Cambridge AS & A Level · Mathematics (9709) · Notes

# Numerical solution of equations A2

Read online at pastopic.com/cambridge/mathematics-9709/notes/numerical-solution-of-equations-p3

Last checked 28 September 2026

Read first: Trigonometry, Logarithmic and exponential functions, Differentiation, Integration



## What you need to know

By the end of this topic you should be able to:

1. **Find roughly where a root is.** Sketch two graphs on the same axes to see how many times they cross (each crossing is a root) and roughly where, and show that a root lies between two given values, such as two consecutive integers, by finding a **change of sign**.
2. **Use the notation for a sequence of approximations**, $x_1, x_2, x_3, \ldots$, that gets closer and closer to a root, and understand what it means for the sequence to **converge**.
3. **Use an iterative formula** $x_{n+1} = F(x_n)$. Understand how it is linked to the equation it solves (a limit $\alpha$ satisfies $\alpha = F(\alpha)$, which rearranges to the equation), use a formula you are given or one you build from a given rearrangement, and iterate to find a root to a stated accuracy. Know that an iteration **may fail to converge**.

You do **not** need the condition that decides whether an iteration converges, and Newton–Raphson is not in the syllabus. The equations usually come out of another Paper 3 topic first: a stationary point, a definite integral or a sector.

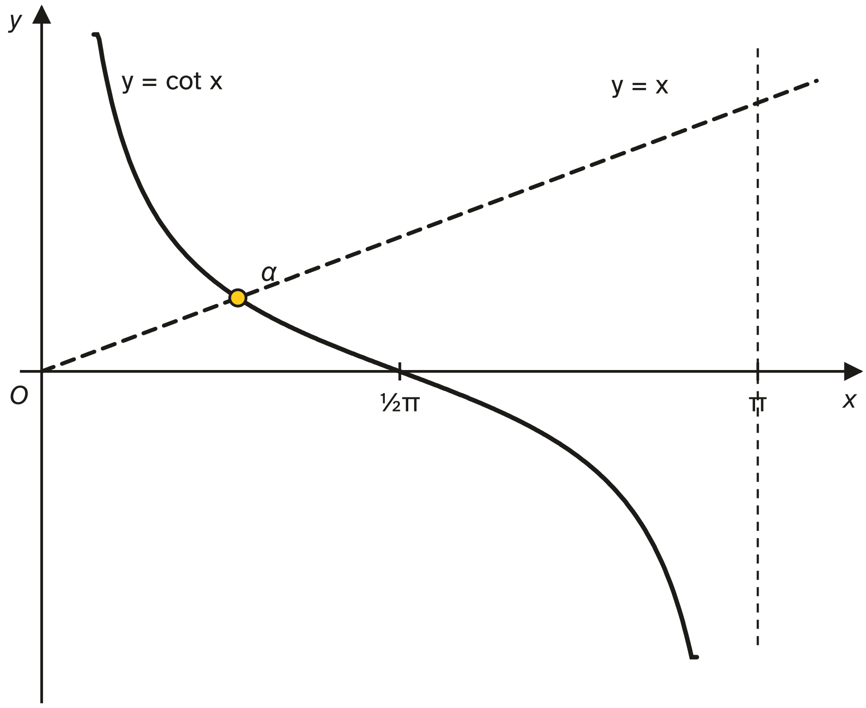
## Understand it

### Why we need numerical methods

Equations such as $\cot x = x$ or $\sec x = 3 - x$ mix trigonometric and polynomial parts, so no algebra will give an exact root. Instead you pin the root down **to a stated accuracy**, such as 2 decimal places, with a method that gets closer and closer to it.

### Locating a root with graphs

Write the equation as "one function $=$ another" and sketch both graphs on the same axes. Each crossing gives one root, so counting crossings counts roots, and the crossing's position shows roughly where each root is.




For $\cot x = x$ with $0 < x < \pi$: $y = \cot x$ falls from very large values near $x = 0$ to $0$ at $x = \frac{1}{2}\pi$ and then goes negative, while $y = x$ rises from $0$. They cross once in $0 < x < \frac{1}{2}\pi$ and never again, because after $\frac{1}{2}\pi$ one is negative and the other positive. So the equation has exactly **one** root, $\alpha$, in $0 < x < \pi$.

## Locating a root with a change of sign

Put everything on one side: $f(x) = \cot x - x$. In **radians**,

$$f(0.8) = 0.1712, \qquad f(0.9) = -0.1064.$$

$f$ is positive at $0.8$ and negative at $0.9$. A graph with no breaks cannot get from above the $x$-axis to below it without crossing, so $f(x) = 0$ somewhere in between: **the root lies between 0.8 and 0.9**. This needs $f$ to be continuous between the two values; $\cot x$ has breaks at $0$ and $\pi$, but not between $0.8$ and $0.9$.

