Cambridge AS & A Level · Mathematics (9709) · Notes

Trigonometry

AS

Read online at pastopic.com/cambridge/mathematics-9709/notes/trigonometry-p2

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Read first: Trigonometry, Algebra

What you need to know

By the end of this topic you should be able to:

1. **Use the three reciprocal ratios.** Know that $\sec\theta = \frac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ and $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$. Sketch and use the graphs of all six functions ($\sin$, $\cos$, $\tan$, $\sec$, cosec, $\cot$) for angles of any size, in degrees or radians, and know their properties: where they are undefined, which values they can never take and how often they repeat.
2. **Use trigonometric identities** to simplify expressions, to find exact values and to solve equations, choosing the identity that fits the question. You need to be fluent with:
   - the two new Pythagorean identities, $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$;
   - the compound-angle formulas for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$;
   - the double-angle formulas for $\sin 2A$, $\cos 2A$ and $\tan 2A$;
   - writing $a\sin\theta + b\cos\theta$ as a single term $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$, and using it to solve equations and find greatest and least values.

The Paper 1 facts ($\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\sin^2\theta + \cos^2\theta = 1$, exact values, the graphs of $\sin$, $\cos$, $\tan$, and how to find every solution in an interval) are used all the time here; see the Paper 1 Trigonometry note. Factor formulas (such as $\sin P + \sin Q$), the $t = \tan\frac{1}{2}\theta$ formulas and general solutions are **not** in this topic. Integrating trigonometric functions is in the Integration topic; this note covers the rewriting that comes before it.

Understand it

Three new ratios

Each new ratio is "one over" an old one:

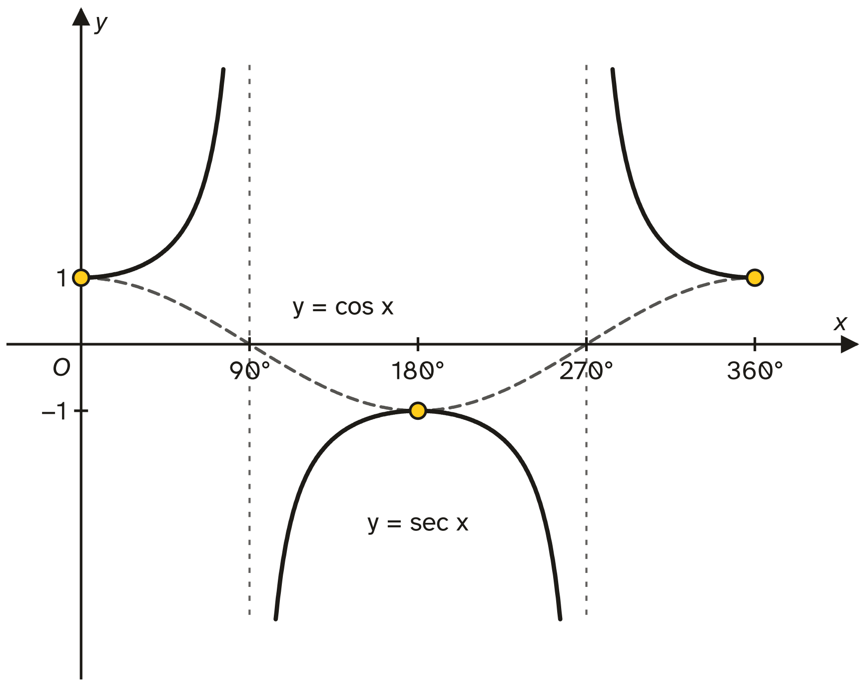
$$\sec\theta = \frac{1}{\cos\theta}, \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta}, \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

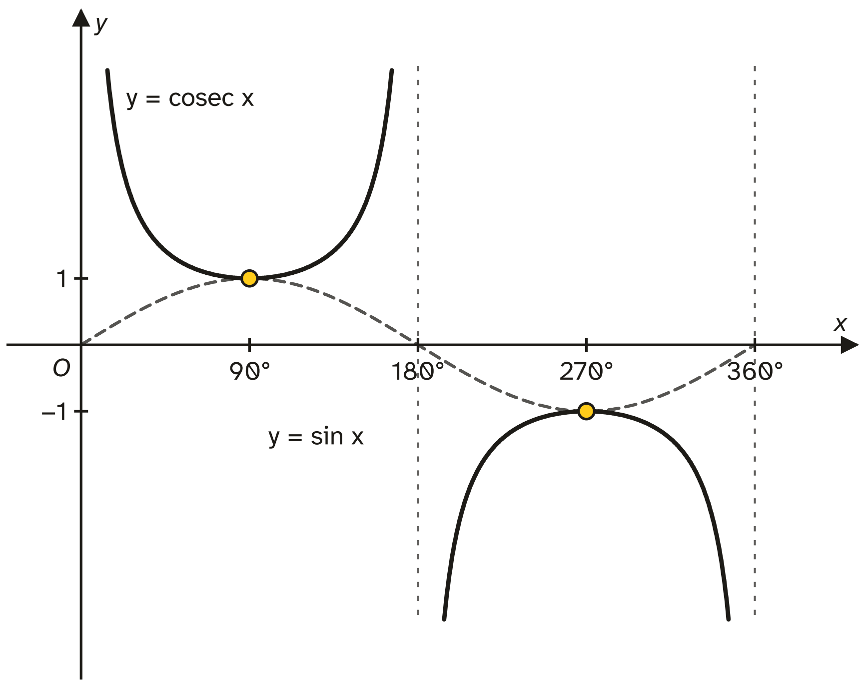
To remember which goes with which, look at the **third letter**: **sec** goes with **c**os, **cos**ec goes with **s**in, **cot** goes with **t**an. Your calculator has no buttons for them, so turn them round first: $\sec 60^\circ = \frac{1}{\cos 60^\circ} = 2$, and $\cot\theta = 3$ means $\tan\theta = \frac{1}{3}$.

