Cambridge AS & A Level · Mathematics (9709) · Notes

Trigonometry

AS

Read online at pastopic.com/cambridge/mathematics-9709/notes/trigonometry-p1

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Read first: Quadratics, Functions, Circular measure

What you need to know

By the end of this topic you should be able to:

1. **Sketch and use the graphs of $\sin x$, $\cos x$ and $\tan x$**, for angles of any size, in degrees or radians. This includes changed versions such as $y = 2\sin x$, $y = 1 + \cos 3x$ or $y = \tan\left(x - \frac{\pi}{6}\right)$: you should know their shape, where they cross the axes, their greatest and least values and how often they repeat.
2. **Use exact values.** Know the sine, cosine and tangent of 30°, 45° and 60° (and $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$) without a calculator, and use them for related angles such as 135°, 240° or $\frac{5\pi}{6}$.
3. **Use the inverse notation $\sin^{-1} x$, $\cos^{-1} x$** and $\tan^{-1} x$ for the *principal value*: the one angle your calculator gives. They are inverse functions, so they undo sine, cosine and tangent on a restricted range.
4. **Use the two identities** $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta \equiv 1$ to prove other identities, simplify expressions and solve equations.
5. **Find every solution of a trigonometric equation in a given interval**, for example $2\cos 3x = 1$ for $-\pi < x < \pi$, or an equation that becomes a quadratic in $\cos\theta$. General formulas for all solutions are not needed.

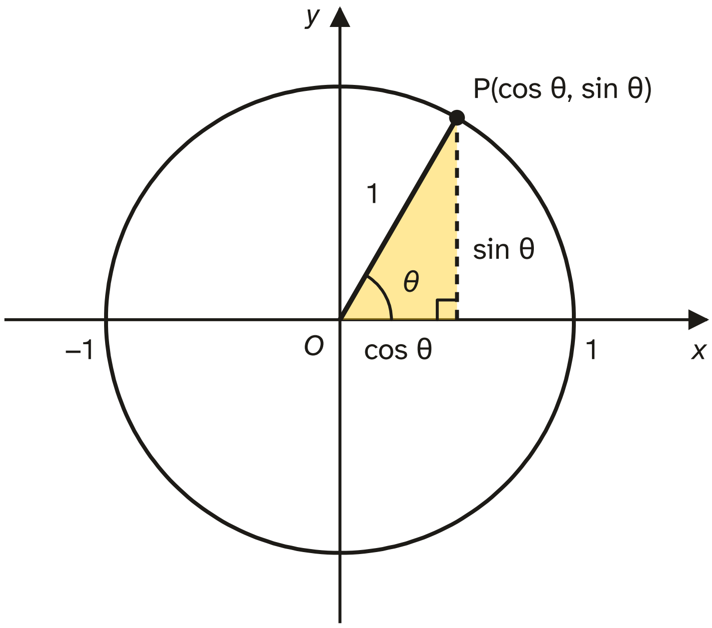
Secant, cosecant and cotangent, the compound-angle and double-angle formulas and $R\cos(\theta - \alpha)$ are **not** in this topic (they are in Paper 2 and Paper 3).

Understand it

Sine, cosine and tangent for any angle

In a right-angled triangle, $\sin$, $\cos$ and $\tan$ are ratios of sides. To make them work for *any* angle, use a circle of radius 1 centred at the origin. Start on the positive x-axis and turn anticlockwise through the angle θ to reach a point P on the circle. Then

- $\cos\theta$ is the x-coordinate of P,
- $\sin\theta$ is the y-coordinate of P,
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, the gradient of OP.



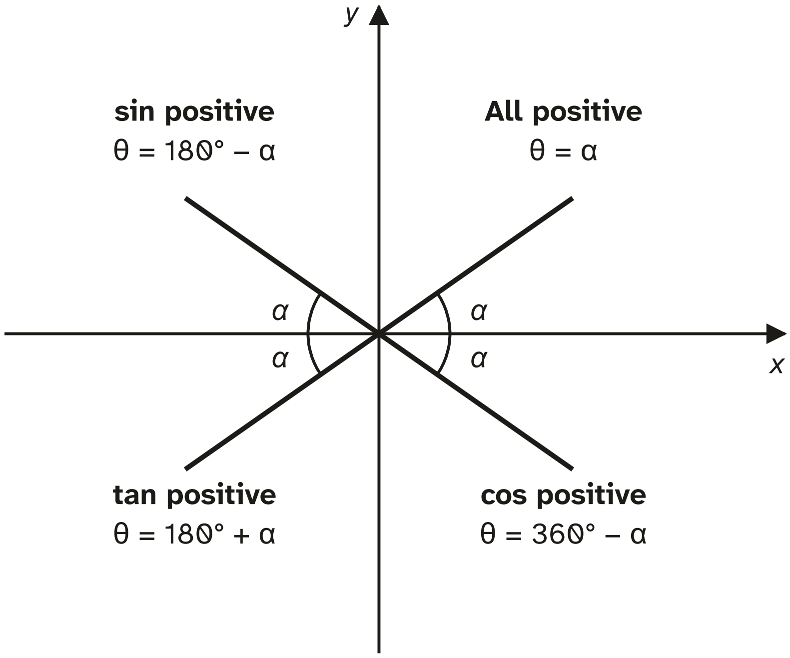
For an acute angle this agrees with SOH CAH TOA, because the hypotenuse is 1. It also gives you the two identities straight away:

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is the gradient: "up" divided by "across".
- Pythagoras in the triangle gives $(\sin\theta)^2 + (\cos\theta)^2 = 1^2$, written $\sin^2\theta + \cos^2\theta = 1$.