## Iteration: the idea

Rearrange the equation so that $x$ is on its own on one side. From $\cot x = x$: $\tan x = \frac{1}{x}$, so $x = \tan^{-1}\left(\frac{1}{x}\right)$. Turn this into a rule that makes a **sequence**:

$$x_{n+1} = \tan^{-1}\left(\frac{1}{x_n}\right).$$

Choose a starting value $x_1$, put it into the right-hand side to get $x_2$, put $x_2$ in to get $x_3$, and so on. With $x_1 = 0.8$ (to 4 decimal places):

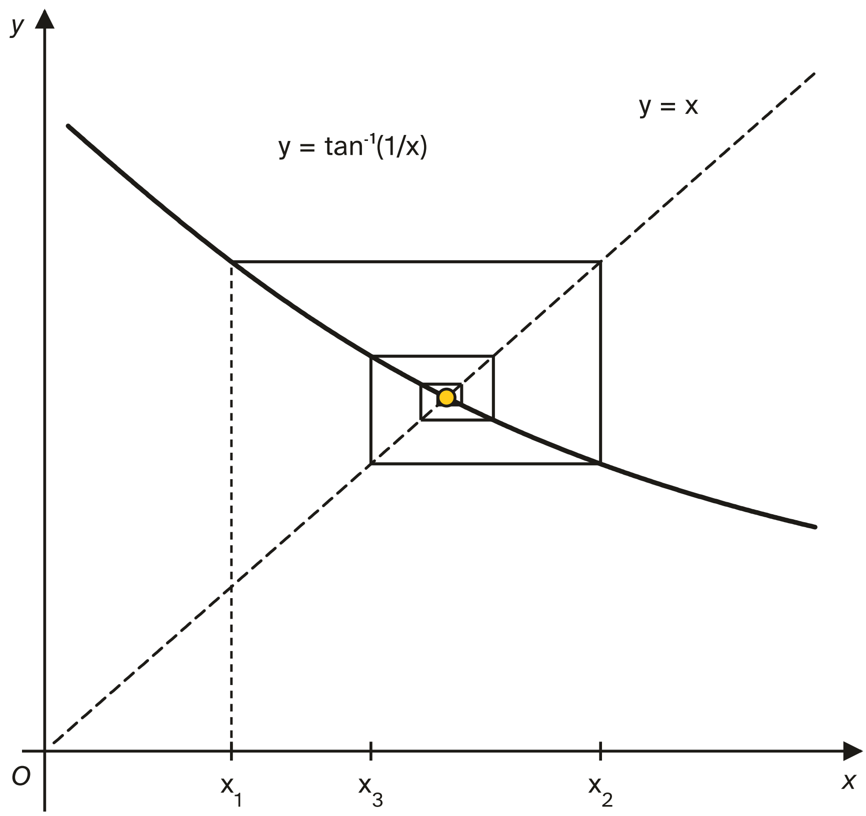
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $x_n$ | 0.8 | 0.8961 | 0.8402 | 0.8720 | 0.8536 | 0.8642 | 0.8581 |

The values settle down: the sequence **converges**. Why to the root? If the values settle on a number $\alpha$, then for large $n$ both $x_n$ and $x_{n+1}$ are (almost exactly) $\alpha$, so $\alpha = \tan^{-1}\left(\frac{1}{\alpha}\right)$. Taking $\tan$ of both sides, $\tan\alpha = \frac{1}{\alpha}$, which is $\cot\alpha = \alpha$. **Whatever an iteration converges to satisfies $x = F(x)$, and so the equation that $F$ came from.** That argument is exactly what "show that if the sequence converges, it converges to the root" asks for.

$x_6$ and $x_7$ both round to $0.86$, so $\alpha = 0.86$ to 2 decimal places. To be sure, check the sign either side of $0.86$ by half a unit in the last place: $f(0.855) = 0.0147$ and $f(0.865) = -0.0128$, so $0.855 < \alpha < 0.865$, which rounds to $0.86$.

## Seeing it on a graph

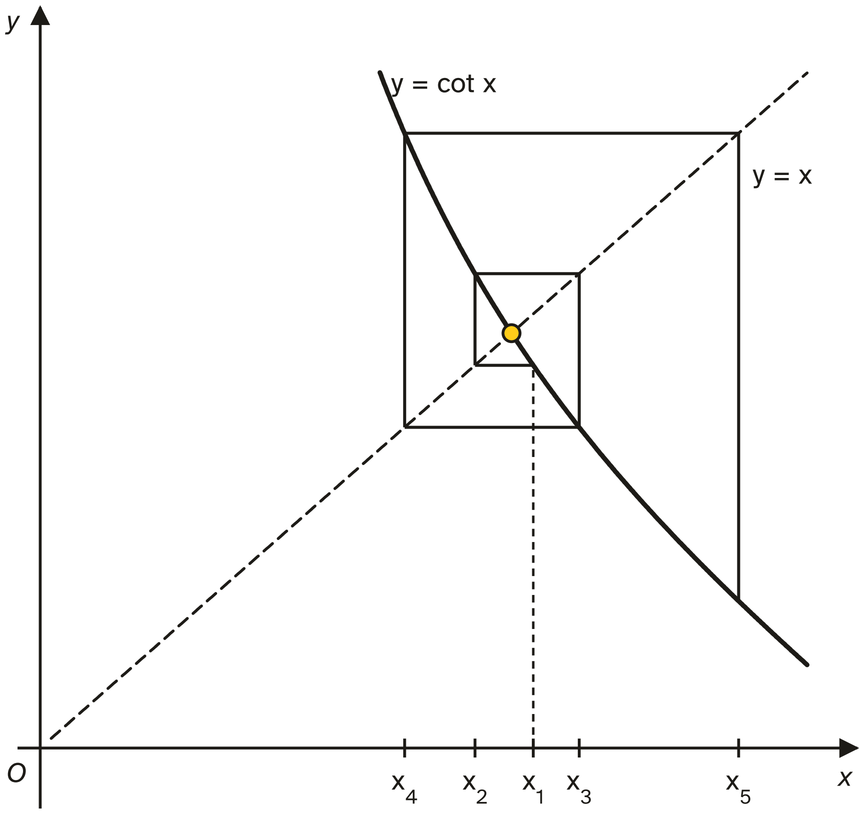
A root of $x = F(x)$ is where the line $y = x$ meets the curve $y = F(x)$. Each step of the iteration is "go up (or down) to the curve, then across to the line", which turns $x_n$ into $x_{n+1}$. You will not be asked to draw this, but it explains how the values behave.




