Cambridge AS & A Level · Mathematics 9709 · Notes

Probability AS

Read online at **pastopic.com/cambridge/mathematics-9709/notes/probability**

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Read first: **Permutations and combinations**

What you need to know

By the end of this topic you should be able to:

1. **Find probabilities by counting equally likely outcomes.** List or draw every outcome (for example the 36 results of throwing two dice) and count the ones you want, or count them with permutations and combinations, for example when several balls are taken from a bag.
2. **Add and multiply probabilities correctly.** Add for "or" when the events cannot happen together; multiply for "and", along the branches of a tree diagram.
3. **Know what exclusive and independent events are**, and decide whether two events A and B are independent by comparing $\mathrm{P}(A \cap B)$ with $\mathrm{P}(A) \times \mathrm{P}(B)$.
4. **Find and use conditional probabilities**, such as "the probability that the first ball was red, given that the second ball is blue", from a list of outcomes, a table, a Venn diagram or a tree diagram, using $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ where needed.

Understand it

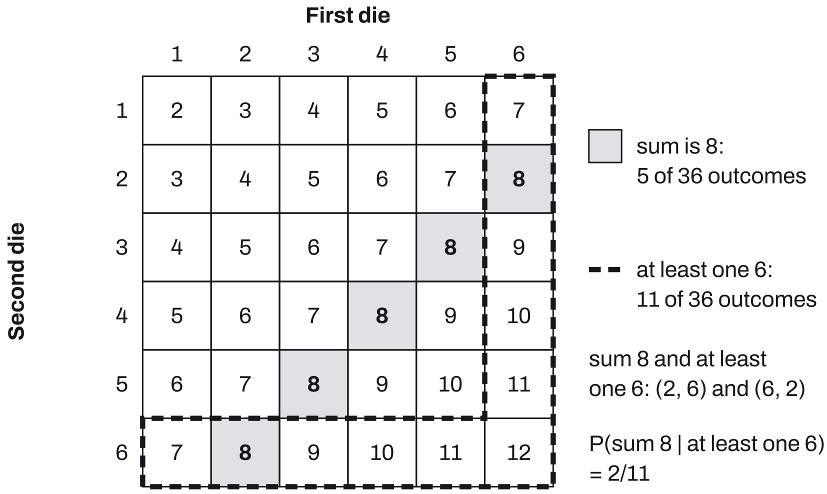
Equally likely outcomes

When every outcome is equally likely,

$$\mathrm{P}(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}}.$$

A probability is always between 0 (impossible) and 1 (certain), and the probabilities of all the possible outcomes add up to 1.

For two dice, list every outcome in a grid (a **sample space diagram**). There are $6 \times 6 = 36$ outcomes, all equally likely. The sum is 8 in 5 of them, so $\mathrm{P}(\text{sum is } 8) = \frac{5}{36}$.



For bigger situations you count instead of listing. If 3 balls are taken from a bag of 4 red, 3 green and 2 yellow, there are ${}^9C_3 = 84$ equally likely selections, and ${}^4C_3 = 4$ of them are all red, so $\mathrm{P}(\text{all red}) = \frac{4}{84} = \frac{1}{21}$. The counting methods are in the note on permutations and combinations.

"Not": the complement

A' means "A does not happen". Since one of them must happen, $\mathrm{P}(A') = 1 - \mathrm{P}(A)$.

This is the quick way to find "**at least one**": the opposite of "at least one six" is "no sixes". With three dice, $\mathrm{P}(\text{no sixes}) = \left(\frac{5}{6}\right)^3 = \frac{125}{216}$, so $\mathrm{P}(\text{at least one six}) = 1 - \frac{125}{216} = \frac{91}{216}$.

"Or": exclusive events

Two events are **mutually exclusive** (or just **exclusive**) if they cannot both happen at once. Then

$$\mathrm{P}(A \text{ or } B) = \mathrm{P}(A) + \mathrm{P}(B).$$

With two dice, "sum is 2" and "sum is 12" are exclusive, so $\mathrm{P}(\text{sum is } 2 \text{ or } 12) = \frac{1}{36} + \frac{1}{36} = \frac{1}{18}$.

If A and B **can** happen together, adding counts the overlap twice. Then it is safest to add up separate, non-overlapping pieces instead, for example the regions of a Venn diagram. (The rule $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ does this in one line, but you are never forced to use it.)

"And": independent events

$A \cap B$ means "A and B both happen". Two events are **independent** if one happening does not change the probability of the other, for example a coin and a die. Then

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B).$$

So $\mathrm{P}(\text{head and a six}) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$. Repeated throws of a coin or a die are independent, and so are draws **with replacement**.

