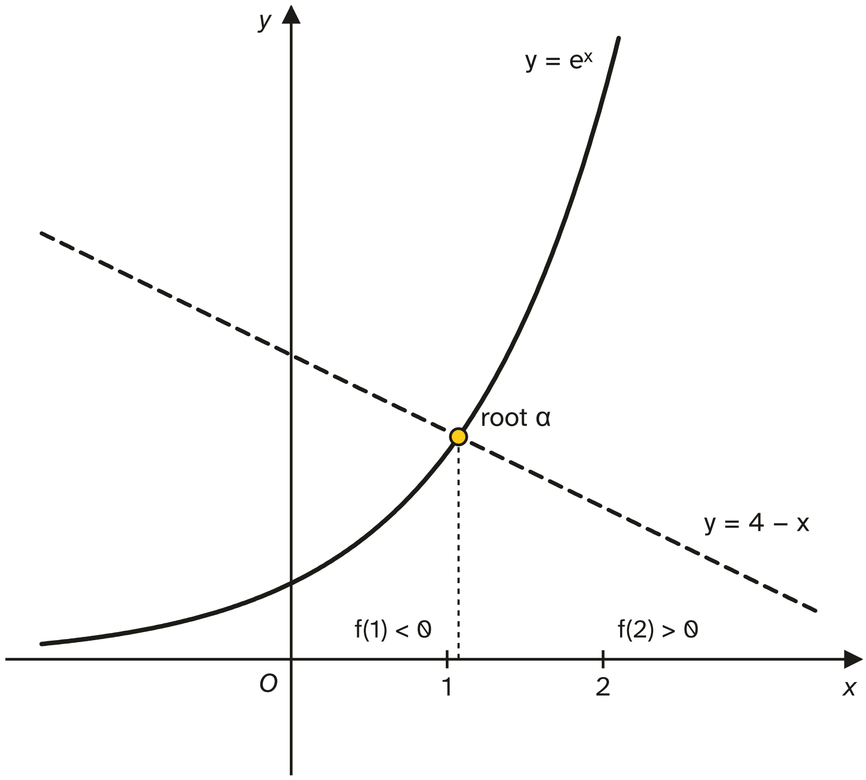
Understand it

Why we need numerical methods

Equations such as $e^x = 4 - x$ or $x = \ln(10x)$ mix different kinds of function, so no algebra will give an exact answer. Instead you find the root **to a given accuracy**, such as 3 significant figures, by a method that gets closer and closer to it.

Locating a root with graphs

Write the equation as "one function $=$ another" and sketch both graphs on the same axes. Every point where the graphs cross gives a root, so the number of crossings is the number of roots, and the crossing's position shows roughly where the root is.



For $e^x = 4 - x$, the curve $y = e^x$ rises and the line $y = 4 - x$ falls, so they cross exactly once: the equation has exactly **one** root, α, somewhere between 1 and 2.

Locating a root with a change of sign

Move everything to one side: $f(x) = e^x + x - 4 = 0$. Then

$$f(1) = e - 3 = -0.282, \qquad f(2) = e^2 - 2 = 5.39.$$

$f(x)$ is negative at $x = 1$ and positive at $x = 2$. A continuous graph cannot get from below the x-axis to above it without crossing, so $f(x) = 0$ somewhere in between: **there is a root between 1 and 2.** This only works for a function whose graph has no breaks between the two values; the functions in these questions have none in the intervals you are given.

Iteration: the idea

Rearrange the equation so that x is on its own on one side: $e^x = 4 - x$ becomes $x = \ln(4 - x)$. Now turn it into a rule that makes a **sequence**:

$$x_{n+1} = \ln(4 - x_n).$$

Start with a first guess x_1, put it into the right-hand side to get x_2, put x_2 in to get x_3, and so on. With $x_1 = 1$ (to 5 decimal places):