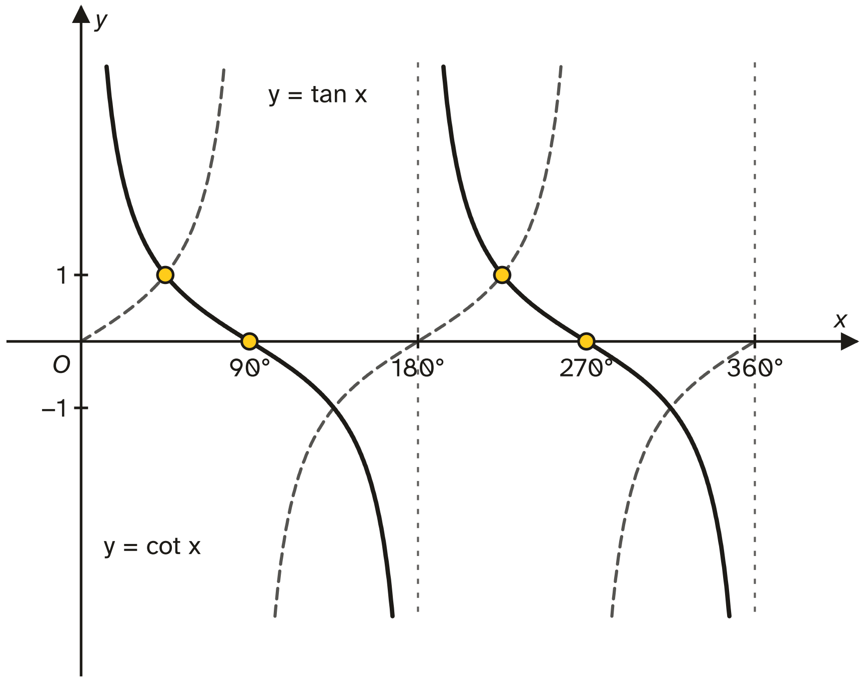
Their graphs

Each graph comes from the graph it is the reciprocal of:

- where the old ratio is 1 or -1, the new one is the same;
- where the old ratio is small, the new one is large, and where the old ratio is 0, the new one is undefined: the graph has a vertical **asymptote** there;
- the sign never changes: where $\cos x$ is negative, so is $\sec x$.







What to take from them:

- $\sec x$ and $\operatorname{cosec} x$ repeat every 360° (2π); $\cot x$ repeats every 180° (π).
- $\sec x$ and $\operatorname{cosec} x$ are **never between** -1 **and** 1: $\sec x \geqslant 1$ or $\sec x \leqslant -1$. So an equation such as $\operatorname{cosec}\theta = \frac{1}{2}$ has no solutions. $\cot x$ takes every value.
- $\sec x$ is undefined at $x = 90^\circ, 270^\circ, \ldots$; $\operatorname{cosec} x$ and $\cot x$ at $x = 0^\circ, 180^\circ, 360^\circ, \ldots$

Two more Pythagorean identities

Start from $\sin^2\theta + \cos^2\theta \equiv 1$ and divide every term by $\cos^2\theta$:

$$\tan^2\theta + 1 \equiv \sec^2\theta.$$

Divide instead by $\sin^2\theta$:

$$1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta.$$

Their main use is to turn an equation with **two different ratios** into one with a single ratio. If an equation has $\tan^2\theta$ and $\sec\theta$, replace $\tan^2\theta$ by $\sec^2\theta - 1$ and you get a quadratic in $\sec\theta$.

Compound angles

$\sin(A+B)$ is **not** $\sin A + \sin B$. Try $A = 30^\circ$, $B = 60^\circ$: $\sin 90^\circ = 1$, but $\sin 30^\circ + \sin 60^\circ = \frac{1+\sqrt{3}}{2}$. The correct expansions are:

$$\begin{aligned}\sin(A \pm B) &\equiv \sin A\cos B \pm \cos A\sin B\\ \cos(A \pm B) &\equiv \cos A\cos B \mp \sin A\sin B\\ \tan(A \pm B) &\equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}\end{aligned}$$

Watch the signs: in the $\cos$ formula the sign **flips** ($\cos(A+B)$ has a minus), and in the $\tan$ formula the bottom sign is the opposite of the top one.

With the exact values of 30°, 45° and 60° these give exact values of other angles. For example,

$$\cos 75^\circ = \cos(45^\circ + 30^\circ) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

The most common use in exams is expanding a bracket such as $\sin(\theta + 60^\circ)$ into $\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta$, so that the equation contains only $\sin\theta$ and $\cos\theta$.

Double angles

Put $B = A$ in the compound-angle formulas:

$$\sin 2A \equiv 2\sin A\cos A, \qquad \tan 2A \equiv \frac{2\tan A}{1-\tan^2 A},$$
$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A.$$

The three forms of $\cos 2A$ come from using $\sin^2 A + \cos^2 A = 1$ to remove one of the squares. Choose the form that leaves you with only one ratio: with $\cos\theta$ elsewhere in the equation, use $2\cos^2\theta - 1$; with $\sin\theta$, use $1 - 2\sin^2\theta$.

Rearranging the last two forms gives the way to get rid of a square, which you need before integrating:

$$\cos^2 A \equiv \tfrac{1}{2}(1+\cos 2A), \qquad \sin^2 A \equiv \tfrac{1}{2}(1-\cos 2A).$$

"Double" means double whatever angle is there: $\sin 6x = 2\sin 3x\cos 3x$ and $\cos x = 1 - 2\sin^2\frac{x}{2}$.