To test for independence, work out $\mathrm{P}(A)$, $\mathrm{P}(B)$ and $\mathrm{P}(A \cap B)$ separately, then compare:

- $\mathrm{P}(A) \times \mathrm{P}(B) = \mathrm{P}(A \cap B)$: independent;
- $\mathrm{P}(A) \times \mathrm{P}(B) \neq \mathrm{P}(A \cap B)$: not independent.

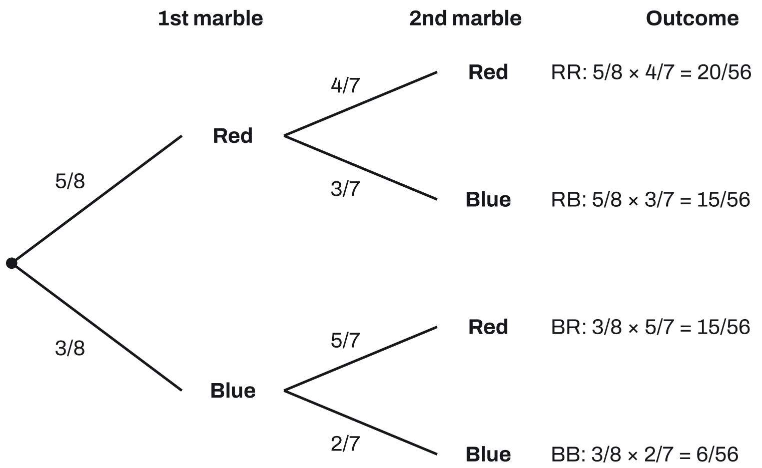
Exclusive and independent are different ideas. Exclusive events cannot happen together, so $\mathrm{P}(A \cap B) = 0$. If $\mathrm{P}(A)$ and $\mathrm{P}(B)$ are both more than 0, their product is more than 0, so **exclusive events like this are never independent**: knowing A happened tells you B did not.

Tree diagrams

When things happen in stages, draw a **tree diagram**. Each set of branches from one point shows the outcomes of one stage, and their probabilities add up to 1.

- **Multiply along a path** to get the probability of that whole route ("this, then that").
- **Add the paths** that give the result you want ("this route or that route").

Without replacement, the second-stage probabilities depend on what happened first. For a bag of 5 red and 3 blue marbles, taking two without replacement:



$$\mathrm{P}(\text{same colour}) = \mathrm{P}(RR) + \mathrm{P}(BB) = \frac{20}{56} + \frac{6}{56} = \frac{26}{56} = \frac{13}{28}.$$

The branch probabilities in the second stage are themselves conditional: $\frac{4}{7}$ is $\mathrm{P}(2\text{nd red} \mid 1\text{st red})$. So multiplying along a path uses

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B \mid A),$$

which is true for any events, independent or not. Check: the four end probabilities must add up to 1.

Conditional probability

$\mathrm{P}(A \mid B)$, read "the probability of A given B", is the probability of A when you already know that B has happened. Knowing B shrinks the outcomes to only those in B:

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

In the dice grid, "given at least one six" leaves the 11 outlined outcomes. Only 2 of them have sum 8, so $\mathrm{P}(\text{sum } 8 \mid \text{at least one six}) = \frac{2}{11}$. The formula gives the same: $\frac{2/36}{11/36} = \frac{2}{11}$.

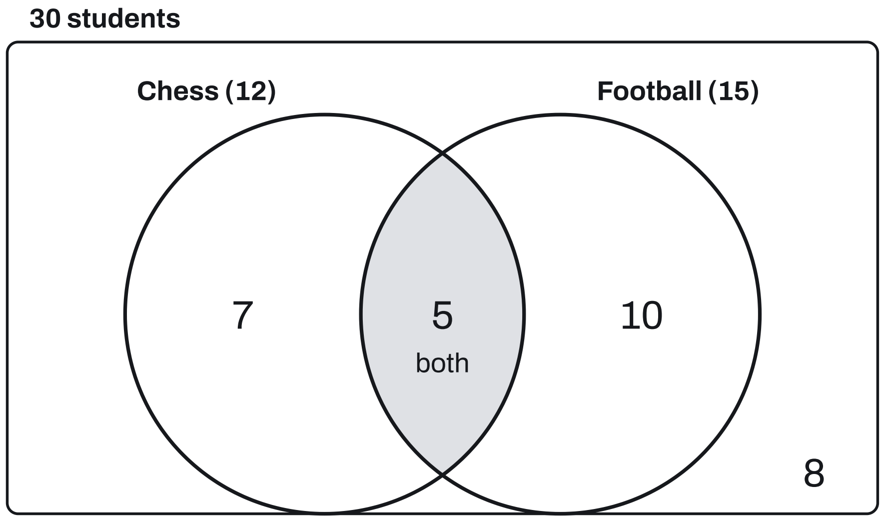
On the tree diagram, this lets you go **backwards**. What is the probability that the first marble was red, given that the second is blue? The second is blue on two paths, RB and BB:

$$\mathrm{P}(\text{2nd blue}) = \frac{15}{56} + \frac{6}{56} = \frac{21}{56}, \qquad \mathrm{P}(\text{1st red} \mid \text{2nd blue}) = \frac{\mathrm{P}(RB)}{\mathrm{P}(\text{2nd blue})} = \frac{15/56}{21/56} = \frac{5}{7}.$$