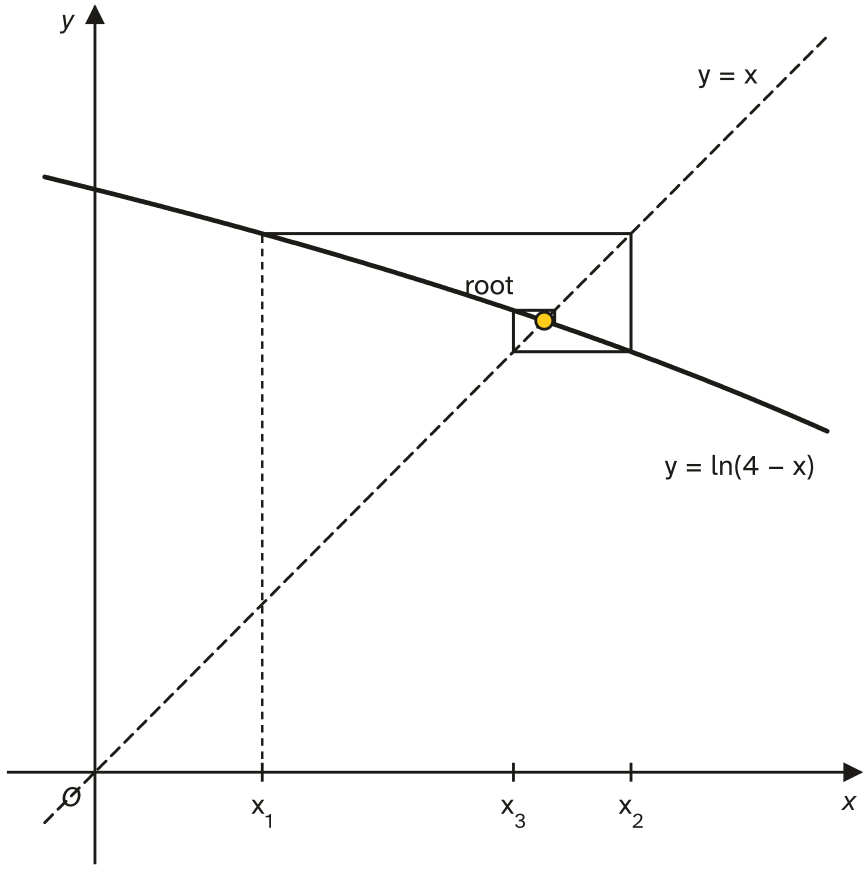
n	1	2	3	4	5	6	7
x_n	1	1.09861	1.06519	1.07664	1.07273	1.07407	1.07361

The values settle down: the sequence **converges**. It converges to the root because of how the formula was made. If the values settle on a number α, then for large n both x_n and x_{n+1} are α, so $\alpha = \ln(4-\alpha)$, which rearranges back to $e^{\alpha} = 4 - \alpha$. **The number an iteration converges to always satisfies the equation $x = F(x)$ it came from.**

x_6 and x_7 both round to 1.074, so $\alpha = 1.074$ to 3 decimal places. To be sure, check the sign of f either side: $f(1.0735) = -0.0009$ and $f(1.0745) = 0.0030$, so $1.0735 < \alpha < 1.0745$, which rounds to 1.074.