The sign $\equiv$ means "true for every value of θ". Turning a negative angle goes clockwise; turning more than 360° goes round again, which is why the graphs repeat.

Where each ratio is positive

As P moves round the circle, its coordinates change sign. Any angle makes the same acute angle α with the x-axis as one angle in the first quadrant, so its sine, cosine and tangent are the same *size* as those of α; only the sign changes.

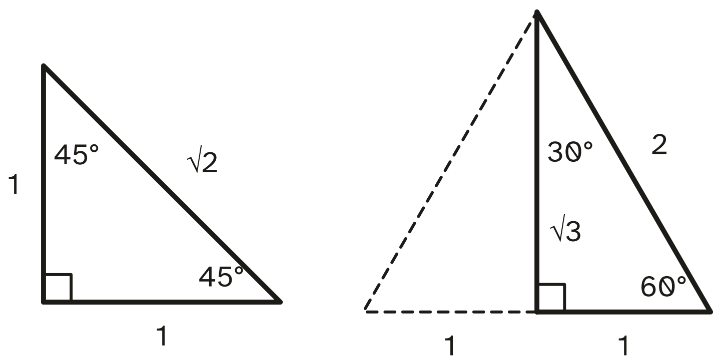


For example, 135° is 45° back from 180°, in the second quadrant where cosine is negative, so $\cos 135^\circ = -\cos 45^\circ = -\frac{\sqrt{2}}{2}$. And $240^\circ = 180^\circ + 60^\circ$ is in the third quadrant where sine is negative, so $\sin 240^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$.

This diagram is also how you find the **second solution** of an equation. If $\sin\theta = 0.4$, the calculator gives $\alpha = 23.6^\circ$; sine is also positive in the second quadrant, so $180^\circ - 23.6^\circ = 156.4^\circ$ is another answer.

Exact values

Two triangles give every exact value you need. Cut a square of side 1 along its diagonal, and cut an equilateral triangle of side 2 in half:



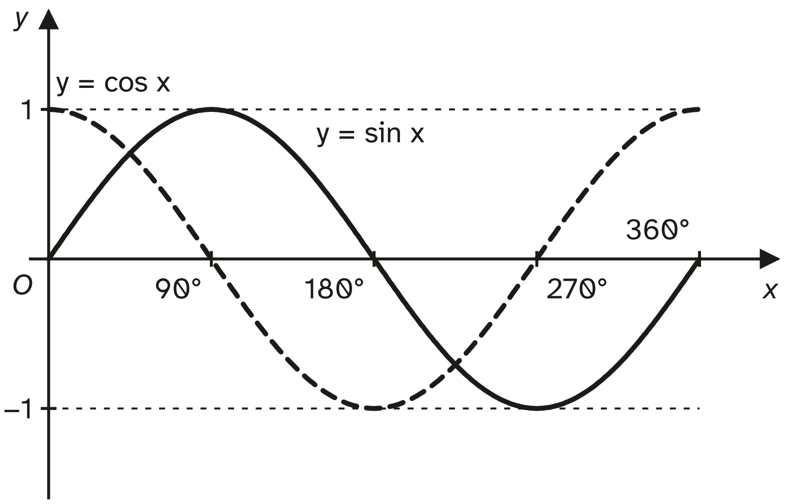
Reading off SOH CAH TOA:

θ	0	$30^\circ = \frac{\pi}{6}$	$45^\circ = \frac{\pi}{4}$	$60^\circ = \frac{\pi}{3}$	$90^\circ = \frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Note $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$: either form is fine.

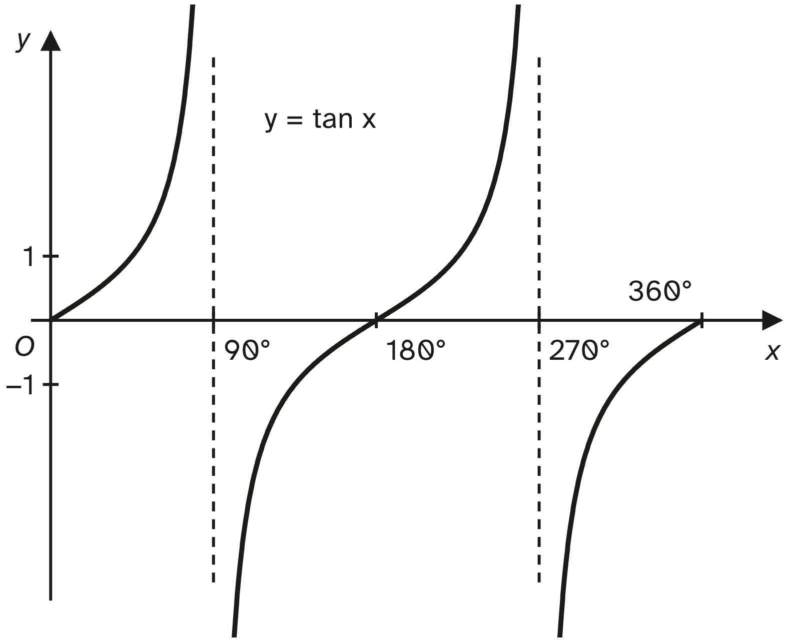
The graphs

Plot the y-coordinate of P as it goes round, and you get the sine wave. The x-coordinate gives the cosine wave, which is the same shape shifted 90° to the left.