Two waves make one: $R\sin(\theta \pm \alpha)$

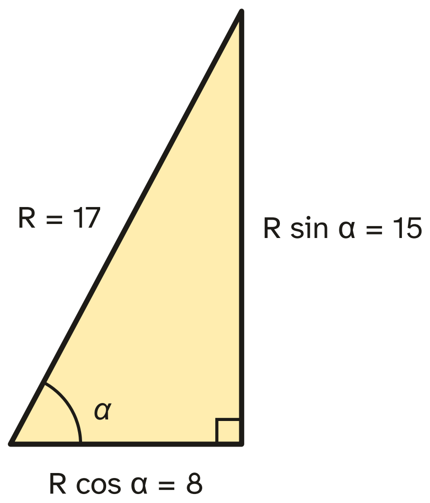
Add a sine wave and a cosine wave with the same period and you get a single, shifted wave. For example, $8\sin\theta - 15\cos\theta$ can be written as $R\sin(\theta - \alpha)$. To find R and α, expand:

$$R\sin(\theta-\alpha) = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha.$$

Match the $\sin\theta$ terms and the $\cos\theta$ terms with $8\sin\theta - 15\cos\theta$:

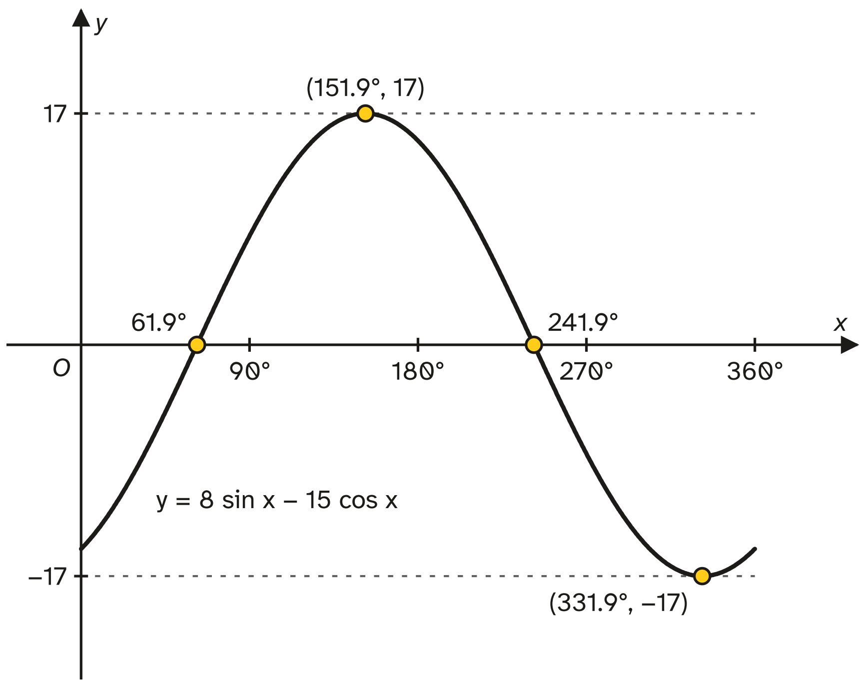
$$R\cos\alpha = 8, \qquad R\sin\alpha = 15.$$

These are the sides of a right-angled triangle with hypotenuse R:



So $R = \sqrt{8^2 + 15^2} = 17$ (Pythagoras) and $\tan\alpha = \frac{15}{8}$, giving $\alpha = 61.93^\circ$. Therefore

$$8\sin\theta - 15\cos\theta \equiv 17\sin(\theta - 61.93^\circ).$$



This single form makes two jobs easy:

- **Solving** $8\sin\theta - 15\cos\theta = 6$ becomes $\sin(\theta - 61.93^\circ) = \frac{6}{17}$, an equation with only one ratio.
- **Greatest and least values.** A sine is between -1 and 1, so $17\sin(\theta - 61.93^\circ)$ is between -17 and 17. The greatest value 17 happens when $\theta - 61.93^\circ = 90^\circ$, that is $\theta = 151.93^\circ$.

The four forms and what to match:

Expression $(a, b > 0)$	**Write as**	**Match**
$a\sin\theta + b\cos\theta$	$R\sin(\theta + \alpha)$	$R\cos\alpha = a$, $R\sin\alpha = b$

Expression $(a, b > 0)$	Write as	Match
$a\sin\theta - b\cos\theta$	$R\sin(\theta-\alpha)$	$R\cos\alpha = a$, $R\sin\alpha = b$
$a\cos\theta + b\sin\theta$	$R\cos(\theta-\alpha)$	$R\cos\alpha = a$, $R\sin\alpha = b$
$a\cos\theta - b\sin\theta$	$R\cos(\theta+\alpha)$	$R\cos\alpha = a$, $R\sin\alpha = b$

In every case $R = \sqrt{a^2+b^2}$ and $\tan\alpha = \dfrac{b}{a}$, with α acute. The question always tells you which form to use.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\sec\theta = \dfrac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$	Learn it
$\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta \equiv 1$	Given
$1+\tan^2\theta \equiv \sec^2\theta$ and $\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$	Given
$\sin(A\pm B) \equiv \sin A\cos B \pm \cos A\sin B$	Given
$\cos(A\pm B) \equiv \cos A\cos B \mp \sin A\sin B$	Given
$\tan(A\pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$	Given
$\sin 2A \equiv 2\sin A\cos A$	Given
$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$	Given
$\tan 2A \equiv \dfrac{2\tan A}{1-\tan^2 A}$	Given
$\cos^2 A \equiv \frac{1}{2}(1+\cos 2A)$ and $\sin^2 A \equiv \frac{1}{2}(1-\cos 2A)$	Learn it
$a\sin\theta \pm b\cos\theta$ or $a\cos\theta \pm b\sin\theta$ in R form: $R=\sqrt{a^2+b^2}$, $\tan\alpha = \dfrac{b}{a}$	Learn it
$R\sin(\theta\pm\alpha)+c$ and $R\cos(\theta\pm\alpha)+c$ lie between $c-R$ and $c+R$	Learn it
sec and cosec: period 360° (2π), values never between -1 and 1; cot: period 180° (π)	Learn it
Exact values of sin, cos, tan at 30°, 45°, 60° (Paper 1 note)	Learn it
Principal values: $-\frac{\pi}{2} \leqslant \sin^{-1}x \leqslant \frac{\pi}{2}$, $0 \leqslant \cos^{-1}x \leqslant \pi$, $-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$	Given

Having a formula in MF19 does not mean you can skip it: write the identity you are using, with the angles put in, before you simplify.