Here the values land on **alternate sides** of the root (a spiral), so the root is always between two consecutive values. With other formulas the values creep towards the root **from one side** (a staircase), as in Example 2 and Example 4 below. That difference matters when you decide you have enough iterations: in a staircase, two values that round the same can both still be short of the root.

### When an iteration fails

$\cot x = x$ can also be written $x = \cot x$ directly. Starting at $x_1 = 0.9$, very close to the root, this gives $0.79355,\ 0.98383,\ 0.66518,\ 1.27480,\ 0.30495, \ldots$ The values jump further and further away: this iteration **does not converge**.




So a correct rearrangement is not always a useful one. In the exam the formula (or the rearrangement it is based on) is chosen for you and converges from any sensible start. If your values do **not** settle, check the calculator first: the usual cause is **degree mode**, with $\sin$, $\cos$, $\tan$ or their inverses in the formula.

### Rearranging with inverse functions

Most Paper 3 formulas undo a function to get $x$ on its own:

- $\tan 2x = k$ gives $x = \frac{1}{2}\tan^{-1} k$; $\cos 2x = k$ gives $x = \frac{1}{2}\cos^{-1} k$; $e^{kx} = A$ gives $x = \frac{1}{k}\ln A$; $\ln(kx) = A$ gives $x = \frac{1}{k}e^{A}$.
- $\sec$, $\operatorname{cosec}$ and $\cot$ are turned into $\cos$, $\sin$ and $\tan$ first: $\sec 2x = B$ means $\cos 2x = \frac{1}{B}$.
- Your calculator's $\tan^{-1}$ gives values between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, and $\cos^{-1}$ between $0$ and $\pi$. So $x = \frac{1}{2}\cos^{-1}(\ldots)$ can only produce values from $0$ to $\frac{1}{2}\pi$; the formula is built to find the root in that range, and checking the range is a quick test that you have the right formula.

## Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

| Formula | Status |
|---|---|
| If $f$ is continuous and $f(a)$, $f(b)$ have opposite signs, then $f(x) = 0$ has a root between $a$ and $b$ | Learn it |
| Iterative formula: $x_{n+1} = F(x_n)$, starting from $x_1$ | Learn it |
| If $x_n \to \alpha$, then $\alpha = F(\alpha)$: the limit is a root of $x = F(x)$ | Learn it |
| Checking an answer $c$: a sign change of $x - F(x)$ between $c - h$ and $c + h$, where $h$ is half a unit in the last place ($h = 0.005$ for 2 decimal places, $0.0005$ for 3) | Learn it |
| Principal values: $-\frac{1}{2}\pi \leqslant \sin^{-1} x \leqslant \frac{1}{2}\pi$, $0 \leqslant \cos^{-1} x \leqslant \pi$, $-\frac{1}{2}\pi < \tan^{-1} x < \frac{1}{2}\pi$ | Given |
| $\sec x = \dfrac{1}{\cos x}$, $\operatorname{cosec} x = \dfrac{1}{\sin x}$, $\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$ | Learn it |
| $\ln e^k = k$, $e^{\ln k} = k$, $\ln a + \ln b = \ln ab$, $k \ln a = \ln a^k$ | Learn it |
| Sector area $\frac{1}{2}r^2\theta$, arc length $r\theta$ ($\theta$ in radians) | Given |
| Triangle area $\frac{1}{2}ab\sin C$, so segment area $= \frac{1}{2}r^2(\theta - \sin\theta)$ | Learn it |
| Product rule $\dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$; integration by parts $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$ | Given |

Your calculator's **Ans** key does the iteration: type $x_1$ and press $=$, type the formula with Ans in place of $x$, then press $=$ again and again. Write down every value it shows, rounded as the question asks.