The numerator is the part of the denominator that also fits the "A" condition, so it can never be bigger than the denominator.

Venn diagrams and two-way tables

Counts of people in groups are often shown in a Venn diagram or a two-way table. In a class of 30 students, 12 play chess, 15 play football and 5 play both.



Fill the overlap first (5), then the rest of each circle ($12 - 5 = 7$ and $15 - 5 = 10$), then the outside ($30 - 7 - 5 - 10 = 8$).

- $\mathrm{P}(\text{chess or football}) = \frac{7+5+10}{30} = \frac{22}{30} = \frac{11}{15}$.
- $\mathrm{P}(\text{chess} \mid \text{football}) = \frac{5}{15} = \frac{1}{3}$: only look inside the football circle.
- $\mathrm{P}(\text{football} \mid \text{chess}) = \frac{5}{12}$: the order matters.
- Independent? $\mathrm{P}(C) \times \mathrm{P}(F) = \frac{12}{30} \times \frac{15}{30} = 0.2$, but $\mathrm{P}(C \cap F) = \frac{5}{30} = \frac{1}{6}$. They are different, so the events are **not** independent.

A two-way table works the same way: "given" means you use only that row or column, and its total goes on the bottom of the fraction.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
Equally likely outcomes: $\mathrm{P}(A)=\dfrac{\text{outcomes in } A}{\text{all outcomes}}$	Learn it
$\mathrm{P}(A')=1-\mathrm{P}(A); \mathrm{P}(\text{at least one})=1-\mathrm{P}(\text{none})$	Learn it
Exclusive events: $\mathrm{P}(A \text{ or } B)=\mathrm{P}(A)+\mathrm{P}(B)$	Learn it
Independent events: $\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)$	Learn it
Any events: $\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B\mid A)$ (multiply along a tree)	Learn it
Conditional probability: $\mathrm{P}(A\mid B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$	Learn it
Exclusive with $\mathrm{P}(A), \mathrm{P}(B)>0$: never independent	Learn it
${}^nC_r=\dfrac{n!}{r!\,(n-r)!}$	Given

Notation: A' is "not A", $A\cap B$ is "A and B", $A\cup B$ is "A or B (or both)", $A\mid B$ is "A given B".

How to do it

In the Paper 5 questions we have tagged for this topic (40 questions, 2021–2025), the types came up this often. 31 of the 40 questions mix two or more types (a tree diagram, then a probability from adding paths, then a conditional probability is the usual chain), so the counts add up to more than 40.

Question type	Questions
Conditional probability $\mathrm{P}(A\mid B)$	26
Adding the products along several paths	21
Drawing or completing a tree diagram	11
Testing whether two events are independent	6
Finding an unknown number or probability	6
Repeated trials and sequences	5
Listing equally likely outcomes	5
Two-way tables	4

1. Conditional probability (26 questions)

1. **Name the two events.** Write the question as $\mathrm{P}(A\mid B)$: the part after "given that" is B, and it goes on the **bottom**.
2. **Find the denominator,** $\mathrm{P}(B)$: add every route (branch path, table cell, list entry) where B happens. It is often an earlier part's answer, or $1-$ an earlier answer, for example $\mathrm{P}(\text{at least one red})=1-\mathrm{P}(\text{both blue})$.

3. **Find the numerator,** $\mathrm{P}(A \cap B)$: add only those routes from step 2 where A happens as well.
4. Divide. Give the answer as a fraction or to 3 significant figures, and check it is between 0 and 1.

Variations you will meet:

- **Going backwards on a tree**: "the first marble was red, given that the second is blue", or "bag Y was chosen, given that both discs are red". The given event is at the **end** of the tree, so the denominator adds every path ending that way.
- **Table or Venn diagram**: use only the row, column or circle you are given; its total is the denominator.
- **Given a "not" event**, such as "given that he is not late" or "given A'": the denominator is $1 - \mathrm{P}(\text{late})$, or $1 - \mathrm{P}(A)$.
- **Several routes on top**, such as "given she finishes, she failed exactly one attempt": list every route that finishes, then pick out the ones with exactly one fail.
- **Given a condition on a random variable** (from a probability distribution table): add the table's probabilities for the given values on the bottom, and the ones that also meet A on top.