How to do it

In the Paper 2 questions we have tagged for this topic (63 questions, 2021–2025; many appear in two or three versions of the same exam, so there are 41 different ones), the types came up this often. 9 of the 63 questions mix two types, so the counts add up to more than 63.

Question type	Questions
Compound-angle formulas: equations and exact values	17
Prove an identity, then use it	16
A trigonometric ratio as the unknown in an algebra or logarithm question	15
Express in the form $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$, then use it	11
Equations with sec, cosec and cot	9
Double-angle formulas to solve an equation or rewrite an expression	4

Every equation ends with one or more lines like $\tan\theta = k$ or $\cos 2\theta = k$. Finish them as in the Paper 1 note: find the calculator angle, use the quadrants to find the others, stretch the interval first for a multiple angle such as 2θ, and give **every** solution in the interval. Give angles in degrees to 1 decimal place and radians to 3 significant figures, unless the question says otherwise.

1. Compound-angle formulas: equations and exact values (17 questions)

Solving an equation with brackets such as $\sin(\theta + 60^\circ)$:

1. **Expand each bracket** with the compound-angle formula and put in the exact values, for example $\cos(\theta - 30^\circ) = \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta$. Any sec, cosec or cot becomes $\frac{1}{\cos}$, $\frac{1}{\sin}$ or $\frac{\cos}{\sin}$.
2. **Collect** the $\sin\theta$ terms on one side and the $\cos\theta$ terms on the other.
3. **Divide by** $\cos\theta$ to get $\tan\theta = k$, keeping k exact if you can.
4. **Solve**, remembering that tan repeats every 180°, so there are usually two answers in $0^\circ < \theta < 360^\circ$.

For example, $\sin(\theta + 60^\circ) = 3\cos\theta$ for $0^\circ < \theta < 360^\circ$:

$$\tfrac{1}{2}\sin\theta + \tfrac{\sqrt{3}}{2}\cos\theta = 3\cos\theta \;\Rightarrow\; \sin\theta = (6 - \sqrt{3})\cos\theta \;\Rightarrow\; \tan\theta = 6 - \sqrt{3} = 4.268\ldots$$

so $\theta = 76.8^\circ$ or $180^\circ + 76.8^\circ = 256.8^\circ$.

Variations you will meet:

- **$\tan(\theta \pm 45^\circ)$ or $\tan(\theta \pm 60^\circ)$ with $\cot\theta$ or $\tan\theta$.** Expand with the tan formula ($\tan 45^\circ = 1$, $\tan 60^\circ = \sqrt{3}$), write $\cot\theta = \frac{1}{\tan\theta}$, and multiply out the fractions. You get a **quadratic in** $\tan\theta$, usually with two roots and so two families of answers. For example, $\tan(\theta - 45^\circ) = 2\cot\theta$ for $0^\circ < \theta < 180^\circ$: with $t = \tan\theta$, $\frac{t-1}{1+t} = \frac{2}{t}$, so $t^2 - t = 2 + 2t$, that is $t^2 - 3t - 2 = 0$. Then $t = \frac{3 \pm \sqrt{17}}{2}$, so $\tan\theta = 3.562$ or -0.562, giving $\theta = 74.3^\circ$ or 150.7°.

- **Double angles inside the brackets**, such as $\sin(2\theta + 30^\circ)$. Expand in the same way to get $\tan 2\theta = k$, then solve for 2θ over the doubled interval before halving. Since $\tan 2\theta$ repeats every 180°, θ repeats every 90°: $0^\circ < \theta < 180^\circ$ has two answers.
- **Exact value from one given ratio.** "θ is acute and $\sin\theta = \frac{5}{13}$; find the exact value of $\cos(\theta + 45^\circ)$." Draw a right-angled triangle (or use $\sin^2\theta + \cos^2\theta = 1$) to get $\cos\theta = \frac{12}{13}$, then expand: $\frac{12}{13} \cdot \frac{\sqrt{2}}{2} - \frac{5}{13} \cdot \frac{\sqrt{2}}{2} = \frac{7\sqrt{2}}{26}$. If you are given $\sec\theta$ or $\cot\theta$, use $\sec^2\theta = 1 + \tan^2\theta$ or the triangle to find $\tan\theta$ first. If θ is not acute, use the quadrant to fix each sign. Examiners give **no marks** for finding θ on a calculator and working out a decimal.
- **Exact value of a number such as** $\cos 75^\circ$: split the angle into two that you know ($45^\circ + 30^\circ$) and expand.

2. Prove an identity, then use it (16 questions)

1. **Start from the more complicated side**, usually the left, and work towards the other side. Never work on both sides at once, and never start from the answer.
2. **Change everything into** $\sin\theta$ **and** $\cos\theta$: $\tan$, $\sec$, cosec and $\cot$ with their definitions, and double angles with their formulas. Expand any compound-angle brackets.
3. **Simplify**: put fractions over a common denominator, multiply out, and use $\sin^2\theta + \cos^2\theta = 1$. When the answer has $\sin 2\theta$ or $\cos 2\theta$ in it, look for $2\sin\theta\cos\theta$ and $\cos^2\theta - \sin^2\theta$ (or a lone $\sin^2\theta$ or $\cos^2\theta$ to replace with the half-angle forms).
4. **Finish on exactly the printed right-hand side.**

For example, to show $\dfrac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta$:

$$\frac{1 - \cos 2\theta}{\sin 2\theta} = \frac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

Putting $\theta = \frac{\pi}{12}$ then gives an exact value: $\tan\frac{\pi}{12} = \dfrac{1 - \cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \dfrac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3}$.