Seeing it on a graph

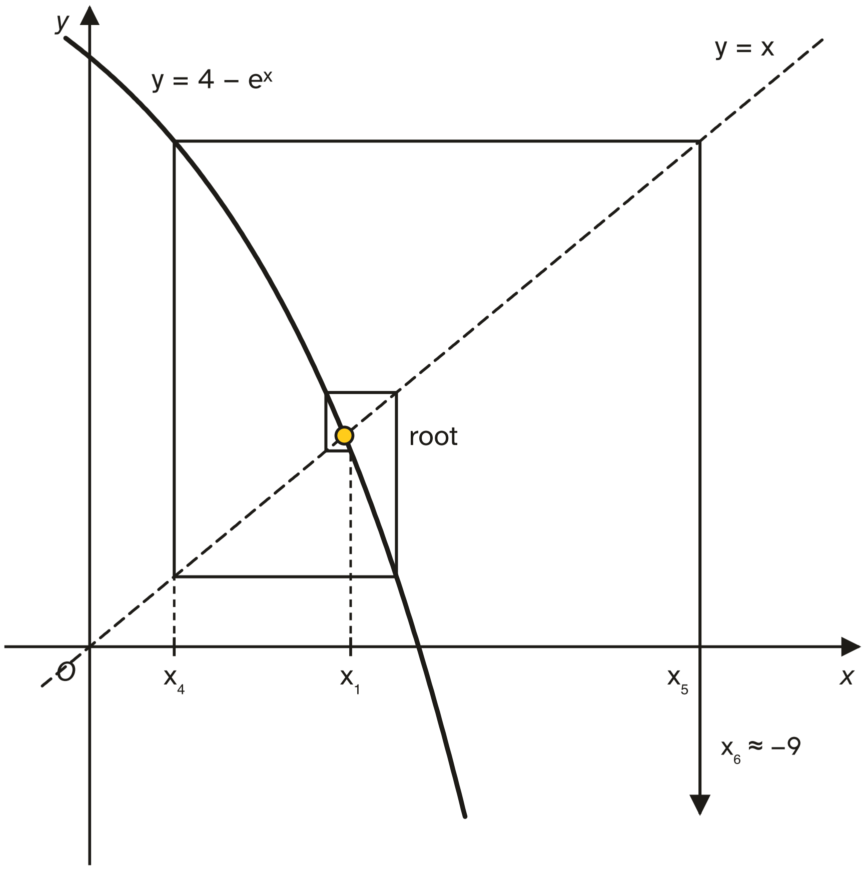
A root of $x = F(x)$ is where the line $y = x$ meets the curve $y = F(x)$. Each step of the iteration is "go up to the curve, then across to the line", which moves x_n to x_{n+1}. You will not be asked to draw this, but it shows why the values behave as they do.



Here the values land on **alternate sides** of the root, so the root is always between two consecutive values. With other formulas the values creep up (or down) towards the root from one side, like a staircase; Example 2 below does this.

When an iteration fails

The same equation can be rearranged another way: $e^x = 4 - x$ gives $x = 4 - e^x$. Starting at $x_1 = 1.1$, very close to the root, the values are $0.99583,\ 1.29302,\ 0.35623,\ 2.57207,\ -9.09283, \ldots$ They jump further and further away: this iteration **does not converge**.



So a correct rearrangement is not always a useful one. In the exam the formula (or the rearrangement to base it on) is always given, and it will converge from a sensible starting value. If your values do **not** settle down, check your working and your calculator first: the most common cause is a calculator in **degrees** when the formula contains $\sin$, $\cos$ or $\tan$.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it. Nothing in this topic is in MF19.

Formula	Status
If f is continuous and $f(a)$, $f(b)$ have opposite signs, then $f(x) = 0$ has a root between a and b	Learn it
Iterative formula: $x_{n+1} = F(x_n)$, starting from x_1	Learn it
If $x_n \to \alpha$, then $\alpha = F(\alpha)$: the limit is a root of $x = F(x)$	Learn it
Checking an answer $\alpha \approx c$: a sign change of $x - F(x)$ between $c - h$ and $c + h$, where h is half a unit in the last place (for example $h = 0.005$ for 2 decimal places)	Learn it
$\ln e^k = k$, $e^{\ln k} = k$, $\ln a + \ln b = \ln ab$, $\ln a - \ln b = \ln \frac{a}{b}$, $k \ln a = \ln a^k$	Learn it

Formula	Status
$\sec x = \dfrac{1}{\cos x}$; $	x - a

Your calculator's **Ans** key does the iteration for you: type x_1, press $=$, type the formula with Ans in place of x, then press $=$ again and again. Write down every value it shows, rounded as the question asks.

How to do it

We have tagged 37 Paper 2 questions for this topic (2021–2025). Many appear word for word in two or three exam variants, which leaves 25 different questions. Every one of the 25 mixes two or more types, so the counts below add up to more than 25. 19 of the 25 are also tagged with another topic, because the equation comes out of it first: Differentiation (10), Integration (6), Algebra (2) or Logarithmic and exponential functions (1).