2. Adding the products along several paths (21 questions)

1. **List every route** that gives the result you want, as a string of outcomes, such as HBB and TBB for "both blue".
2. **Multiply along each route**, using the branch probabilities.
3. **Add the routes.** For "at least one", $1 - \mathrm{P}(\text{none})$ is often quicker.
4. Check that you have no route twice and none missing: the routes you did not use make up the opposite event.

Variations you will meet:

- **A coin, die or spinner picks the bag first**: every route starts with its probability, such as $\frac{1}{2} \times \ldots$.
- **Outcomes in any order**, such as "one red and one blue": RB and BR are different routes, and both count. For three items, "one of each colour" has $3! = 6$ orders and "two the same and one different" has 3.
- **Several items taken without replacement from one bag.** Either use combinations, or multiply fractions in one order and then multiply by the number of orders. For 4 books taken at random from a shelf of 3 maths, 4 science and 5 history books, there are ${}^{12}C_4 = 495$ selections, and $\mathrm{P}(2 \text{ maths and } 2 \text{ history}) = \frac{{}^3C_2 \times {}^5C_2}{495} = \frac{30}{495} = \frac{2}{33}$. In order: $\frac{3}{12} \times \frac{2}{11} \times \frac{5}{10} \times \frac{4}{9} \times 6 = \frac{2}{33}$, since $MMHH$ can be arranged in $\frac{4!}{2!\,2!} = 6$ ways.
- **"At least one of each"**: sort the selections by how many of each kind. For at least one book of each subject, one subject gives 2 books and the other two give 1 each, so count each case: ${}^3C_2 \times 4 \times 5 = 60$, $3 \times {}^4C_2 \times 5 = 90$ and $3 \times 4 \times {}^5C_2 = 120$. That is $\frac{270}{495} = \frac{6}{11}$.
- **"Show that"** parts: write every product, then the total.

3. Drawing or completing a tree diagram (11 questions)

1. **Draw the tree** in the order things happen: the coin or die that picks a bag first, then the first item, then the second. Label every branch with its outcome and its probability.
2. **Work out the bag for each second-stage branch.** Without replacement, the total on the bottom goes down by one on **every** branch, but only the colour that was taken has one fewer on top; the other colours keep their counts. After a red from 5 red and 3 blue, it is red $\frac{4}{7}$ and blue $\frac{3}{7}$. If an item is **added** to a bag (moved in from another bag, or a new one put in), that bag's total and the count of that item's colour both go **up** by one.
3. Each set of branches from one point adds to 1, and all the path ends add to 1. Use this to check your tree.

Variations you will meet:

- **"Draw a tree diagram"** is often worth 2 or 3 marks on its own: every branch needs a label and a probability. A branch that is certain has probability 1, and a branch that cannot happen can be left out or shown with 0; any other extra branch loses marks.
- **Different rules in different branches**: with replacement from one bag and without from the other, or "if blue, replace it; if red, keep it out". Work out each branch's bag contents separately.
- **Stages that stop early**, such as "at most three attempts" or "a second test only if the first is failed". A route ends as soon as the stopping outcome happens.
- **Three tiers**, such as a counter that picks the box, then two marbles from that box.
- **Probabilities in terms of x**, such as $\frac{4}{5+x}$ when a bag has x yellow balls: they go on the branches like numbers.

4. Testing whether two events are independent (6 questions)

1. **Define** each event in words if it has no letter.
2. **Find $\mathrm{P}(A)$, $\mathrm{P}(B)$ and $\mathrm{P}(A \cap B)$ separately**, from the table, list or diagram. $\mathrm{P}(A \cap B)$ must be found directly, not by multiplying.
3. **Compare** $\mathrm{P}(A) \times \mathrm{P}(B)$ with $\mathrm{P}(A \cap B)$, showing both numbers.
4. **Conclude** in words: "equal, so independent" or "not equal, so not independent".

Variations you will meet:

- **Events from two dice or spinners**: use the grid and count each event, then the overlap (see Example 1).
- **Events from arrangements**, such as marbles placed at random in boxes: count the arrangements for each event and for both together.
- **"Are the events exclusive?"** Check whether they can happen together: exclusive only if $\mathrm{P}(A \cap B) = 0$.

5. Finding an unknown number or probability (6 questions)

1. Put the unknown (x) on the tree, in the table or in the bag contents, like any other number.
2. Write the probability you are told in terms of x: add the right paths, exactly as if you were finding it.