## How to do it

We have tagged 35 different Paper 3 questions for this topic (2021–2025). Every one of them mixes two or more types, so the counts below add up to more than 35. 17 of the 35 are also tagged with another topic, because the equation is set up there first: Differentiation (11), Integration (5) or Differential equations (1).

| Question type | Questions |
|---|---|
| Use an iterative formula to find a root to a stated accuracy | 34 |
| Show by calculation that a root lies between two values | 30 |
| Show that a root satisfies a given equation (from a curve, an integral or a shape) | 19 |
| Sketch two graphs to show how many roots an equation has | 13 |
| Show that if the iteration converges, it converges to the root | 11 |

## 1. Use an iterative formula to find a root to a stated accuracy (34 questions)

1. **Write the formula** as $x_{n+1} = F(x_n)$: the one printed, or the equation from an earlier part with $x_{n+1}$ on the left and $x_n$ on the right, whatever letter it uses ($a$, $p$, $\alpha$ or $\theta$).
2. **Choose $x_1$.** Use the initial value if one is given; you must. Otherwise start inside the interval from the earlier part, for example its midpoint.
3. **Set radians** if the formula has any trigonometric function.
4. **Iterate** with Ans and write down **every** value to the number of decimal places asked for the iterates, usually two more than the answer (4 d.p. for a 2 d.p. answer, 5 d.p. for 3 d.p., 6 d.p. for 4 d.p.).
5. **Stop** when two consecutive values round to the same answer **and** you are sure the root is not beyond them (see below).
6. **State the answer** to exactly the accuracy asked, rounded correctly (not truncated).

For $\cot x = x$ with $x_1 = 0.8$: $0.8961, 0.8402, 0.8720, 0.8536, 0.8642, 0.8581$. The last two both give $0.86$, and because the values alternate the root is between them, so $\alpha = 0.86$.

**Variations you will meet:**

- **Staircase convergence.** When the values keep moving the same way, two that round the same are not always enough: in Example 4, $5.3494$ and $5.3540$ both round to $5.35$, but the root is $5.36$. Keep going until the values stop changing in the place that matters, or finish with a sign change over the interval $[c - 0.005,\ c + 0.005]$ around your answer $c$.
- **"State the minimum number of iterations needed"**: count the values you **calculated** (not $x_1$) up to the first pair of consecutive values that both round to the answer. From $0.8$ above, that is $6$ (from $x_2$ to $x_7$).
- **Different accuracy**: 2, 3 or 4 decimal places. Read which one, for both the answer and the iterates.
- **Formulas with** $\exp$: $\exp(x)$ means $e^x$. A formula such as $x_{n+1} = \frac{1}{3}\exp\left(\frac{1}{2x_n}\right)$ needs brackets typed carefully: from $x_1 = 0.7$ it gives $x_2 = 0.6809$.

## 2. Show by calculation that a root lies between two values (30 questions)

1. **Make one function that is $0$ at the root**: take everything to one side of the equation, or use $x - F(x)$ with the equation $x = F(x)$ from the question.
2. **Substitute both values** (in radians) and write each result as a decimal to at least 2 significant figures.
3. **Conclude**: "$f(0.8) > 0$ and $f(0.9) < 0$; there is a change of sign, so the root lies between $0.8$ and $0.9$."

**Variations you will meet:**

- **Comparing the two sides** instead: for $x = \tan^{-1}(1/x)$, at $0.8$ the right side is $0.8961 > 0.8$, and at $0.9$ it is $0.8380 < 0.9$. The inequality turns round, so the root is between them. Say this in words; don't talk about a "sign change" when you compared two values.
- **The equation is the answer to an earlier "show that"**: work with that equation, even if you did not manage to show it.

- **A root that is a stationary point**: you may also use the sign of $\frac{dy}{dx}$ at the two values, unless the question tells you to use the equation from part (a).
- **"Verify by calculation"** means the same as "show by calculation".

## 3. Show that a root satisfies a given equation (19 questions)

1. **Set up the starting equation** from the context (below).
2. **Rearrange step by step to the printed equation**, exactly, with the letter it uses. It is a given answer, so show every step. Work forwards; don't substitute into the printed answer.
3. **Look at the target first.** If it has $\tan^{-1}$, $\cos^{-1}$ or $\ln$ on one side, get $\tan$, $\cos$ or the exponential on its own first, then apply the inverse.

**Where the starting equation comes from:**

- **A stationary point** (8 questions): differentiate with the product or quotient rule, set $\frac{dy}{dx} = 0$, and rearrange. For $y = x^2 \cos x$: $2x \cos x - x^2 \sin x = 0$; divide by $x \cos x$ (both non-zero at the point) to get $\tan x = \frac{2}{x}$, so $x = \tan^{-1}\left(\frac{2}{x}\right)$. Example 2 does a full one.
- **A definite integral equal to a value** (5 questions): integrate (usually **by parts**), put in the limits, set equal to the value, then isolate the unknown limit. Example 4.
- **A shape** (3 questions): write the areas with sector $\frac{1}{2}r^2\theta$, triangle $\frac{1}{2}r^2 \sin\theta$ and segment $\frac{1}{2}r^2(\theta - \sin\theta)$, form the equation the question describes, and cancel $r^2$. Example 3.
- **A tangent with a given gradient** (2 questions): set $\frac{dy}{dx}$ equal to that gradient and rearrange.
- **A differential equation** (1 question): solve it, substitute the given value of $t$, and rearrange for $x$.