Question type	Questions
Use an iterative formula to find a root to a stated accuracy	25
Show that a root satisfies a given equation $x = F(x)$	23
Show by calculation that a root lies between two values	14
Sketch two graphs to show how many roots an equation has	7

1. Use an iterative formula to find a root to a stated accuracy (25 questions)

1. **Write the formula** as $x_{n+1} = F(x_n)$, using the equation from the earlier part (or the formula printed in the question).
2. **Choose x_1.** Use the initial value if one is given. Otherwise start inside the interval you were given or found, for example its midpoint.
3. **Iterate** with the Ans key. Write down **every** value to the number of figures the question asks for. This is usually two more than the answer: 5 significant figures for a 3 s.f. answer, 6 s.f. for a 4 s.f. answer, or 5 decimal places for a 3 d.p. answer. The mark schemes want enough iterations shown to that accuracy.
4. **Stop** when two consecutive values round to the same answer at the stated accuracy.
5. **State the answer** to exactly the accuracy asked for, rounded correctly, not truncated.

For example, with $x_{n+1} = \ln(4 - x_n)$ and $x_1 = 1$, for 3 decimal places: $1.09861, 1.06519, 1.07664, 1.07273, 1.07407, 1.07361$. The last two both give 1.074, so the root is 1.074.

Variations you will meet:

- **"Based on the equation in part (b)"** means you make the formula yourself: put x_{n+1} on the left and x_n on the right of that equation, whatever letter it uses (a, p, q or t as well as x).
- **A given formula**, such as $x_{n+1} = e^{-x_n/10}$: use it exactly as printed.

- **An initial value given**: you must start there. A mark scheme will not give the last mark for a different start.
- **Different accuracy**: 3 or 4 significant figures, 3 decimal places, or occasionally 5 significant figures. Read which one, for both the answer and the iterates.
- **Radians.** Any formula with $\sin$, $\cos$, $\tan$ or $\sec$ needs radian mode.
- **One of two roots.** The question tells you which root the formula is for (for example "the larger root" or β where $\alpha < \beta$). Start near that root.

2. Show that a root satisfies a given equation $x = F(x)$ (23 questions)

1. **Get the starting equation** from the context, then rearrange step by step until you reach the printed equation **exactly**. The answer is given, so every step must be shown.
2. **Look at the target** before you start. If it has $\ln$ on one side, isolate the exponential term first, then take logs. If it has e, isolate the $\ln$ term first. If it has a cube root or a fraction, collect the other terms first.

For example, in Example 3 the stationary point gives $e^x - 10x = 0$, so $e^x = 10x$, and taking logs gives $x = \ln(10x)$.

Variations you will meet (where the starting equation comes from):

- **Where two graphs meet** (5 questions): set the two expressions equal. With a modulus, choose the right branch first: for the root where $x > a$ use $|x - a| = x - a$; where $x < a$ use $a - x$. The sketch shows you which branch meets the other graph.
- **A stationary point or a given gradient** (9 questions): differentiate (product, quotient, implicit or parametric rules from Differentiation), set $\frac{dy}{dx}$ equal to 0 or to the given gradient, then rearrange.
- **A definite integral equal to a value** (5 questions): integrate, put in the limits, set it equal to the value, then rearrange for the unknown limit.
- **A polynomial** (1 question): divide by the factor you found, then rearrange "quotient $= 0$".
- **A point on a parametric curve** (1 question): put the given coordinate equal to its expression in t and rearrange for the parameter.
- **A logarithm equation** (1 question): combine the logs, remove them, then rearrange.
- **"Show that if the sequence converges, it converges to the root"** (1 question): replace both x_{n+1} and x_n by x (or α) and rearrange the result back into the original equation. Example 1(b) does this.

3. Show by calculation that a root lies between two values (14 questions)

1. **Make one function that is 0 at the root.** Best: $x - F(x)$, using the equation $x = F(x)$ from the question. Or take everything in the original equation to one side.
2. **Substitute both values** and write each result as a decimal, for example -0.0553 and 0.0165.
3. **Conclude in words**: "there is a change of sign, so the root lies between 3.5 and 3.6".