- Both repeat every 360° (2π): their **period** is 360°.
- Both stay between -1 and 1.
- $\sin x$ is symmetrical about $x = 90^\circ$, which is why $\sin\theta = \sin(180^\circ - \theta)$. $\cos x$ is symmetrical about $x = 180^\circ$ (and about the y-axis), which is why $\cos\theta = \cos(360^\circ - \theta)$ and $\cos(-\theta) = \cos\theta$.

The tangent graph is different. $\tan x = \frac{\sin x}{\cos x}$ is undefined where $\cos x = 0$, at 90° and 270°, so the graph has vertical **asymptotes** there. It repeats every 180° and takes every value.



Changing the graphs

Each number in $y = a\sin(bx) + c$ does one job (the same for $\cos$):

- a stretches the graph vertically: the wave goes a above and a below its middle line. A negative a also flips it upside down.
- b squashes it horizontally: there are b complete waves where there used to be one, so the period becomes $\frac{360^\circ}{b}$ or $\frac{2\pi}{b}$.
- c moves it up by c: the middle line is $y = c$.

So the greatest value is $c + a$ and the least is $c - a$ (for positive a). For example $y = 3\sin 2x + 1$ has greatest value 4, least value -2 and period 180°. A term added inside the bracket, such as $\sin(x + 30^\circ)$, slides the graph 30° to the left.

Inverse notation

$\sin x = 0.5$ has infinitely many solutions, so "the angle whose sine is 0.5" needs a rule to pick one. $\sin^{-1} 0.5$ means the **principal value**, the one in a fixed range:

- $\sin^{-1} x$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (-90° to 90°),
- $\cos^{-1} x$ is between 0 and π (0° to 180°),
- $\tan^{-1} x$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

So $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$, $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ and $\tan^{-1}(-1) = -\frac{\pi}{4}$. These are the inverse functions of sine, cosine and tangent with their domains cut down so that they are one-one. (Note $\sin^{-1} x$ is not $\frac{1}{\sin x}$.)

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$	Given
$\sin^2\theta + \cos^2\theta \equiv 1$, so $\sin^2\theta \equiv 1 - \cos^2\theta$ and $\cos^2\theta \equiv 1 - \sin^2\theta$	Given
Principal values: $-\frac{\pi}{2} \leqslant \sin^{-1} x \leqslant \frac{\pi}{2}$, $0 \leqslant \cos^{-1} x \leqslant \pi$, $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$	Given
Exact values of $\sin$, $\cos$, $\tan$ at $0, 30^\circ, 45^\circ, 60^\circ, 90^\circ$ (table above)	Learn it
$\sin(180^\circ - \theta) = \sin\theta$, $\cos(360^\circ - \theta) = \cos\theta$, $\tan(180^\circ + \theta) = \tan\theta$	Learn it
Signs: all positive (first quadrant), $\sin$ (second), $\tan$ (third), $\cos$ (fourth)	Learn it
$\sin x$, $\cos x$ have period 360° (2π) and lie between -1 and 1; $\tan x$ has period 180° (π) and asymptotes at $x = 90^\circ, 270^\circ, \ldots$	Learn it
$y = a\sin(bx) + c$: greatest $c + a$, least $c - a$ (for $a > 0$), period $\frac{360^\circ}{b}$ or $\frac{2\pi}{b}$	Learn it
$180^\circ = \pi$ radians	Learn it

Useful algebra: $\sin^2\theta$ means $(\sin\theta)^2$, and $\sin^4\theta - \cos^4\theta = (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)$, a difference of two squares.

How to do it

In the Paper 1 questions we have tagged for this topic (59 questions, 2021–2025), the types came up this often. Many questions mix two types, such as an identity followed by "hence solve".

Question type	Questions
Solve an equation that becomes a quadratic in one ratio	25
Prove an identity (and then use it)	14
Sketch or use a graph	12
Exact values and inverse notation	9
Solve by factorising	7
Transformations of a trigonometric graph	6
Trigonometry inside a progression	6

First, how to finish any equation

Every trigonometric equation ends with one or more lines like $\sin\theta = k$. To get all the solutions:

1. **Check k.** $\sin\theta$ and $\cos\theta$ must be between -1 and 1; reject any other value and say so. $\tan\theta$ can be anything.
2. **Find the key angle** α using the size of k (ignore its sign): $\alpha = \sin^{-1}|k|$, and so on. For $\sin\theta = -0.4$, $\alpha = \sin^{-1} 0.4 = 23.6^\circ$.
3. **Use the quadrant diagram.** Sine is negative in the third and fourth quadrants, so $\theta = 180^\circ + 23.6^\circ = 203.6^\circ$ and $\theta = 360^\circ - 23.6^\circ = 336.4^\circ$.
4. **Add or subtract** 360° (or 2π; 180° for $\tan$) to reach every solution in the interval, including negative ones if the interval allows them.

When k is 0, 1 or -1, read the angles straight off the graph instead: $\sin\theta = -1$ only at 270° (or $\frac{3\pi}{2}$), and $\cos\theta = 0$ at 90° and 270°.