## 4. Sketch two graphs to show how many roots an equation has (13 questions)

1. **Split the equation** into two graphs you know, usually the two sides as printed (for example $y = \sec x$ and $y = 3 - x$ in Example 1). You may rearrange to easier graphs.
2. **Sketch both on the same axes, over the whole interval given**, with the key features:

| Graph | Key features |
|---|---|
| $y = \cot x$ | asymptotes $x = 0$ and $x = \pi$; crosses the $x$-axis at $\frac{1}{2}\pi$; always falling |
| $y = \sec x$ | $(0, 1)$; asymptote $x = \frac{1}{2}\pi$; negative between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$ |
| $y = \operatorname{cosec} x$ | asymptotes $x = 0$ and $x = \pi$; lowest point $\left(\frac{1}{2}\pi, 1\right)$ |
| $y = \sec 2x$, $\cot 2x$, $\sin 2x$ | the same shapes squashed sideways by 2: asymptotes of $\sec 2x$ at $x = \frac{1}{4}\pi, \frac{3}{4}\pi$; $\cot 2x$ at $x = 0, \frac{1}{2}\pi$ |
| $y = e^{-x}$, $y = a + e^{-x}$ | through $(0, 1)$ or $(0, a + 1)$; asymptote $y = 0$ or $y = a$ |
| $y = -e^x$ | through $(0, -1)$; falls ever more steeply |
| $y = \ln x$, $y = \ln(kx)$ | asymptote the $y$-axis; crosses the $x$-axis at $1$ or $\frac{1}{k}$ |

| Graph | Key features |
|---|---|
| $y = \dfrac{k}{x}$ $(k > 0)$ | for $x > 0$, falls from very large values; asymptotes both axes, never touching them |
| $y = \|x - a\|$ | a sharp V with its vertex at $(a, 0)$ |

3. **Mark each crossing** with a dot and **state the conclusion**: "the graphs cross exactly once in $0 < x < \frac{1}{2}\pi$, so the equation has exactly one root there."

**Variations you will meet:**

- **"Only one root" with no interval** (for example **9709/31/O/N/24 Q5**, a log curve against an exponential curve): show the long-term behaviour, such as one curve levelling off at an asymptote while the other keeps rising.
- **Trigonometric graphs with a restricted interval**: draw only that interval, and make sure the asymptotes and axis crossings are in the right places, because they decide the number of crossings.

### 5. Show that if the iteration converges, it converges to the root (11 questions)

1. **Replace both $x_{n+1}$ and $x_n$ by $x$** (or $\alpha$): write "if the sequence converges to $\alpha$, then $\alpha = F(\alpha)$".
2. **Rearrange that equation** step by step back to the equation in the question. No numbers are needed.

For $x_{n+1} = \frac{1}{2}\cos^{-1}\left(\frac{-2}{3+x_n}\right)$: $x = \frac{1}{2}\cos^{-1}\left(\frac{-2}{3+x}\right)$ gives $\cos 2x = \frac{-2}{3+x}$, so $\sec 2x = -\frac{3+x}{2}$. You may also go the other way, from the equation to the formula, then say the limit satisfies it.

## Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **DM1** method mark that needs the M mark before it.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

### Example 1

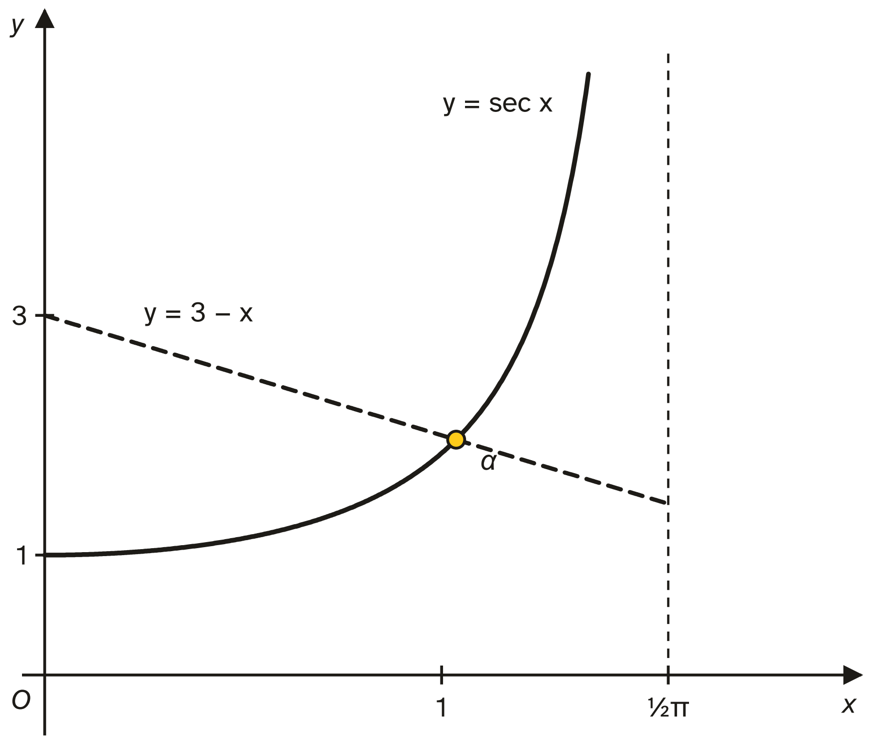
(a) By sketching a suitable pair of graphs, show that the equation $\sec x = 3 - x$ has exactly one root in the interval $0 \leqslant x < \frac{1}{2}\pi$. **[2]**