Then use it ("hence"):

- **Hence solve.** Replace the long expression by the short one from the identity and solve that. Watch for a **changed angle**: if the identity is in θ but the equation has $2x$ (or $\frac{\pi}{8}$, or 4α) in the same places, the identity holds with θ replaced by that angle, so the short side has $2x$ too. Examiners see many students miss this.
- **Hence find an exact value.** Choose θ so that the expression matches, say so ("putting $\theta = \frac{\pi}{12}$"), and use exact values. Sometimes you use the identity twice with two values of θ and add, noticing that $\sin(-A) = -\sin A$ makes two terms cancel.
- **Hence find the set of values of** k for which an equation has no solutions (or has solutions): the identity shows the range of the expression (for example $3\sin^2 x$ is between 0 and 3), and k must lie outside (or inside) that range.
- **Hence integrate.** The identity turns an expression you cannot integrate into one you can, such as $a + b\cos 2x$. The integrating itself is in the Integration topic.

- **Triple angles.** To write $\cos 3\theta$ or $\sin 3\theta$ in terms of $\cos\theta$ or $\sin\theta$, expand $\cos(2\theta+\theta)$, then use the double-angle formulas and $\sin^2\theta = 1-\cos^2\theta$.

3. A trigonometric ratio as the unknown in an algebra or logarithm question (15 questions)

The first parts are from another topic: find a polynomial's unknown coefficients and factorise it, or solve a modulus or logarithm equation. The last part replaces x by a ratio, such as $p(\operatorname{cosec}\theta)=0$ or $|2\sec\theta-3| = \dots$

1. **Use your earlier answer**: the roots of $p(x)=0$ are the values the ratio can take. Don't start again.
2. **Set the ratio equal to each root** and turn it into $\sin$, $\cos$ or $\tan$.
3. **Reject impossible values.** $\sec\theta$ and $\operatorname{cosec}\theta$ must be $\geqslant 1$ or $\leqslant -1$; $\sin\theta$ and $\cos\theta$ must be between -1 and 1. $\tan\theta$ and $\cot\theta$ can be anything.
4. **Solve in the given interval**, in radians if the interval uses π.

For example, if $p(x)=0$ has roots -3, $\frac{1}{2}$ and 2, then $p(\operatorname{cosec}\theta)=0$ for $-90^\circ<\theta<90^\circ$ gives:

- $\operatorname{cosec}\theta=-3$, so $\sin\theta=-\frac{1}{3}$ and $\theta=-19.5^\circ$;
- $\operatorname{cosec}\theta=\frac{1}{2}$ is impossible (it would need $\sin\theta=2$);
- $\operatorname{cosec}\theta=2$, so $\sin\theta=\frac{1}{2}$ and $\theta=30^\circ$.

Variations you will meet:

- **A multiple angle**, such as $\cot 2\theta=5$: $\tan 2\theta=\frac{1}{5}$, so $2\theta=0.1974$ and $\theta=0.0987$ is the least positive value.
- **A scaled ratio**, such as $p(2\operatorname{cosec}\theta)=0$: set the whole of $2\operatorname{cosec}\theta$ equal to each root, then divide, and only then reject impossible values. With roots -3, 1 and 4 and $0^\circ<\theta<360^\circ$: $\operatorname{cosec}\theta=-\frac{3}{2}$ gives $\sin\theta=-\frac{2}{3}$, so $\theta=221.8^\circ$ or 318.2°; $\operatorname{cosec}\theta=\frac{1}{2}$ is impossible; $\operatorname{cosec}\theta=2$ gives $\theta=30^\circ$ or 150°.
- **A squared ratio**, such as $p(\operatorname{cosec}^2\theta)=0$: a square can't be negative, so reject negative roots; then take **both** square roots, so the answers come in pairs. With roots -2 and 3 and $-90^\circ<\theta<90^\circ$: $\operatorname{cosec}^2\theta=-2$ is impossible; $\operatorname{cosec}^2\theta=3$ gives $\sin\theta=\pm\frac{1}{\sqrt{3}}$, so $\theta=\pm 35.3^\circ$.
- **Modulus or logarithm first**: solve the equation in x, then use the root with the ratio, for example $\cot y=\frac{1}{3}$ gives $\tan y=3$.

4. Express in R form, then use it (11 questions)

1. **Write the expansion** of the form you are given, for example $R\cos(\theta-\alpha)=R\cos\theta\cos\alpha+R\sin\theta\sin\alpha$.
2. **Match coefficients** to get $R\cos\alpha$ and $R\sin\alpha$ (both positive when the form is chosen to fit, as in the table in "Understand it").
3. $R=\sqrt{a^2+b^2}$, exact: $\sqrt{45}=3\sqrt{5}$ is fine if the question says "exact".

4. $\tan\alpha = \dfrac{R\sin\alpha}{R\cos\alpha}$, and give α in the units of the question's interval ($0^\circ < \alpha < 90^\circ$ means degrees; $0 < \alpha < \frac{\pi}{2}$ means radians) to the accuracy asked.

Then use it:

- **Solve** $a\sin\theta + b\cos\theta = k$: write $R\sin(\theta+\alpha) = k$, so $\sin(\theta+\alpha) = \frac{k}{R}$. Change the interval: for $0^\circ < \theta < 360^\circ$, the bracket runs from α to $360^\circ + \alpha$. Find **every** value of the bracket in that interval (a negative calculator value may need 360° added), then subtract α.
- **Greatest and least values** of $R\sin(\theta+\alpha)+c$ are $c+R$ and $c-R$. The greatest happens when $\sin(\theta+\alpha) = 1$ (bracket $= 90^\circ$), the least when $\sin(\theta+\alpha) = -1$ (bracket $= 270^\circ$, or -90°). For $R\cos(\theta-\alpha)$: greatest when the bracket is 0° (or 360°), least when it is 180°.
- **A fraction** such as $\dfrac{40}{\ldots+25}$ is **greatest when its denominator is least**, and the other way round.
- **A square**, such as $(a\sin\theta + b\cos\theta)^2 + c$, is greatest when the bracket is R or $-R$, giving $R^2 + c$.
- **A related expression with a different angle or a factor**, such as $3\sin 4\beta - 4\cos 4\beta$ after you have done $6\sin\theta - 8\cos\theta$: it is half of the part (a) expression with $\theta = 4\beta$, so it equals $\frac{R}{2}\sin(4\beta - \alpha)$. Solve for 4β, then divide by 4.
- **Expand first.** If the expression is a product such as $10\sin\theta\cos\theta - 24\sin^2\theta$ and the form has 2θ in it, use double angles: $5\sin 2\theta - 12(1-\cos 2\theta) = 5\sin 2\theta + 12\cos 2\theta - 12$, which is $13\sin(2\theta + 67.38^\circ) - 12$. Brackets like $\sin(\theta + 30^\circ)$ need the compound-angle formula first. The form has a constant k added when a lone number appears.
- **Hence integrate** $\dfrac{1}{(a\cos x + b\sin x)^2} = \dfrac{1}{R^2}\sec^2(x-\alpha)$: the integrating is in the Integration topic.