A multiple angle such as $\cos 2x = 0.3$ for $0^\circ \leqslant x \leqslant 180^\circ$: first stretch the interval, $0^\circ \leqslant 2x \leqslant 360^\circ$. Solve for $2x$: $2x = 72.5^\circ$ or $360^\circ - 72.5^\circ = 287.5^\circ$. Only then divide: $x = 36.3^\circ$ or 143.7°. With $\sin(x - 30^\circ)$ or $\tan(5x - 3)$, change the interval the same way, solve for the whole bracket, then rearrange.

Units: if the interval is in radians (it contains π, or has no degree sign), work and answer in radians; otherwise use degrees. Give degrees to 1 decimal place and radians to 3 significant figures unless an exact answer is asked for.

1. Solve an equation that becomes a quadratic in one ratio (25 questions)

The equation mixes $\sin$, $\cos$ and $\tan$, or has fractions. Often part (a) says "show that the equation can be written as $a\cos^2\theta + b\cos\theta + c = 0$" and part (b) says "hence solve".

1. **Clear fractions.** Replace $\tan\theta$ by $\frac{\sin\theta}{\cos\theta}$ (and $\frac{1}{\tan\theta}$ by $\frac{\cos\theta}{\sin\theta}$), then multiply every term by the denominators. Include the right-hand side: every term gets multiplied.
2. **Get one ratio only.** Decide which ratio the answer should be in (the question usually tells you). Replace $\sin^2\theta$ by $1 - \cos^2\theta$, or $\cos^2\theta$ by $1 - \sin^2\theta$. Put brackets round the replacement: $6\sin^2\theta = 6(1 - \cos^2\theta)$.
3. **Collect terms** into $as^2 + bs + c = 0$, where s is the ratio. In a "show that", this line must match the given one exactly, including "$= 0$".
4. **Solve the quadratic** by factorising, the formula or completing the square, and show it. Calculator equation-solver answers with no working lose the method mark.
5. **Finish each root** as above: reject impossible values (like $\cos\theta = 2$), then find every angle.

Variations you will meet:

- **Quartics.** $2\cos^4\theta + 5\cos^2\theta - 3 = 0$ is a quadratic in $\cos^2\theta$: $(2\cos^2\theta - 1)(\cos^2\theta + 3) = 0$. Reject the negative value, then take $\pm\sqrt{\ }$: $\cos^2\theta = \frac{1}{2}$ gives $\cos\theta = +\frac{1}{\sqrt{2}}$ **and** $\cos\theta = -\frac{1}{\sqrt{2}}$, so four angles ($45^\circ, 135^\circ, 225^\circ, 315^\circ$) instead of two.
- **A quadratic in $\tan\theta$.** This happens when the equation has $\tan^2\theta$ or after an identity like $\frac{\tan\theta + a}{\tan^2\theta - b}$. Solve for $\tan\theta$; each value gives one angle per 180°.
- **"Hence" with a quadratic from part (a).** If part (a) solved $3y^2 - 5y - 2 = 0$ (so $y = 2$ or $-\frac{1}{3}$), part (b) with $\tan x$ in place of y uses the same roots: $\tan x = 2$ or $\tan x = -\frac{1}{3}$.

2. Prove an identity, then use it (14 questions)

"Prove" or "show that" an expression $\equiv$ another.

1. **Start from the more complicated side**, usually the left, and work towards the other side. Don't work on both sides at once, and never start from the result you're proving.
2. **Change to** $\sin$ **and** $\cos$**.** Replace $\tan\theta$ by $\frac{\sin\theta}{\cos\theta}$.
3. **Combine fractions** over a common denominator. Keep brackets: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.
4. **Use** $\sin^2\theta + \cos^2\theta = 1$ to simplify, and **factorise** where you can. Look for $(1 - \sin\theta)(1 + \sin\theta) = 1 - \sin^2\theta = \cos^2\theta$ and the difference of two squares.
5. **Look at the target.** If it has only $\cos\theta$ in it, replace every $\sin^2\theta$. If it has $\tan\theta$, divide the top and bottom by $\cos\theta$ or $\cos^2\theta$. Going the other way, from a fraction in $\tan\theta$ and $\tan^2\theta$ to $\sin$ and $\cos$, multiply the top and bottom by $\cos^2\theta$.
6. **Finish** with the exact expression you were given.

Then "hence solve". Replace the complicated side of the equation by the simple side from the identity, then solve as in type 1. Watch for:

- **Values that make a denominator zero.** If the identity has $\cos\theta$ in a denominator, $\cos\theta = 0$ can't be a solution.
- **Common factors on both sides.** If you reach $\tan^3\theta = 4\tan\theta$, don't divide by $\tan\theta$; write $\tan\theta(\tan^2\theta - 4) = 0$ and factorise (type 5), otherwise you lose $\tan\theta = 0$.

3. Sketch or use a graph (12 questions)

Sketching $y = a\sin(bx) + c$ or $y = a\cos(bx) + c$:

1. Work out the greatest value $c + a$, least value $c - a$, middle line $y = c$ and period $\frac{2\pi}{b}$ (or $\frac{360^\circ}{b}$).
2. Mark the start: $\sin$ starts on its middle line and goes up (down if $a < 0$); $\cos$ starts at its greatest value (least if $a < 0$).
3. Draw the right number of complete waves across the given interval, smooth and curved (not straight lines), flat only at the tops and bottoms. Label the greatest and least values on the y-axis and the key x-values.