(b) Show by calculation that this root lies between $1$ and $1.1$. **[2]**

(c) Show that if a sequence of values given by the iterative formula $x_{n+1} = \cos^{-1}\left(\dfrac{1}{3 - x_n}\right)$ converges, then it converges to the root of the equation in part (a). **[2]**

(d) Use this iterative formula with $x_1 = 1$ to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. **[3]**

**(a)**




**B1** for $y = \sec x$ from $(0, 1)$ rising to the asymptote $x = \frac{1}{2}\pi$. **B1** for the line from $(0, 3)$, the crossing marked, and "exactly one point of intersection, so exactly one root".

**(b)** Let $f(x) = \sec x - (3 - x)$. Then $f(1) = 1.851 - 2 = -0.149$ and $f(1.1) = 2.205 - 1.9 = 0.305$. **M1** for both values, in radians.

The sign changes, so the root lies between $1$ and $1.1$. **A1**

**(c)** If the sequence converges to $\alpha$, then $\alpha = \cos^{-1}\left(\dfrac{1}{3 - \alpha}\right)$. **M1**

So $\cos\alpha = \dfrac{1}{3 - \alpha}$, and taking reciprocals, $\sec\alpha = 3 - \alpha$: $\alpha$ is the root. **A1**

**(d)** $x_1 = 1$, $x_2 = 1.04720$, $x_3 = 1.03319$, $x_4 = 1.03743$, $x_5 = 1.03615$, $x_6 = 1.03654$, $x_7 = 1.03642$, $x_8 = 1.03646$. **M1** for using the formula correctly at least once.

$x_7$ and $x_8$ both round to $1.036$, and the values alternate, so the root is $1.036$ (3 d.p.). **A1** for $1.036$, **A1** for enough iterations to 5 d.p.

## Example 2

The curve $y = \dfrac{\sin x}{x+1}$, for $0 < x < \pi$, has a maximum point where $x = a$.

(a) Show that $a$ satisfies the equation $a = \tan^{-1}(1+a)$. **[4]**

(b) Verify by calculation that $a$ lies between $1$ and $1.2$. **[2]**

(c) Use an iterative formula based on the equation in part (a) to determine $a$ correct to 3 decimal places. Give the result of each iteration to 5 decimal places. **[3]**

**(a)** Quotient rule: $\dfrac{dy}{dx} = \dfrac{(x+1)\cos x - \sin x}{(x+1)^2}$. **M1** for the quotient (or product) rule, **A1** for the correct derivative.

At the maximum, $(a+1)\cos a - \sin a = 0$, so $\sin a = (a+1)\cos a$. **M1** for setting the derivative to $0$ and rearranging.

Divide by $\cos a$: $\tan a = 1 + a$, so $a = \tan^{-1}(1+a)$. **A1** (given answer, every step shown).

**(b)** Let $h(x) = x - \tan^{-1}(1+x)$. $h(1) = -0.107$ and $h(1.2) = 0.0558$. **M1**

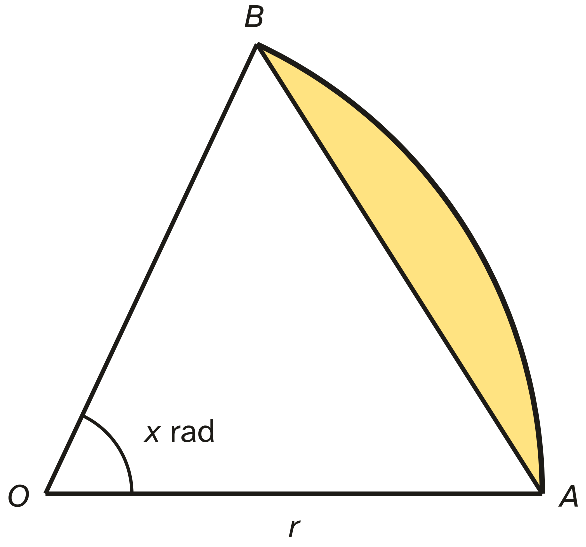
The sign changes, so $a$ lies between $1$ and $1.2$. **A1**

**(c)** $a_{n+1} = \tan^{-1}(1+a_n)$ with $a_1 = 1$: $a_2 = 1.10715$, $a_3 = 1.12769$, $a_4 = 1.13144$, $a_5 = 1.13212$, $a_6 = 1.13224$, $a_7 = 1.13226$. **M1**

The values rise steadily (a staircase). $a_6$ and $a_7$ differ by only $0.00002$ and both round to $1.132$, so $a = 1.132$ (3 d.p.). **A1 A1**

(Check: $h(1.1315) = -0.00063$ and $h(1.1325) = 0.00019$, a sign change.)