5. Equations with sec, cosec and cot (9 questions)

1. **Pick the identity that leaves one ratio.** $\tan^2\theta$ with $\sec\theta$: replace $\tan^2\theta$ by $\sec^2\theta - 1$. $\cot^2\theta$ with $\operatorname{cosec}\theta$: replace $\cot^2\theta$ by $\operatorname{cosec}^2\theta - 1$. Keep any number in front: $5\tan^2\theta$ becomes $5\sec^2\theta - 5$.
2. **Rearrange to a three-term quadratic** in that ratio and solve it (factorise or use the formula).
3. **Turn each root round**: $\sec\theta = k$ means $\cos\theta = \frac{1}{k}$. Reject values with $|\sec\theta| < 1$ or $|\operatorname{cosec}\theta| < 1$.
4. **Find every angle** in the interval.

For example, $3\cot^2\theta - 5\operatorname{cosec}\theta + 1 = 0$ becomes $3(\operatorname{cosec}^2\theta - 1) - 5\operatorname{cosec}\theta + 1 = 0$, that is $3\operatorname{cosec}^2\theta - 5\operatorname{cosec}\theta - 2 = 0$, so $(3\operatorname{cosec}\theta + 1)(\operatorname{cosec}\theta - 2) = 0$. $\operatorname{cosec}\theta = -\frac{1}{3}$ is impossible, and $\operatorname{cosec}\theta = 2$ gives $\sin\theta = \frac{1}{2}$: $\theta = 30^\circ$ or 150° in $0^\circ < \theta < 360^\circ$.

Variations you will meet:

- **First powers of several ratios**, such as $\tan\theta$, $\cot\theta$ and $\sec\theta$ together. Write them all with $\sin\theta$ and $\cos\theta$, multiply through by $\sin\theta\cos\theta$, then use $\cos^2\theta = 1 - \sin^2\theta$ to get a quadratic in $\sin\theta$.
- **One ratio equal to a multiple of another**, such as $\sec\theta = 4\operatorname{cosec}\theta$: $\frac{1}{\cos\theta} = \frac{4}{\sin\theta}$ gives $\tan\theta = 4$ straight away.
- **An identity from an earlier part** may turn a messier term (with $\operatorname{cosec} 2\theta$, say) into $\sec\theta$ or $\cos\theta$ first; then carry on as above.

6. Double-angle formulas to solve an equation or rewrite an expression (4 questions)

- $\sin 2x$ **with** $\sin x$ **or** $\cos x$: write $\sin 2x = 2\sin x\cos x$ and **factorise**; never divide by $\sin x$ or $\cos x$, or you lose the solutions where it is 0. For example, $\sin 2x + \cos x = 0$ gives $\cos x(2\sin x + 1) = 0$, so $\cos x = 0$ ($x = 90^\circ, 270^\circ$) or $\sin x = -\frac{1}{2}$ ($x = 210^\circ, 330^\circ$) in $0^\circ \leqslant x \leqslant 360^\circ$.
- $\cos 2x$ **with** $\sin x$ **or** $\cos x$: choose the form of $\cos 2x$ with the same ratio, then solve the quadratic.
- **Reciprocals multiplied together**, such as $\sec\theta\,\mathrm{cosec}\,\theta = 5$: $\frac{1}{\sin\theta\cos\theta} = 5$, so $\sin\theta\cos\theta = \frac{1}{5}$ and $\sin 2\theta = \frac{2}{5}$. For $0^\circ < \theta < 90^\circ$: $2\theta = 23.6^\circ$ or 156.4°, so $\theta = 11.8^\circ$ or 78.2°.
- **Rewriting before integrating**: squares of sin and cos become double angles, for example $6\cos^2 3x = 3 + 3\cos 6x$ and $4\sin^2 x\cos^2 x = \sin^2 2x = \frac{1}{2}(1 - \cos 4x)$; $\frac{1}{\cos^2 x}$ is $\sec^2 x$.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **DM1** method mark that needs the M mark before it.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

Solve the equation $2\tan^2\theta + 9 = 9\sec\theta$ for $-180^\circ < \theta < 180^\circ$. **[5]**

Replace $\tan^2\theta$ by $\sec^2\theta - 1$: **M1** for using the identity to get an equation in $\sec\theta$ only.

$$2(\sec^2\theta - 1) + 9 = 9\sec\theta \;\Rightarrow\; 2\sec^2\theta - 9\sec\theta + 7 = 0.$$

A1 for the correct three-term quadratic.

Factorise: $(2\sec\theta - 7)(\sec\theta - 1) = 0$, so $\sec\theta = \frac{7}{2}$ or $\sec\theta = 1$. **DM1** for solving the quadratic and turning sec into cos to find an angle.

- $\sec\theta = \frac{7}{2}$: $\cos\theta = \frac{2}{7}$, so $\theta = 73.4^\circ$. Cosine is also positive in the fourth quadrant, and -73.4° is in the interval. **A1** for $\pm 73.4^\circ$.
- $\sec\theta = 1$: $\cos\theta = 1$, so $\theta = 0^\circ$. **A1** for 0°, with no other solutions in the interval.

$\theta = -73.4^\circ$, 0° or 73.4°.