Reading a**,** b **and** c **from a graph:**

- $a = \dfrac{\text{greatest} - \text{least}}{2}$ and $c = \dfrac{\text{greatest} + \text{least}}{2}$.
- $b = \dfrac{2\pi}{\text{period}}$ (or $\frac{360^\circ}{\text{period}}$): find how far along the x-axis one complete wave takes.
- For $\tan(x - b)$, the graph is shifted right by b: find where the asymptote or the zero has moved to.

Greatest and least values of an expression like $1 + 4\sin x$: use the fact that $\sin x$ goes from -1 to 1, giving -3 and 5, but only if the interval includes the x-values where $\sin x = 1$ and $\sin x = -1$. On a restricted interval that misses one of them, also work out the values at the ends of the interval and take the largest and smallest: for $1 + 4\sin x$ on $0 \leqslant x \leqslant \frac{\pi}{6}$ they are 1 and 3.

"Determine the number of solutions of $a\sin(bx)+c=(\text{line})$": draw the line on your sketch (find its value at each end of the interval), then count the crossings. You don't solve anything; the answer is a number.

4. Exact values and inverse notation (9 questions)

- **"Exact"** means surds, fractions and π. Use the table and the quadrant signs; never decimals from a calculator.
- **Solving with an inverse**, such as $\cos^{-1}(2x+1)=\frac{2\pi}{3}$: apply the ordinary function to both sides, $2x+1=\cos\frac{2\pi}{3}=-\frac{1}{2}$, then solve: $x=-\frac{3}{4}$. There is only one answer, because the inverse gives only principal values.
- **Evaluating a mixture** such as $\cos\left(\frac{\pi}{6}\right)+\tan^{-1}(\ldots)$: work out each exact part, rearrange, then use the principal range.
- **Other ratios from one ratio.** Given an acute angle with $\cos\alpha=\frac{5}{13}$ and no calculator, use $\sin^2\alpha+\cos^2\alpha=1$ (or draw a right-angled triangle with Pythagoras) to get $\sin\alpha=\frac{12}{13}$, then $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$. For an obtuse angle, cosine and tangent are **negative**: choose the sign from the quadrant, and show the working.

5. Solve by factorising (7 questions)

If a ratio appears in every term, or the question says "by factorising", move everything to one side and factorise. **Never divide by $\sin\theta$, $\cos\theta$ or $\tan\theta$**: it throws away solutions.

For example, $2\sin\theta\cos\theta=\cos\theta$ becomes $\cos\theta(2\sin\theta-1)=0$, so $\cos\theta=0$ (90°, 270°) or $\sin\theta=\frac{1}{2}$ (30°, 150°). Dividing by $\cos\theta$ would lose 90° and 270°.

Four-term expressions factorise in pairs: $4\sin\theta\cos\theta+2\sin\theta-2\cos\theta-1=2\sin\theta(2\cos\theta+1)-(2\cos\theta+1)=(2\cos\theta+1)(2\sin\theta-1)$. Each bracket then gives its own simple equation.

An equation like $\sin\theta+2\cos\theta=0$ becomes $\tan\theta=-2$ after dividing by $\cos\theta$. That's allowed here, because $\cos\theta=0$ would force $\sin\theta=0$ too, which is impossible. It gives $180^\circ-63.4^\circ=116.6^\circ$ and 296.6°.

Squaring can add false roots. If you square both sides, for example to turn $\sin\theta-\cos\theta=1$ into $1-2\sin\theta\cos\theta=1$, the new equation also has the roots of $\sin\theta-\cos\theta=-1$. Here $\sin\theta\cos\theta=0$ gives $0^\circ,90^\circ,180^\circ,270^\circ,360^\circ$ for $0^\circ\leqslant\theta\leqslant360^\circ$, but substituting each into the original equation leaves only 90° and 180°. Always check every root in the original equation.

Showing there are no solutions. Solve the quadratic in the ratio and show that every root lies outside -1 to 1 (a negative root is also impossible for $\cos^2\theta$ or $\sin^2\theta$). For example $\cos^2\theta+\cos\theta-6=0$ gives $\cos\theta=2$ or -3, so there are no solutions. If the quadratic has a letter k in it, use the quadratic formula and show each root is outside the range for all the k-values you are given.

6. Transformations of a trigonometric graph (6 questions)

These use the rules from the Functions topic, applied to sin and cos:

Change	Transformation
$y = a\sin x$	Stretch, parallel to the y-axis, factor a
$y = \sin(bx)$	Stretch, parallel to the x-axis, factor $\frac{1}{b}$
$y = \sin x + c$	Translation by $\begin{pmatrix} 0 \\ c \end{pmatrix}$
$y = \sin(x + d)$	Translation by $\begin{pmatrix} -d \\ 0 \end{pmatrix}$
$y = -\sin x$	Reflection in the x-axis

1. Name **every** part: "stretch, factor 3, parallel to the y-axis"; "translation by the vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$".
2. **Order matters** when a stretch and a translation act in the same direction. For $y = 3\sin x + 2$, stretch first, then translate. For $y = \sin(2x - 40^\circ)$, rewrite it as $\sin\left(2(x - 20^\circ)\right)$: either stretch parallel to the x-axis by factor $\frac{1}{2}$ and then translate by $\begin{pmatrix} 20^\circ \\ 0 \end{pmatrix}$, or translate by $\begin{pmatrix} 40^\circ \\ 0 \end{pmatrix}$ first and then stretch.
3. To follow a point through the changes, apply each transformation to its coordinates in the same order. A maximum $\left(\frac{\pi}{2}, k\right)$ on $y = k\sin x$, reflected in the x-axis and then translated 3 up, goes to $\left(\frac{\pi}{2}, -k\right)$ and then $\left(\frac{\pi}{2}, 3 - k\right)$; the new curve is $y = 3 - k\sin x$.