3. Set it equal to the given value and solve. It is usually a linear equation; with x items in a bag you may get a quadratic.
4. Reject values that are impossible, such as a probability outside 0 to 1 or a number of items that is not a positive whole number.

For example, a bag holds 5 red counters and n blue ones. Two are taken with replacement, and the probability that they are different colours is $\frac{4}{9}$. Then $2 \times \frac{5}{n+5} \times \frac{n}{n+5} = \frac{4}{9}$ (the 2 is for RB and BR), so $90n = 4(n+5)^2$, which gives $2n^2 - 25n + 50 = 0$ and $(2n-5)(n-10) = 0$. A number of counters must be a whole number, so $n = 10$.

Variations you will meet:

- **The given probability is for "not"**, such as "not late 0.48": either use $\mathrm{P}(\text{late}) = 1 - 0.48$ or add the "not late" paths.
- **One probability is a multiple of another**, such as "P(both red) is twice P(both blue)": write both in terms of x and form the equation.
- **Probabilities in a table that must add to** 1: form the sum and solve.
- **Independent events with an unknown**: $\mathrm{P}(\text{neither}) = (1 - \mathrm{P}(A))(1 - \mathrm{P}(B))$ gives a quadratic (see Example 3).

6. Repeated trials and sequences (5 questions)

The same experiment is repeated: throws, turns, days or independent people, each with the same probability p of "success".

1. **One exact sequence** (such as fail, fail, success): multiply, $(1-p)(1-p)p$.
2. **At least one success in n trials**: $1 - (1-p)^n$. **All n fail**: $(1-p)^n$.
3. **First success on the nth trial**: $(1-p)^{n-1}p$. "First success **before** the nth trial" means at least one success in the first $n-1$ trials: $1 - (1-p)^{n-1}$.
4. **Different orders**: if the outcomes you want can come in several orders, multiply one order's probability by the number of orders.
5. **Exactly r successes in n trials**: the r successes can be in ${}^nC_r$ different places, so the probability is ${}^nC_r\, p^r (1-p)^{n-r}$. For "**fewer than** k", add the cases $0, 1, \ldots, k-1$, or take the rest away from 1.

For example, each of the 4 smoke alarms in a house works with probability 0.9, independently. All four work with probability $0.9^4 = 0.6561$, so at least one fails with probability 0.3439. Exactly three work with probability $4 \times 0.9^3 \times 0.1 = 0.2916$, so fewer than three work with probability $1 - 0.6561 - 0.2916 = 0.0523$.

A caller gets through to a busy helpline with probability 0.25 on each call, independently. The first call to get through is the 3rd with probability $0.75^2 \times 0.25 = 0.141$ (3 s.f.), and at least one of the first 4 calls gets through with probability $1 - 0.75^4 = 0.684$. The **second** call to get through is the 4th when exactly one of the first three gets through (in any of 3 places) and then the 4th does: $3 \times 0.25 \times 0.75^2 \times 0.25 = 0.105$.

Variations you will meet:

- **A probability from an earlier part becomes p**: if each of three people independently passes with the probability you found, the probability that none of them passes is $(1 - \mathrm{P}(\text{pass}))^3$.
- **Two players**, such as "player A scores 2 more points than player B": list the score pairs that work, find each pair's probability as a product, and add.
- **Each day depends on the day before** ("0.7 if it rained yesterday, 0.2 if not"): draw a tree over the days and multiply along the chain.

7. Listing equally likely outcomes (5 questions)

1. **List or draw every outcome** in a grid (two dice or spinners) or a systematic list (three dice).
2. Count the outcomes in the event, then divide by the total.
3. If one spinner is **biased**, outcomes are not equally likely: find each outcome's probability as a product and add.

Variations you will meet:

- **Three dice with a small total**: list the ordered triples. With three fair four-sided dice, a total of 5 comes from $(1, 1, 3)$ and $(1, 2, 2)$ in 3 orders each, so the probability is $\frac{6}{64} = \frac{3}{32}$.
- **A score defined by a rule** (a product, or "higher minus lower"): make a grid of scores, then count.
- **Conditional from the grid**: count only inside the given event, as in the dice diagram above.

8. Two-way tables (4 questions)

1. Add the row and column totals if they are not given, and the grand total.
2. "$\mathrm{P}(A \text{ and } B)$" uses one cell over the **grand total**.
3. "$\mathrm{P}(A \mid B)$" uses one cell over B**'s row or column total**.
4. For independence, compare the cell over the grand total with the product of the two totals over the grand total, as in section 4.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

A fair spinner has five sectors numbered 1 to 5, and a fair four-sided die has faces numbered 1 to 4. The spinner is spun and the die is thrown, and the two numbers are multiplied. The events A and B are:

- A: the product is even;
- B: the spinner's number is greater than the die's number.