Example 2

(a) The angle θ is obtuse and $\tan\theta = -\frac{3}{4}$. Find the exact value of $\cos(\theta + 60^\circ)$. **[3]**

(b) Solve the equation $2\sin(x+60^\circ) = 5\cos(x+30^\circ)$ for $0^\circ < x < 360^\circ$. **[4]**

(a) A 3–4–5 triangle gives the sizes. θ is in the second quadrant, where sine is positive and cosine negative, so $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$. **B1**

$$\cos(\theta+60^\circ) = \cos\theta\cos 60^\circ - \sin\theta\sin 60^\circ = -\frac{4}{5}\cdot\frac{1}{2} - \frac{3}{5}\cdot\frac{\sqrt{3}}{2}.$$

M1 for the correct expansion with the values put in.

$$= -\frac{4+3\sqrt{3}}{10}.$$

A1

(b) Expand both sides: **B1** for one correct expansion.

$$2\left(\tfrac{1}{2}\sin x + \tfrac{\sqrt{3}}{2}\cos x\right) = 5\left(\tfrac{\sqrt{3}}{2}\cos x - \tfrac{1}{2}\sin x\right) \Rightarrow \sin x + \sqrt{3}\cos x = \tfrac{5\sqrt{3}}{2}\cos x - \tfrac{5}{2}\sin x.$$

Collect terms: $\frac{7}{2}\sin x = \frac{3\sqrt{3}}{2}\cos x$, so dividing by $\cos x$,

$$\tan x = \frac{3\sqrt{3}}{7} = 0.7423\ldots$$

M1 for expanding both sides and reaching $\tan x = k$; **A1** for $\tan x = \frac{3\sqrt{3}}{7}$.

$x = 36.6^\circ$ or $180^\circ + 36.6^\circ = 216.6^\circ$. **A1** for both, and no others.

Example 3

(a) Prove that $2\operatorname{cosec} 2\theta - \cot\theta \equiv \tan\theta$. **[3]**

(b) Hence solve the equation $2\operatorname{cosec} 4x - \cot 2x = 3\cot 2x$ for $0 < x < \pi$, giving your answers in terms of π. **[4]**

(a) Start from the left-hand side and write everything with $\sin\theta$ and $\cos\theta$:

$$2\operatorname{cosec} 2\theta - \cot\theta = \frac{2}{2\sin\theta\cos\theta} - \frac{\cos\theta}{\sin\theta}.$$

B1 for $\sin 2\theta = 2\sin\theta\cos\theta$ (with $\operatorname{cosec} = \frac{1}{\sin}$).

$$= \frac{1-\cos^2\theta}{\sin\theta\cos\theta}$$

M1 for a common denominator.

$$= \frac{\sin^2\theta}{\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

A1, the given answer with the use of $1-\cos^2\theta = \sin^2\theta$ shown.

(b) The left-hand side is the identity with θ replaced by $2x$, so it equals $\tan 2x$: **M1** for using the identity with $2x$.

$$\tan 2x = 3\cot 2x = \frac{3}{\tan 2x} \;\Rightarrow\; \tan^2 2x = 3 \;\Rightarrow\; \tan 2x = \pm\sqrt{3}.$$

A1

For $0 < x < \pi$, the bracket $2x$ runs over $0 < 2x < 2\pi$. $\tan 2x = \sqrt{3}$ gives $2x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$; $\tan 2x = -\sqrt{3}$ gives $2x = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$. **M1** for finding values of $2x$ over the doubled interval.

$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}.$$

A1 for all four, and no others.

Example 4

(a) Express $8\sin\theta - 15\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. Give the value of α correct to 2 decimal places. **[3]**

(b) Hence solve the equation $8\sin\theta - 15\cos\theta = 6$ for $0^\circ < \theta < 360^\circ$. **[4]**

(c) Find the greatest value of $\dfrac{40}{8\sin\theta - 15\cos\theta + 25}$ as θ varies, and the value of θ in $0^\circ < \theta < 360^\circ$ at which it occurs. **[3]**

(a) $R = \sqrt{8^2 + 15^2} = 17$. **B1**

$R\sin(\theta - \alpha) = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$, so $R\cos\alpha = 8$ and $R\sin\alpha = 15$, and $\tan\alpha = \frac{15}{8}$. **M1**

$\alpha = 61.93^\circ$, so $8\sin\theta - 15\cos\theta = 17\sin(\theta - 61.93^\circ)$. **A1**

(b) $17\sin(\theta - 61.93^\circ) = 6$, so $\sin(\theta - 61.93^\circ) = \frac{6}{17}$. The bracket runs from -61.93° to 298.07°.

$\theta - 61.93^\circ = 20.67^\circ$, so $\theta = 82.6^\circ$. **M1 A1**

Sine is also positive in the second quadrant: $\theta - 61.93^\circ = 180^\circ - 20.67^\circ = 159.33^\circ$, so $\theta = 221.3^\circ$. **M1 A1**, with no others in the interval.

(c) The fraction is greatest when the denominator $17\sin(\theta - 61.93^\circ) + 25$ is least, which is $25 - 17 = 8$. The greatest value is $\frac{40}{8} = 5$. **B1**

This happens when $\sin(\theta - 61.93^\circ) = -1$, that is $\theta - 61.93^\circ = 270^\circ$ (the value -90° gives a negative θ). **M1**

$\theta = 331.9^\circ$. **A1**

The second diagram in "Two waves make one" shows this wave: its lowest point is at 331.9°.

Common mistakes

- **Mixing up the reciprocals.** $\sec\theta$ is $\frac{1}{\cos\theta}$, not $\frac{1}{\sin\theta}$. Examiners saw this swap in a question where everything else was right. Use the third-letter rule.