Variations you will meet:

- **Comparing F with x** is also accepted: for $x = \ln(10x)$, $F(3.5) = \ln 35 = 3.5553 > 3.5$ and $F(3.6) = \ln 36 = 3.5835 < 3.6$. The order changes, so the root is between them. Say this in words.
- **Using the gradient**: when the root is the x-coordinate of a stationary point, a change of sign of $\frac{dy}{dx}$ between the two values also works, **unless the question says to use the equation from an earlier part**. Then you must use that equation, for example $x - F(x)$.
- **A root on a modulus graph**: use the branch that the root is on, for example $4 - 5x$ rather than $5x - 4$ for a root where $x < 0.8$.
- **Negative values**, such as "show that $-1.4 < \beta < -1.0$": the method is the same; take care with signs on the calculator.
- **The other root**: a question may ask you to iterate for one root and check the interval for the other one.

4. Sketch two graphs to show how many roots an equation has (7 questions)

1. **Split the equation** into two functions you can sketch: often it is already "graph $=$ graph", such as $|x - 4| = 2\ln x$.
2. **Sketch both on the same axes** with their key features: where they cross the axes, the vertex of a modulus V, asymptotes, and the right shape (for example $\ln x$ rising ever more slowly, e^{-x} falling towards the x-axis).
   - **Draw each graph on both sides of the y-axis** unless the domain is restricted. For example, $y = e^{-2x}$ goes on into the 2nd quadrant, getting steeper; $y = x^3$ goes on into the 3rd quadrant; $y = \ln(x+2)$ crosses the y-axis at $\ln 2$ and carries on down to its asymptote $x = -2$.
   - **Show which graph is higher where each crosses the y-axis.** For $y = |x - 3|$ and $y = 1 + e^x$, the V meets the y-axis at 3, above the curve at 2. This decides how many times the graphs meet.
3. **Count the crossings** and say so: "the graphs meet at exactly two points, so the equation has exactly two roots". Mark the crossings on the sketch.

Variations you will meet:

- **A modulus and a log or exponential curve** (4 questions): the V's two arms can each meet the curve. Check that the slopes you draw give the number of crossings the question says.
- **An exponential and** $\sec x$ (1 question), for $0 \leqslant x < \frac{1}{2}\pi$: $\sec x$ starts at 1 and rises steeply towards the asymptote $x = \frac{1}{2}\pi$.
- **An exponential and a power of** x (1 question), such as $y = e^{-x/2}$ and $y = x^5$.
- $\ln x$ **and an exponential** (1 question), such as $y = \ln x$ and $y = 2e^{-x}$.
- **One graph drawn for you**: you add the other on the diagram and still state the number of roots.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

The equation $x^3 + 3x - 9 = 0$ has one real root, α.

(a) Show by calculation that α lies between 1.5 and 1.7. **[2]**

(b) Show that if a sequence of values given by the iterative formula $x_{n+1} = \sqrt[3]{9 - 3x_n}$ converges, then it converges to α. **[1]**

(c) Use this iterative formula with $x_1 = 1.6$ to find α correct to 3 significant figures. Give the result of each iteration to 5 significant figures. **[3]**

(a) Let $f(x) = x^3 + 3x - 9$. Then $f(1.5) = -1.125$ and $f(1.7) = 1.013$. **M1** for substituting both values.

The sign changes, so α lies between 1.5 and 1.7. **A1** (both values and the conclusion).

(b) If the sequence converges to α, then x_n and x_{n+1} both tend to α, so $\alpha = \sqrt[3]{9 - 3\alpha}$. Cubing, $\alpha^3 = 9 - 3\alpha$, so $\alpha^3 + 3\alpha - 9 = 0$: α is the root of the equation. **B1**

(c) $x_1 = 1.6$, $x_2 = 1.6134$, $x_3 = 1.6083$, $x_4 = 1.6103$, $x_5 = 1.6095$. **M1** for using the formula correctly at least once.

x_4 and x_5 both round to 1.61, so $\alpha = 1.61$ (3 s.f.). **A1** for 1.61, **A1** for enough iterations shown to 5 s.f.

(A sign change also confirms it: $f(1.605) = -0.050$ and $f(1.615) = 0.057$.)