7. Trigonometry inside a progression (6 questions)

The terms of an arithmetic or geometric progression are given as trigonometric expressions. Use the progression condition to get a trigonometric equation, solve it, then carry on with the progression formulas from the Series topic.

- Three terms u_1, u_2, u_3 in arithmetic progression: $2u_2 = u_1 + u_3$ (or $u_2 - u_1 = u_3 - u_2$).
- In geometric progression: $u_2^2 = u_1 u_3$, and the common ratio is $r = \frac{u_2}{u_1}$. A ratio like $\frac{\sin\theta}{\tan\theta}$ simplifies to $\cos\theta$.
- A sum to infinity $\frac{a}{1-r}$ needs $-1 < r < 1$.

Keep exact values (surds) until the end if the question asks for an exact sum.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

(a) The angle θ is obtuse and $\sin\theta = \frac{2}{5}$. Find the exact values of $\cos\theta$ and $\tan\theta$. **[3]**

(b) Find the exact solution of the equation $3\tan^{-1}(2x) + \frac{\pi}{2} = 0$. **[2]**

(a) $\cos^2\theta = 1 - \sin^2\theta = 1 - \frac{4}{25} = \frac{21}{25}$. **M1** for using $\sin^2\theta + \cos^2\theta = 1$.

θ is obtuse (second quadrant), where cosine is negative, so $\cos\theta = -\dfrac{\sqrt{21}}{5}$. **A1**, with the negative sign.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2}{5} \div \left(-\frac{\sqrt{21}}{5}\right) = -\frac{2}{\sqrt{21}} = -\frac{2\sqrt{21}}{21}$$

A1

(b) $\tan^{-1}(2x) = -\dfrac{\pi}{6}$, so $2x = \tan\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{\sqrt{3}}$. **M1** for making $\tan^{-1}(2x)$ the subject and taking $\tan$.

$$x = -\frac{1}{2\sqrt{3}} = -\frac{\sqrt{3}}{6}$$

A1. There is exactly one answer, because $\tan^{-1}$ gives only principal values.

Example 2

(a) Show that the equation $2\tan\theta = 3\cos\theta$ can be written as $3\sin^2\theta + 2\sin\theta - 3 = 0$. **[2]**

(b) Hence solve the equation $2\tan\theta = 3\cos\theta$ for $0^\circ \leqslant \theta \leqslant 360^\circ$. **[4]**

(a) Replace $\tan\theta$ and multiply by $\cos\theta$:

$$\frac{2\sin\theta}{\cos\theta} = 3\cos\theta \;\Rightarrow\; 2\sin\theta = 3\cos^2\theta$$

M1 for using $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and clearing the fraction.

$$2\sin\theta = 3(1 - \sin^2\theta) \;\Rightarrow\; 3\sin^2\theta + 2\sin\theta - 3 = 0$$

A1, the given answer reached with every step shown.

(b) The quadratic doesn't factorise, so use the formula:

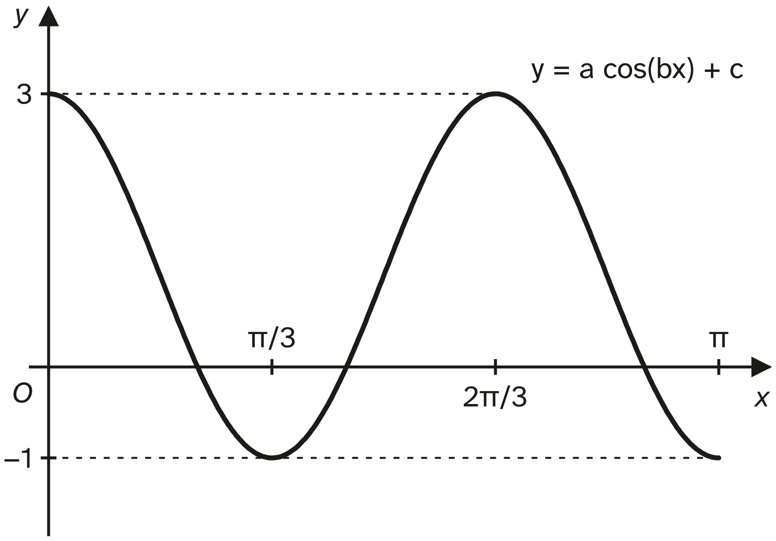
$$\sin\theta = \frac{-2 \pm \sqrt{4+36}}{6} = \frac{-1 \pm \sqrt{10}}{3}$$

M1 for solving the quadratic with visible working.

$\sin\theta = 0.7208$ or $\sin\theta = -1.387$. The second is less than -1, so it has no solutions. **A1** for 0.7208 (and rejecting the other).

$\alpha = \sin^{-1} 0.7208 = 46.1^\circ$. Sine is positive in the first and second quadrants: $\theta = 46.1^\circ$ **B1**, and $\theta = 180^\circ - 46.1^\circ = 133.9^\circ$ **B1**, with no others in the interval.