(a) Show that $\mathrm{P}(A) = \frac{7}{10}$. **[2]**

(b) Determine whether A and B are independent. **[4]**

(c) The event C is "the spinner and the die show the same number". State, with a reason, whether B and C are exclusive, and whether they are independent. **[2]**

(d) Find the probability that the spinner shows 5, given that the product is even. **[2]**

There are $5 \times 4 = 20$ equally likely outcomes; list them in a grid with the spinner across and the die down.

(a) The product is odd only when both numbers are odd: 3 odd spinner numbers and 2 odd die numbers give $3 \times 2 = 6$ outcomes. **M1**

$\mathrm{P}(A) = 1 - \frac{6}{20} = \frac{14}{20} = \frac{7}{10}$. **A1**

(b) B: die 1 with spinner 2 to 5 (4), die 2 (3), die 3 (2), die 4 (1): 10 outcomes, $\mathrm{P}(B) = \frac{10}{20} = \frac{1}{2}$. **B1**

$A \cap B$: $(2,1)$, $(4,1)$, $(3,2)$, $(4,2)$, $(5,2)$, $(4,3)$, $(5,4)$, written (spinner, die): 7 outcomes, so $\mathrm{P}(A \cap B) = \frac{7}{20}$. **B1**

$\mathrm{P}(A) \times \mathrm{P}(B) = \frac{7}{10} \times \frac{1}{2} = \frac{7}{20}$. **M1** for the product compared with $\mathrm{P}(A \cap B)$.

They are equal, so A and B **are independent**. **A1**

(c) If the two numbers are the same, the spinner's number cannot be greater, so $\mathrm{P}(B \cap C) = 0$: B and C **are exclusive**. **B1**

$\mathrm{P}(C) = \frac{4}{20}$, so $\mathrm{P}(B) \times \mathrm{P}(C) = \frac{1}{2} \times \frac{1}{5} = \frac{1}{10} \neq 0$: **not independent**. **B1**

(d) Of the 14 outcomes with an even product, the spinner shows 5 only with die 2 or 4: 2 outcomes. **M1** for counting inside A only.

$$\mathrm{P}(\text{spinner } 5 \mid A) = \frac{2}{14} = \frac{1}{7}.$$

A1

Example 2

Ten cards are numbered 1 to 10. Four cards are chosen at random, without replacement.

(a) Find the probability that all four numbers are even. **[2]**

(b) Find the probability that the largest number chosen is 8. **[2]**

(c) Find the probability that exactly two of the numbers are greater than 7. **[2]**

(d) Find the probability that card 10 is chosen, given that exactly two of the numbers are greater than 7. **[3]**

There are ${}^{10}C_4 = 210$ equally likely selections.

(a) All four from the 5 even cards: ${}^{5}C_4 = 5$. **M1**

$$\mathrm{P}(\text{all even}) = \frac{5}{210} = \frac{1}{42}.$$

A1. Check in order: $\frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} = \frac{120}{5040} = \frac{1}{42}$.

(b) Card 8 must be chosen, and the other three all come from 1 to 7: ${}^{7}C_3 = 35$. **M1**

$$\mathrm{P}(\text{largest is } 8) = \frac{35}{210} = \frac{1}{6}.$$

A1

(c) Two of the three cards $8, 9, 10$ and two of the seven cards 1 to 7: ${}^{3}C_2 \times {}^{7}C_2 = 3 \times 21 = 63$. **M1**

$$\mathrm{P}(\text{exactly two above } 7) = \frac{63}{210} = \frac{3}{10}.$$

A1

(d) Card 10 and exactly two above 7: card 10, one of 8 and 9, and two from 1 to 7: $1 \times 2 \times 21 = 42$ selections. **B1**

$$\mathrm{P}(10 \text{ chosen} \mid \text{exactly two above } 7) = \frac{42/210}{63/210} = \frac{42}{63} = \frac{2}{3}.$$

M1 for the fraction with (c) as the denominator, **A1**.

Example 3

Two machines, P and Q, on a production line break down independently of each other. In any week, the probability that P breaks down is x and the probability that Q breaks down is $x + 0.1$. The probability that neither machine breaks down in a week is 0.42.

(a) Find the value of x. **[4]**

(b) Find the probability that exactly one of the machines breaks down in a week. **[2]**

(c) Find the probability that it was machine P that broke down, given that exactly one machine broke down in a week. **[2]**

(d) Find the probability that, in 3 weeks, there is at least one week in which neither machine breaks down. **[2]**

(a) The machines are independent, so $\mathrm{P}(\text{neither}) = (1 - x)(1 - x - 0.1) = (1 - x)(0.9 - x)$. **B1**

$(1-x)(0.9-x)=0.42$, so $x^2-1.9x+0.9=0.42$ and $x^2-1.9x+0.48=0$. **M1** for forming and expanding the equation.