Example 2

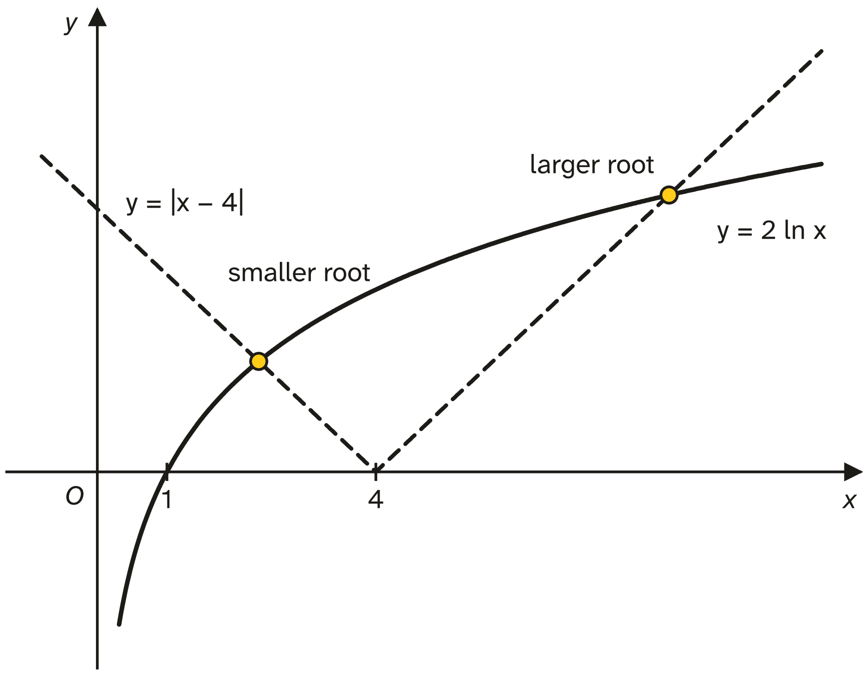
(a) By sketching the graphs of $y = |x - 4|$ and $y = 2\ln x$ on the same diagram, show that the equation $|x - 4| = 2\ln x$ has exactly two roots. **[3]**

(b) Show that the larger root satisfies the equation $x = 4 + 2\ln x$. **[1]**

(c) Show by calculation that the larger root lies between 8 and 9. **[2]**

(d) Use an iterative formula, based on the equation in part (b), to find the larger root correct to 3 significant figures. Give the result of each iteration to 5 significant figures. **[3]**

(a)



B1 for the V with its vertex at $(4, 0)$ on the positive x-axis. **B1** for the $\ln$ curve through $(1, 0)$, rising more and more slowly. **B1** for "the graphs meet at exactly two points, so the equation has exactly two roots".

(b) At the larger root $x > 4$, so $|x - 4| = x - 4$. Then $x - 4 = 2\ln x$, so $x = 4 + 2\ln x$. **B1**

(c) Let $g(x) = x - 4 - 2\ln x$. $g(8) = -0.159$ and $g(9) = 0.606$. **M1**

The sign changes, so the larger root lies between 8 and 9. **A1**

(d) $x_{n+1} = 4 + 2\ln x_n$. With $x_1 = 8$: $x_2 = 8.1589$, $x_3 = 8.1982$, $x_4 = 8.2078$, $x_5 = 8.2102$. **M1**

x_4 and x_5 both round to 8.21, so the larger root is 8.21 (3 s.f.). **A1 A1**

The values rise steadily towards the root from below: a staircase rather than a spiral.

Example 3

A curve has equation $y = e^x - 5x^2$. It has a stationary point P whose x-coordinate is greater than 1.

(a) Show that the x-coordinate of P satisfies the equation $x = \ln(10x)$. **[3]**

(b) Show by calculation that the x-coordinate of P lies between 3.5 and 3.6. **[2]**

(c) Use an iterative formula, based on the equation in part (a), to find the x-coordinate of P correct to 4 significant figures. Use an initial value of 3.5 and give the result of each iteration to 6 significant figures. **[3]**

(a) $\dfrac{dy}{dx} = e^x - 10x$. **B1**

At a stationary point $e^x - 10x = 0$, so $e^x = 10x$. **M1** for setting the derivative to 0 and isolating e^x.