## Example 3




The diagram shows a sector $OAB$ of a circle with centre $O$ and radius $r$. The angle $AOB$ is $x$ radians. The area of the shaded segment is one quarter of the area of triangle $OAB$.

(a) Show that $x = \frac{5}{4}\sin x$. **[3]**

(b) Show by calculation that the root of this equation lies between $1$ and $1.2$. **[2]**

(c) Use an iterative formula based on the equation in part (a), with $x_1 = 1.1$, to calculate $x$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. **[3]**

**(a)** Triangle $OAB = \frac{1}{2}r^2 \sin x$; sector $= \frac{1}{2}r^2 x$; segment $= \frac{1}{2}r^2 x - \frac{1}{2}r^2 \sin x$. **B1**

$\frac{1}{2}r^2(x - \sin x) = \frac{1}{4} \times \frac{1}{2}r^2 \sin x$. **M1** for forming the equation.

Cancel $\frac{1}{2}r^2$: $x - \sin x = \frac{1}{4}\sin x$, so $x = \frac{5}{4}\sin x$. **A1** (given answer).

**(b)** Let $g(x) = x - \frac{5}{4}\sin x$. $g(1) = -0.0518$ and $g(1.2) = 0.0350$. Sign change, so the root is between $1$ and $1.2$. **M1 A1**

**(c)** $x_{n+1} = \frac{5}{4}\sin x_n$: $x_2 = 1.1140$, $x_3 = 1.1218$, $x_4 = 1.1261$, $x_5 = 1.1284$, $x_6 = 1.1297$. **M1**

$x_5$ and $x_6$ both round to $1.13$. The values are still creeping up, so check: $g(1.125) = -0.0028$ and $g(1.135) = 0.0018$, a sign change, so $x = 1.13$ (2 d.p.). **A1 A1**

## Example 4

The constant $a$, where $a > 1$, is such that $\displaystyle\int_1^a \frac{\ln x}{x^2}\,dx = \frac{1}{2}$.

(a) Show that $a = 2(1 + \ln a)$. **[5]**

(b) Verify by calculation that $a$ lies between $5$ and $6$. **[2]**

(c) Use an iterative formula based on the equation in part (a) to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. **[3]**

**(a)** By parts with $u = \ln x$ and $\frac{dv}{dx} = x^{-2}$, so $v = -x^{-1}$: **M1**

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x}.$$

**A1** for the first step, **A1** for the complete integral.

Limits: $\left(-\frac{\ln a}{a} - \frac{1}{a}\right) - (0 - 1) = \frac{1}{2}$. **DM1** for using both limits and equating to $\frac{1}{2}$.

So $\frac{\ln a + 1}{a} = \frac{1}{2}$, giving $a = 2(1 + \ln a)$. **A1** (given answer).

**(b)** Let $k(a) = a - 2(1 + \ln a)$. $k(5) = -0.219$ and $k(6) = 0.416$. Sign change, so $5 < a < 6$. **M1 A1**

**(c)** $a_{n+1} = 2(1 + \ln a_n)$, $a_1 = 5$: $5.2189$, $5.3046$, $5.3371$, $5.3494$, $5.3540$, $5.3557$, $5.3563$, $5.3566$. **M1**

**Careful:** $5.3494$ and $5.3540$ both round to $5.35$, but the values are still rising. Carrying on, $5.3557$, $5.3563$ and $5.3566$ all round to $5.36$, and the steps are now too small to reach $5.365$. A sign check confirms it: $k(5.355) = -0.0011$ and $k(5.365) = 0.0052$. So $a = 5.36$ (2 d.p.). **A1 A1** ($k(5.345) = -0.0073$ is also negative, which shows $5.35$ is wrong.)

## Common mistakes

- **Calculator in degrees.** Examiner reports mention it almost every year: with any trigonometric function, degree mode gives wrong values, and mark schemes give no marks for that part. Set radians before part (b).
- **Values with no comparison or conclusion.** Writing two numbers is not enough. Say what you compared them with, and conclude: "change of sign, so the root lies between ...".
- **Mixing the two methods in "show by calculation".** If you compare $x$ with $F(x)$, look for the inequality turning round, not a sign change. If you use $x - F(x)$ or "everything on one side", look for a sign change.
- **Stopping too early**, especially when the values creep one way. Keep going until two consecutive values round to the same answer **and** are no longer drifting, or confirm with a sign change.
- **Rounding the final answer wrongly**: for example $0.80$ instead of $0.79$, or giving 3 decimal places when 2 were asked. Round correctly, to exactly the accuracy asked.
- **Iterates to too few places.** Every value must be shown to the stated number of decimal places, including the last two.
- **Ignoring a given starting value, or starting outside the interval.** If the question gives $x_1$, use it. If not, start inside the interval from the earlier part.
- **Using the calculator's equation solver.** A "starting value" that is already the answer to many decimal places earns nothing; the method must be seen.