$(x-0.3)(x-1.6)=0$, so $x=0.3$ or $x=1.6$. **M1**

A probability cannot be more than 1, so $x=0.3$. **A1**

(b) $\mathrm{P}(P\text{ only})=0.3\times0.6=0.18$ and $\mathrm{P}(Q\text{ only})=0.7\times0.4=0.28$. **M1** for the two products added.

$\mathrm{P}(\text{exactly one})=0.18+0.28=0.46$. **A1**

(c)

$$\mathrm{P}(P\mid\text{exactly one})=\frac{0.18}{0.46}=\frac{9}{23}=0.391\ (3\text{ s.f.})$$

M1 for $\frac{0.18}{\text{their (b)}}$, **A1**.

(d) In one week, $\mathrm{P}(\text{not "neither"})=1-0.42=0.58$. None of the 3 weeks has neither breaking down with probability $0.58^3=0.195112$. **M1**

$\mathrm{P}(\text{at least one such week})=1-0.195112=0.805$ (3 s.f.). **A1**

Example 4

A spinner points to A, B or C with probabilities 0.5, 0.3 and 0.2. Bag A holds 3 red and 2 blue counters, bag B holds 1 red and 4 blue, and bag C holds 4 red and 1 blue. Lina spins the spinner and takes two counters at random, without replacement, from the bag it points to.

(a) Find the probability that both counters are blue. **[3]**

(b) Find the probability that exactly one of the counters is red. **[3]**

(c) Find the probability that the spinner pointed to A, given that at least one of the counters is red. **[3]**

Draw a tree: three branches for the spinner, then two stages for the counters from each bag. Each bag has 5 counters, then 4.

(a) Both blue from A: $\frac{2}{5}\times\frac{1}{4}=\frac{1}{10}$. From B: $\frac{4}{5}\times\frac{3}{4}=\frac{3}{5}$. From C there is only one blue counter, so this is impossible. **M1** for one correct product for a bag.

$$\mathrm{P}(\text{both blue})=0.5\times\frac{1}{10}+0.3\times\frac{3}{5}+0.2\times0=0.05+0.18=0.23.$$

M1 for adding the paths, each multiplied by its spinner probability. **A1**

(b) Exactly one red can be RB or BR. From A: $\frac{3}{5}\times\frac{2}{4}+\frac{2}{5}\times\frac{3}{4}=\frac{3}{5}$. From B: $\frac{1}{5}\times\frac{4}{4}+\frac{4}{5}\times\frac{1}{4}=\frac{2}{5}$. From C: $\frac{4}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{4}{4}=\frac{2}{5}$. **M1** for both orders in one bag.

$$\mathrm{P}(\text{exactly one red})=0.5\times\frac{3}{5}+0.3\times\frac{2}{5}+0.2\times\frac{2}{5}=0.3+0.12+0.08=0.5.$$

M1 A1

(c) "At least one red" is the opposite of "both blue": $\mathrm{P}(\text{at least one red}) = 1 - 0.23 = 0.77$. **B1**

Spinner on A and at least one red: $0.5 \times \left(1 - \frac{1}{10}\right) = 0.45$.

$$\mathrm{P}(A \mid \text{at least one red}) = \frac{0.45}{0.77} = \frac{45}{77} = 0.584 \text{ (3 s.f.)}$$

M1 for the conditional probability fraction with the right denominator, **A1**.

Common mistakes

- **Finding $\mathrm{P}(A \cap B)$ and stopping.** "Given that" always means dividing by the probability of the given event. Examiners see many answers that just give the numerator.
- **Turning the condition round.** $\mathrm{P}(\text{late} \mid \text{bus})$ and $\mathrm{P}(\text{bus} \mid \text{late})$ are different. The event after "given" goes on the bottom.
- **Forgetting the different orders.** Two white and one black can come out as WWB, WBW or BWW. Using fractions in one order, multiply by the number of orders; with combinations the orders are already counted.
- **Leaving out routes.** List the routes systematically (or mark them on the tree) and check the ones you did not use make up the rest.
- **Using "with replacement" numbers for a "without replacement" question**, or not changing the bag after a marble is moved or added. Rewrite the bag's contents on each branch.
- **Multiplying two alternative routes.** Different routes to the same result are added; only the stages along one route are multiplied.
- **Misreading "always" and "never".** "When he takes the train he is always on time" gives a branch with probability 1 (and its other branch 0), not a number borrowed from the next sentence.
- **An independence answer with no numbers or no notation.** State what you compare ($\mathrm{P}(A) \times \mathrm{P}(B)$ with $\mathrm{P}(A \cap B)$), give both values and write a conclusion. Say what A and B stand for.
- **Mixing up exclusive and independent.** Exclusive means they cannot both happen; independent means one does not affect the other.
- **Rounding too early, or too far.** Keep fractions exact until the last line, and give decimals to 3 significant figures, not 2. Don't turn an exact fraction into a rounded decimal unless asked.
- **A total that is not 1.** If the ends of your tree or your probability table do not add to 1, a branch is wrong.