Taking natural logs: $x = \ln(10x)$. **A1** (given answer, every step shown).

(b) Consider $x - \ln(10x)$: at 3.5 it is $3.5 - \ln 35 = -0.0553$; at 3.6 it is $3.6 - \ln 36 = 0.0165$. **M1**

The sign changes, so the x-coordinate of P lies between 3.5 and 3.6. **A1**

(c) $x_{n+1} = \ln(10x_n)$, $x_1 = 3.5$: $x_2 = 3.55535$, $x_3 = 3.57104$, $x_4 = 3.57544$, $x_5 = 3.57667$, $x_6 = 3.57702$. **M1**

x_5 and x_6 both round to 3.577, so the x-coordinate of P is 3.577 (4 s.f.). **A1 A1**

The curve has a second stationary point, at $x = 0.112$ (3 d.p.), which also satisfies $x = \ln(10x)$. This formula cannot find it: starting at $x_1 = 0.5$ gives $1.6094, 2.7785, 3.3245, \ldots$, heading for 3.577 instead. That is why questions tell you which root, and where to start.

Example 4

It is given that $\displaystyle\int_0^a (3 - \cos 2x)\,dx = 2$, where a is a positive constant.

(a) Show that $a = \frac{1}{6}(4 + \sin 2a)$. **[3]**

(b) Use an iterative formula, based on the equation in part (a), to find the value of a correct to 4 significant figures. Use an initial value of 0.8 and give the result of each iteration to 6 significant figures. **[3]**

(a) $\displaystyle\int (3 - \cos 2x)\,dx = 3x - \frac{1}{2}\sin 2x$. **B1**

Limits: $\left(3a - \frac{1}{2}\sin 2a\right) - (0 - 0) = 2$. **M1** for using both limits and setting the result equal to 2.

Multiply by 2: $6a - \sin 2a = 4$, so $6a = 4 + \sin 2a$ and $a = \frac{1}{6}(4 + \sin 2a)$. **A1** (given answer).

(b) $a_{n+1} = \frac{1}{6}(4 + \sin 2a_n)$, in **radians**. $a_1 = 0.8$, $a_2 = 0.833262$, $a_3 = 0.832570$, $a_4 = 0.832592$, $a_5 = 0.832591$. **M1**

a_4 and a_5 both round to 0.8326, so $a = 0.8326$ (4 s.f.). **A1 A1**

In degree mode the values settle on 0.6706 instead (the first step gives 0.671320), which is wrong.

Common mistakes

- **Rounding the final answer wrongly.** Examiner reports from 2021 to 2023 keep noting students who do all the iterations correctly, then truncate the answer or give it to the wrong accuracy (for example 0.356 instead of 0.357). Round the final answer correctly, to exactly the accuracy asked for.
- **Not showing the iterations properly.** Write down every iterate, to the accuracy the question asks for, including the last one. An answer with too few iterations, or with values rounded to fewer figures than asked, loses marks.
- **Calculator in degrees.** With $\sin$, $\cos$, $\tan$ or $\sec$ in the formula, degree mode gives values that settle on the wrong number or do not settle at all. If they do not settle, check the mode.

- **Testing the wrong function for a sign change.** Substituting the two values into $F(x)$ alone proves nothing. Use $x - F(x)$ (or the original equation with everything on one side), or compare $F(x)$ with x and say what the comparison shows.
- **Giving signs without values, or values without a conclusion.** Write the decimal values, then a sentence: "change of sign, so the root lies between ...".
- **Using the wrong branch of a modulus.** For the larger root of $|x-4| = 2\ln x$ the root is where $x > 4$, so use $x - 4$; for a root on the left arm use $4 - x$. Look at your sketch.
- **Sketches without the conclusion.** A correct pair of graphs is not enough: say how many times they meet and so how many roots there are. Draw a modulus graph as a sharp V with its vertex on the x-axis, and don't let a $\ln$ curve bend back down.
- **Taking logs too early** when rearranging. Get the exponential term on its own first ($e^{2a+1} = \ldots$), then take logs of both sides; $\ln(A + B)$ is not $\ln A + \ln B$.
- **Starting somewhere else.** If an initial value is given, use it. If not, start inside the interval you found.
- **Misreading "show that if it converges, it converges to the root"**: this needs no numbers. Replace x_{n+1} and x_n by x and rearrange back to the original equation.