- **Not saying which identity you used.** Jumping straight to a quadratic in $\sec\theta$ with no sign of $\tan^2\theta = \sec^2\theta - 1$ or $\sec\theta = \frac{1}{\cos\theta}$ can lose marks, even though the identity is in MF19. Write it down.
- **Dropping the number in front.** $5\tan^2\theta$ becomes $5(\sec^2\theta - 1) = 5\sec^2\theta - 5$, not $5\sec^2\theta - 1$. Sign slips here are common too.
- **Using a calculator for an "exact value".** Finding θ on a calculator and putting it into $\tan\left(\theta + \frac{\pi}{4}\right)$ earns nothing; the question tests the identities. Get $\tan\theta$ exactly, then expand.
- **Multiplying every bracket by the number in front.** $4\sin(\ldots)\cos(\ldots)$ means $4\times$ (first bracket) $\times$ (second bracket), not 4 times each bracket.
- **Missing solutions.** With $\tan 2\theta$ or a bracket like $\theta - 53.13^\circ$, students often give one answer. List the angles for the bracket over the **whole** changed interval, including negative ones, before rearranging.
- **Giving an answer outside the interval**, such as 372.5° for $0^\circ < \theta < 360^\circ$. Check each final answer against the interval.
- **Degrees and radians mixed.** If the interval is in degrees, α must be in degrees; if it uses π, work in radians. Several students gave α in radians when degrees were asked for.
- **Ignoring the double angle in the form.** If you must write an expression with $\sin\theta\cos\theta$ and $\cos^2\theta$ as $R\cos(2\theta - \alpha) + k$, the 2θ tells you to change to $\sin 2\theta$ and $\cos 2\theta$ first.
- **Starting again instead of using "hence".** When part (b) puts $\cot y$ or $\operatorname{cosec}\theta$ where x was, the roots from part (a) are the values of $\cot y$ or $\operatorname{cosec}\theta$. Likewise, a greatest-value part needs the R form from part (a).
- **Forgetting that the angle changed.** Using an identity proved for θ on an expression in $2x$ gives an answer in $2x$, not x.
- **Truncating instead of rounding.** $41.87\ldots^\circ$ is 41.9°, not 41.8°.

How to get the marks

- **Show the identity step.** Method marks are for "use of a correct identity": write the formula with your angles, then the simplified form.
- **"Prove" or "show that"**: the answer is printed, so every step must be there, including the use of $\sin^2\theta + \cos^2\theta = 1$ or $2\sin\theta\cos\theta = \sin 2\theta$. Work from one side to the other. Working backwards from the given answer, or changing lines to make them fit, loses marks.
- **"Hence"** means use the previous part. Show how: say which value of θ you substitute, or write the equation in the form from part (a).
- **"Exact"** means surds, fractions and π, with no calculator decimals. R as a surd ($\sqrt{45}$ or $3\sqrt{5}$) is exact.
- **"Solve ... for $0^\circ < \theta < 360^\circ$"**: the last accuracy mark needs **all** the solutions in the interval **and no others**. Extra answers inside the interval lose it; answers outside it are ignored but waste time.
- **Accuracy:** angles in degrees to 1 decimal place, radians to 3 significant figures, unless the question asks for more (such as α to 2 decimal places). Keep more figures in your working and round only at the end.
- **Greatest or least value questions** usually have a mark for the value and one for the angle: find both, and give the angle asked for (for example the smallest positive one).
- **Follow-through.** Later parts are often marked on your R and α, so use them even if you doubt them.

Quick check

1. The angle θ is obtuse and $\sin\theta = \frac{2}{3}$. Find the exact values of $\cot\theta$ and $\sec 2\theta$.
2. Solve $\operatorname{cosec}^2\theta = 3\cot\theta + 5$ for $0^\circ < \theta < 180^\circ$.
3. Prove that $\sin(x + 60^\circ) + \sin(x - 60^\circ) \equiv \sin x$.
4. Express $4\cos x - 3\sin x$ in the form $R\cos(x+\alpha)$, with $R > 0$ and $0^\circ < \alpha < 90^\circ$. Hence find the least value of $10 + 4\cos x - 3\sin x$ and the value of x in $0^\circ < x < 360^\circ$ where it occurs.
5. Solve $\cos 2x + 5\sin x = 3$ for $0 < x < 2\pi$.

Answers

1. $\cos\theta$ is negative (second quadrant): $\cos\theta = -\sqrt{1-\frac{4}{9}} = -\frac{\sqrt{5}}{3}$. So $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = -\dfrac{\sqrt{5}}{2}$. $\cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{8}{9} = \frac{1}{9}$, so $\sec 2\theta = 9$.
2. $1+\cot^2\theta = 3\cot\theta + 5$, so $\cot^2\theta - 3\cot\theta - 4 = 0$ and $(\cot\theta - 4)(\cot\theta + 1) = 0$. $\cot\theta = 4$ gives $\tan\theta = \frac{1}{4}$, $\theta = 14.0^\circ$; $\cot\theta = -1$ gives $\tan\theta = -1$, $\theta = 135^\circ$.
3. $\sin(x+60^\circ) + \sin(x-60^\circ) = \left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right) + \left(\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right) = \sin x$.
4. $R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$, so $R\cos\alpha = 4$, $R\sin\alpha = 3$: $R = 5$, $\tan\alpha = \frac{3}{4}$, $\alpha = 36.87^\circ$. So $10 + 5\cos(x+36.87^\circ)$ has least value $10-5=5$, when $x + 36.87^\circ = 180^\circ$, that is $x = 143.1^\circ$.
5. $1 - 2\sin^2 x + 5\sin x = 3$, so $2\sin^2 x - 5\sin x + 2 = 0$ and $(2\sin x - 1)(\sin x - 2) = 0$. $\sin x = 2$ is impossible; $\sin x = \frac{1}{2}$ gives $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

Past questions to try

- **9709/22/O/N/22 Q1**: one reciprocal ratio equal to a multiple of another.
- **9709/23/M/J/21 Q3**: compound angles with 2θ inside the brackets, two answers from $\tan 2\theta$.
- **9709/21/M/J/23 Q7**: $R\cos(\theta-\alpha)$, solving, and the greatest value of a fraction.
- **9709/21/O/N/24 Q4**: a polynomial's roots used as values of $\operatorname{cosec}^2\theta$, with answers in $\pm$ pairs.