Example 3



The diagram shows the curve $y = a\cos(bx) + c$ for $0 \leqslant x \leqslant \pi$, where a, b and c are positive constants.

(a) State the values of a, b and c. **[3]**

(b) Determine the number of solutions of the equation $a\cos(bx) + c = x - 1$ for $0 \leqslant x \leqslant \pi$. **[1]**

(c) Find the exact solutions of the equation $a\cos(bx) + c = 0$ for $0 \leqslant x \leqslant \pi$. **[3]**

(a) Greatest 3, least -1: $a = \frac{3-(-1)}{2} = 2$ **B1**, and $c = \frac{3+(-1)}{2} = 1$ **B1**.

One complete wave (top to top) runs from $x = 0$ to $x = \frac{2\pi}{3}$, so the period is $\frac{2\pi}{3}$ and $b = 2\pi \div \frac{2\pi}{3} = 3$. **B1**

The curve is $y = 2\cos 3x + 1$.

(b) The line $y = x - 1$ goes from $(0, -1)$ to $(\pi, 2.14)$. At $x = 0$ the curve (3) is above it; at $x = \frac{\pi}{3}$ the curve (-1) is below it (0.05); at $x = \frac{2\pi}{3}$ the curve (3) is above it (1.09); at $x = \pi$ the curve (-1) is below it (2.14). It changes sides three times: **3 solutions. B1**

(c) $2\cos 3x + 1 = 0$, so $\cos 3x = -\frac{1}{2}$.

Since $0 \leqslant x \leqslant \pi$, the interval for $3x$ is $0 \leqslant 3x \leqslant 3\pi$. **M1** for working with $3x$ over the stretched interval.

The key angle is $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$. Cosine is negative in the second and third quadrants: $3x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and $3x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$. Adding 2π: $3x = \frac{8\pi}{3}$ (the next, $\frac{10\pi}{3}$, is too big). **A1**

$$x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$$

A1. Three solutions, matching the three places where the graph crosses the x-axis.

Example 4

(a) Prove the identity $\dfrac{1+\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta} \equiv \dfrac{2}{\cos\theta}$. **[3]**

(b) Hence solve the equation $\dfrac{1+\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{1+\sin\theta}=3\tan\theta+4\cos\theta$ for $0^\circ\leqslant\theta\leqslant 360^\circ$. **[5]**

(a) Start from the left-hand side and use a common denominator:

$$\frac{(1+\sin\theta)^2+\cos^2\theta}{\cos\theta\,(1+\sin\theta)}=\frac{1+2\sin\theta+\sin^2\theta+\cos^2\theta}{\cos\theta\,(1+\sin\theta)}$$

M1 for the common denominator and a correctly expanded numerator.

$$=\frac{2+2\sin\theta}{\cos\theta\,(1+\sin\theta)}=\frac{2(1+\sin\theta)}{\cos\theta\,(1+\sin\theta)}$$

M1 for using $\sin^2\theta+\cos^2\theta=1$.

$$=\frac{2}{\cos\theta}$$

A1, after cancelling the common factor.

(b) Using part (a):

$$\frac{2}{\cos\theta}=\frac{3\sin\theta}{\cos\theta}+4\cos\theta\;\Rightarrow\;2=3\sin\theta+4\cos^2\theta$$

B1 for replacing the left-hand side and clearing the fraction.

$$2=3\sin\theta+4(1-\sin^2\theta)\;\Rightarrow\;4\sin^2\theta-3\sin\theta-2=0$$

M1 for a three-term quadratic in $\sin\theta$.

$$\sin\theta=\frac{3\pm\sqrt{9+32}}{8}=\frac{3\pm\sqrt{41}}{8}=1.175\text{ or }-0.4254$$

M1 for solving it. $1.175>1$ is impossible, so $\sin\theta=-0.4254$. **A1**

Key angle $\alpha=\sin^{-1}0.4254=25.2^\circ$. Sine is negative in the third and fourth quadrants:

$$\theta=180^\circ+25.2^\circ=205.2^\circ,\qquad\theta=360^\circ-25.2^\circ=334.8^\circ$$

A1 for both and no others. Neither makes $\cos\theta$ or $1+\sin\theta$ zero, so both are valid.

Common mistakes

- **Missing the second (or third, or fourth) angle.** After the calculator's answer, always use the quadrant diagram and add 360° to find every solution in the interval. Examiners often see only one of the two angles.
- **Including angles outside the interval, or a wrong extra one.** An extra wrong answer inside the interval loses the final mark. Check each answer against the interval, and the quadrant against the sign of k.
- **Dividing by $\sin\theta$, $\cos\theta$ or $\tan\theta$.** You lose the solutions where that ratio is zero. Factorise instead.
- **Forgetting the** $\pm$ when taking the square root of $\sin^2\theta=k$ or $\tan^2\theta=k$. $\tan^2\theta=3$ means $\tan\theta=\sqrt{3}$ **or** $-\sqrt{3}$.