- **Plotting instead of sketching.** A sketch shows the whole shape over the whole interval: asymptotes (for example $y = 1$ for $1 + e^{-x}$ and the $y$-axis for $\ln x$), axis crossings in the right places, and the right curvature. Joining a few plotted points misses these.
- **No statement after the sketch.** Mark the crossing and say how many roots there are.
- **Answering "if it converges, it converges to the root" with numbers.** No iteration is needed: replace $x_{n+1}$ and $x_n$ by one letter and rearrange to the equation. Working from the formula towards the equation is usually easiest.
- **Keeping the subscripts** in that argument. Once you say the limit is $\alpha$, write $\alpha = F(\alpha)$, not $x_{n+1} = F(x_n)$.

## How to get the marks

- **The iteration is usually 3 marks:** one for using the formula correctly at least once (in radians), one for the final answer to the stated accuracy, and one for enough iterations shown to the stated number of places, **or** a sign change over the interval that rounds to your answer, such as $[0.855, 0.865]$ for $0.86$.
- **"Show by calculation" or "verify by calculation"** is 2 marks: correct values at both ends (2 significant figures or better), then a clear conclusion.
- **"By sketching a suitable pair of graphs"** is 2 marks: one per correct graph with its key features, and the second needs the crossing marked and the number of roots stated.
- **"Show that if ... converges, then it converges to ..."** is 1 or 2 marks: write $\alpha = F(\alpha)$, then rearrange fully to the printed equation.
- **"Show that $a$ satisfies ..."** is a given answer. Every step must be there, ending in exactly the printed form and letter; a slip that you "fix" to reach it loses the last mark.
- **"Hence" or "based on the equation in part (a)"** means use that equation to build the formula.
- **Accuracy words:** "correct to 2 decimal places" applies to the answer; "give the result of each iteration to 4 decimal places" applies to every iterate. Give the answer with the question's own letter (for example "$p = 0.35$" when the root is $p$) or just the number "$0.35$"; a different letter, or an iterate label such as $p_7$, can lose the mark.

## Quick check

1. The equation $\cos x = x^2$ has one positive root, $\alpha$. Show by calculation that $\alpha$ lies between $0.8$ and $0.9$.
2. Show that if the sequence given by $x_{n+1} = \sqrt{\cos x_n}$ converges, it converges to $\alpha$. Use it with $x_1 = 0.8$ to find $\alpha$ correct to 3 decimal places, giving each iteration to 5 decimal places.
3. The same equation can be written $x = \cos^{-1}(x^2)$. Find $x_2$, $x_3$, $x_4$ and $x_5$ to 4 decimal places for $x_{n+1} = \cos^{-1}(x_n^2)$ with $x_1 = 0.82$, and say what happens.
4. A student uses $x_{n+1} = \sqrt{\cos x_n}$ with $x_1 = 0.8$ and gets $x_2 = 0.99995$. What has gone wrong?
5. By sketching $y = \operatorname{cosec} x$ and $y = x$ for $0 < x < \pi$, find how many roots the equation $\operatorname{cosec} x = x$ has in that interval.

## Answers

1. $f(x) = \cos x - x^2$: $f(0.8) = 0.0567$ and $f(0.9) = -0.188$. The sign changes, so $\alpha$ lies between $0.8$ and $0.9$.
2. If the values converge to $\alpha$, then $\alpha = \sqrt{\cos \alpha}$, so $\alpha^2 = \cos \alpha$. Iterating: $0.83469$, $0.81939$, $0.82623$, $0.82319$, $0.82455$, $0.82395$, $0.82422$. The values alternate and the last two both round to $0.824$, so $\alpha = 0.824$ (3 d.p.).
3. $x_2 = 0.8333$, $x_3 = 0.8031$, $x_4 = 0.8698$, $x_5 = 0.7128$. The values move further from $\alpha = 0.824$ each time: this iteration does not converge.
4. The calculator is in degrees. In radians, $x_2 = \sqrt{\cos 0.8} = 0.83469$.
5. $y = \operatorname{cosec} x$ comes down from very large values near $0$ to its lowest point $\left(\frac{1}{2}\pi, 1\right)$ and then rises towards the asymptote $x = \pi$. The line $y = x$ goes from $0$ to $\pi$. At $\frac{1}{2}\pi$ the curve (1) is below the line (1.57), but near both ends it is above, so the graphs cross **twice**: two roots, one in $0 < x < \frac{1}{2}\pi$ and one in $\frac{1}{2}\pi < x < \pi$.

## Past questions to try

- **9709/32/M/J/22 Q5**: sketch a log curve and a quadratic, a sign change, then iterate.
- **9709/31/M/J/22 Q10**: the equation comes from a stationary point, then why the formula converges to it.
- **9709/32/O/N/22 Q9**: the equation comes from a segment in a semicircle.
- **9709/33/M/J/22 Q10**: the equation comes from a definite integral by parts.