How to get the marks

- **The iteration is usually 3 marks:** a method mark for using the formula correctly at least once, a mark for the final answer to the exact accuracy asked for, and a mark for showing enough iterations to that many figures (or for a sign change over the interval that rounds to your answer, such as $[1.605, 1.615]$ for 1.61).
- **"Show that" rearrangements are given answers.** Every step must be visible and your last line must match the printed equation exactly, including which letter it uses. Work forwards from the starting equation: substituting into the printed answer to check it is often not accepted.
- **"Show by calculation"** needs both values written as decimals (2 significant figures is plenty) and a conclusion that mentions the change of sign.
- **"By sketching ... show that ... has exactly n roots"**: marks for each correct graph, and full marks need the statement linking the crossings to the number of roots. A graph earns its mark only if it is drawn over its whole range: on both sides of the y-axis unless the domain is restricted, with the right one higher where they cross the y-axis.
- **"Hence"** in these questions usually means use your root in the next part, for example as a limit when finding an area.
- **Accuracy words:** "3 significant figures" and "3 decimal places" are different. The paper's rule for answers not otherwise specified is 3 significant figures.

Quick check

1. Show by calculation that the equation $x^3 - 4x - 2 = 0$ has a root α between 2.2 and 2.3.
2. Show that if the iterative formula $x_{n+1} = \sqrt[3]{4x_n + 2}$ converges, it converges to α. Use it with $x_1 = 2.2$ to find α correct to 3 significant figures, giving each iteration to 5 significant figures.

3. The same equation can be written as $x = \frac{1}{4}(x^3 - 2)$. Find x_2, x_3 and x_4 for $x_{n+1} = \frac{1}{4}(x_n^3 - 2)$ with $x_1 = 2.2$, and say what happens.
4. A student uses $x_{n+1} = 1 + \frac{1}{2}\cos x_n$ with $x_1 = 1.2$ and gets $x_2 = 1.4999$. What has gone wrong? Find the correct x_2 and x_3 to 5 significant figures.
5. Show that the root of $3e^x + x^2 = 8$ satisfies $x = \ln\left(\frac{8 - x^2}{3}\right)$. Use this with $x_1 = 0.9$ to find the root correct to 3 significant figures.

Answers

1. $f(x) = x^3 - 4x - 2$: $f(2.2) = -0.152$ and $f(2.3) = 0.967$. The sign changes, so there is a root between 2.2 and 2.3.
2. If the values converge to α, then $\alpha = \sqrt[3]{4\alpha + 2}$, so $\alpha^3 = 4\alpha + 2$ and $\alpha^3 - 4\alpha - 2 = 0$. Iterating: $x_2 = 2.2104$, $x_3 = 2.2133$, $x_4 = 2.2140$. The last two both round to 2.21, so $\alpha = 2.21$ (3 s.f.).
3. $x_2 = 2.162$, $x_3 = 2.0264$, $x_4 = 1.5803$. The values move further from $\alpha = 2.21$ each time, so this iteration does not converge to α.
4. The calculator is in degrees. In radians, $x_2 = 1 + \frac{1}{2}\cos 1.2 = 1.1812$ and $x_3 = 1.1899$.
5. $3e^x = 8 - x^2$, so $e^x = \dfrac{8 - x^2}{3}$, and taking logs gives $x = \ln\left(\dfrac{8 - x^2}{3}\right)$. Iterating from 0.9: $0.87408, 0.88045, 0.87891, 0.87928$. The last two both round to 0.879, so the root is 0.879 (3 s.f.).

Past questions to try

- **9709/22/O/N/21 Q6**: sketch, sign change, why the formula converges to the root, then iterate.
- **9709/21/M/J/22 Q5**: a modulus graph and a log curve, the larger root.
- **9709/23/O/N/23 Q7**: the equation comes from a stationary point found by implicit differentiation.
- **9709/22/O/N/25 Q6**: the equation comes from a definite integral.