- **Wrong identities.** $\sin\theta = 1 - \cos\theta$ is false; the identity is about the squares. And $\sin^2\theta$ is $(\sin\theta)^2$, not $\sin(\theta^2)$.
- **Dropping brackets or the right-hand side** when clearing fractions. Every term, on both sides, is multiplied.
- **Multiple angles done in the wrong order.** For $\cos 2x = k$, find all values of $2x$ in the doubled interval first, then halve. Halving first loses solutions.
- **Mixing units.** If the interval is in radians, answers in degrees score at most a special-case mark, and the reverse. Don't round radians to 1 decimal place, or degrees to whole numbers.
- **Unsupported calculator answers.** Solving a quadratic with the equation solver, or quoting $\sin\theta$ when the question says "without using the trigonometric functions on your calculator", scores little.
- **Proofs that skip steps or start from the answer.** In a "show that", each line must follow from the one before, with the identity you've used visible. Writing only the first and last lines loses marks.
- **Sketches with the wrong period or start.** Count the waves: $\cos 2x$ on 0 to 2π has two. $\sin$ starts on its middle line, $\cos$ at its top. Don't draw straight-line zigzags.
- **Reading b wrongly from a graph.** b is the number of waves in 2π (or 360°), not the period itself; $y = \sin\frac{1}{2}x$ has $b = \frac{1}{2}$ and a period of 4π.
- **Greatest and least values from the interval's ends only.** For $1 + 4\sin x$ over a full wave, the extremes come from $\sin x = \pm 1$, not from the ends of the interval. Use the ends as well only when the interval doesn't reach $\sin x = 1$ or -1.

How to get the marks

- **"Show that" and "prove"** mean the answer is given: every step must be written out, including $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$ where you use them, and the final line must match exactly (with "$= 0$" if it has one).
- **"Hence"** means use the previous part. A "hence solve" after an identity expects you to replace one side with the other; after a "show that", solve the quadratic you were given.
- **Show how you solved the quadratic**: factors, formula or completed square. Then write the values of $\sin\theta$ (or $\cos\theta$, $\tan\theta$) before the angles: those values often earn a mark on their own.
- **Verify your roots** whenever you squared both sides: substitute each one into the original equation and say which ones fail.
- **All answers, and no extras.** Most final marks need every solution in the interval and nothing else in it. Answers outside the interval are ignored.
- **Accuracy:** angles in degrees to 1 decimal place (37.4°, not 37°); radians to 3 significant figures, unless the question asks for something else. Keep more figures in the working.
- **"Exact"** means $\frac{\pi}{6}$, $\frac{\sqrt{3}}{2}$ and so on, no decimals.
- **"State"** (for example the values of a, b, c, or the greatest value) needs only the answer. **"Determine the number of solutions"** needs a number, which you get from a sketch, not by solving.
- **"Sketch"** means the correct shape, with the key features labelled: the greatest and least values on the y-axis, where it starts and ends, and the right number of waves. It doesn't need to be to scale.

- **"Describe fully"** for a transformation needs every part: the type (stretch, translation, reflection), the factor or vector, the direction, and the order when there's more than one.

Quick check

1. Without a calculator, find the exact values of $\sin 240^\circ$, $\cos \frac{3\pi}{4}$ and $\tan\left(-\frac{\pi}{3}\right)$.
2. Solve $4\sin\theta\cos\theta = 3\sin\theta$ for $0^\circ \leqslant \theta \leqslant 360^\circ$.
3. Find the exact solutions of $\sqrt{3}\tan 2x = 1$ for $0 \leqslant x \leqslant \pi$.
4. Solve $2\cos^2\theta + 3\sin\theta = 3$ for $-180^\circ < \theta < 180^\circ$.
5. Describe fully a sequence of transformations that maps the graph of $y = \sin x$ onto the graph of $y = 3 - 2\sin x$, and state the greatest and least values of $3 - 2\sin x$.

Answers

1. $\sin 240^\circ = -\sin 60^\circ = -\dfrac{\sqrt{3}}{2}$; $\cos\dfrac{3\pi}{4} = -\cos\dfrac{\pi}{4} = -\dfrac{\sqrt{2}}{2}$; $\tan\left(-\dfrac{\pi}{3}\right) = -\tan\dfrac{\pi}{3} = -\sqrt{3}$.
2. $\sin\theta(4\cos\theta - 3) = 0$. $\sin\theta = 0$ gives $0^\circ, 180^\circ, 360^\circ$; $\cos\theta = \frac{3}{4}$ gives 41.4° and $360^\circ - 41.4^\circ = 318.6^\circ$. Five solutions: $0^\circ, 41.4^\circ, 180^\circ, 318.6^\circ, 360^\circ$.
3. $\tan 2x = \frac{1}{\sqrt{3}}$ with $0 \leqslant 2x \leqslant 2\pi$: $2x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$, so $x = \frac{\pi}{12}$ or $\frac{7\pi}{12}$.
4. $2(1 - \sin^2\theta) + 3\sin\theta = 3$, so $2\sin^2\theta - 3\sin\theta + 1 = 0$ and $(2\sin\theta - 1)(\sin\theta - 1) = 0$. $\sin\theta = \frac{1}{2}$ gives $30^\circ, 150^\circ$; $\sin\theta = 1$ gives 90°.
5. Reflection in the x-axis; stretch parallel to the y-axis, factor 2 (these two can be swapped); then translation by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$, which must come last. Greatest value $3 + 2 = 5$ (when $\sin x = -1$), least value $3 - 2 = 1$.

Past questions to try

- 9709/13/M/J/25 Q5: an equation that becomes a quadratic in $\cos\theta$, with negative angles.
- 9709/12/M/J/25 Q5: greatest and least values, a sketch, and counting solutions.
- 9709/11/M/J/24 Q5: prove an identity, then use it to solve an equation.
- 9709/11/O/N/21 Q3: solve by factorising.