How to get the marks

- **Show each product before you add.** A method mark is often for one correct path written out, such as $\frac{1}{3} \times \frac{5}{8} \times \frac{4}{7}$, and another for adding the right paths. Writing only the final number risks losing both if it is wrong.
- **Name what each product is**, such as $\mathrm{P}(HBB)$, or mark it on the tree, so that follow-through marks can be given.

- **Conditional probability:** write the fraction with the numerator and denominator both visible. The denominator can come from an earlier part ("their (b)"), and method marks follow through from a wrong earlier answer.
- **"Draw a tree diagram"** earns marks for the right structure and labels (every branch named) as well as the probabilities.
- **"Show that"** means the answer is given: write the full calculation that reaches it; working back from the answer earns nothing.
- **"Determine whether … independent"** needs three numbers, a comparison and a conclusion.
- **Accuracy:** exact fractions are always fine; otherwise give 3 significant figures (for example 0.105, 0.584). Work with at least 4 significant figures before the final rounding.
- **Answers must be probabilities:** a value above 1 or below 0 shows an error, and some method marks need your value to be between 0 and 1.

Quick check

1. A whole number is chosen at random from 1 to 30. A is "the number is a multiple of 3" and B is "the number is a multiple of 5". Determine whether A and B are independent, and find $\mathrm{P}(A \mid B)$.
2. A and B are independent events with $\mathrm{P}(A) = 0.4$ and $\mathrm{P}(B) = 0.5$. Find $\mathrm{P}(A \cap B)$ and the probability that neither happens. Are A and B exclusive?
3. A player scores each free throw with probability 0.7, independently. Find the probability that she (a) misses her first two throws and scores with her third, (b) scores at least once in four throws.
4. In a group of 40 students, 18 study French, 22 study Spanish and 6 study both. Find the probability that a student chosen at random studies neither, and the probability that a student studies French, given that they study Spanish.
5. A box holds 5 batteries, of which 2 work and 3 are flat. Batteries are tested one at a time, in a random order, until both working batteries have been found. Find the probability that exactly 3 batteries are tested, and the probability that the first battery tested works, given that exactly 3 are tested.

Answers

1. A has 10 numbers, B has 6, and $A \cap B$ (multiples of 15) has 2. $\mathrm{P}(A) \times \mathrm{P}(B) = \frac{10}{30} \times \frac{6}{30} = \frac{1}{15}$ and $\mathrm{P}(A \cap B) = \frac{2}{30} = \frac{1}{15}$: equal, so **independent**. $\mathrm{P}(A \mid B) = \frac{2}{6} = \frac{1}{3}$.
2. $\mathrm{P}(A \cap B) = 0.4 \times 0.5 = 0.2$. Neither: $0.6 \times 0.5 = 0.3$. Not exclusive, because $\mathrm{P}(A \cap B) = 0.2 \neq 0$.
3. (a) $0.3 \times 0.3 \times 0.7 = 0.063$. (b) $1 - 0.3^4 = 1 - 0.0081 = 0.9919$.
4. French only 12, both 6, Spanish only 16, neither $40 - 34 = 6$. $\mathrm{P}(\text{neither}) = \frac{6}{40} = 0.15$. $\mathrm{P}(\text{French} \mid \text{Spanish}) = \frac{6}{22} = \frac{3}{11}$.
5. Exactly one working battery in the first two tests (WF or FW), then a working one: $\left(\frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4}\right) \times \frac{1}{3} = \frac{12}{20} \times \frac{1}{3} = \frac{1}{5}$. First works and exactly 3 tested is WFW: $\frac{2}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{10}$. So $\mathrm{P}(\text{first works} \mid 3 \text{ tested}) = \frac{1/10}{1/5} = \frac{1}{2}$.

Past questions to try

- **9709/52/O/N/24 Q1**: a two-way table, conditional probability and a test for independence.
- **9709/51/O/N/24 Q4**: a biased coin chooses the bag; with and without replacement; going backwards with a condition.
- **9709/53/M/J/24 Q3**: a ball moved between boxes, then an unknown number of balls from a given probability.
- **9709/51/M/J/24 Q4**: repeated throws that stop early, "at least one", and two players' scores.