Cambridge AS & A Level · Mathematics (9709) · Notes

Trigonometry

A2

Read online at pastopic.com/cambridge/mathematics-9709/notes/trigonometry-p3

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Read first: Trigonometry, Trigonometry

What you need to know

By the end of this topic you should be able to:

1. **Work with all six trigonometric functions.** Know that $\sec\theta = \frac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ and $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$, and use the graphs and properties of $\sin$, $\cos$, $\tan$, $\sec$, cosec and $\cot$ for angles of any size, positive or negative, in degrees or radians.
2. **Use identities to simplify expressions, find exact values and solve equations**, choosing the identity (or several) that suits the question. In particular:
   - $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$;
   - the compound-angle formulas for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$;
   - the double-angle formulas for $\sin 2A$, $\cos 2A$ and $\tan 2A$;
   - writing $a\sin\theta + b\cos\theta$ as $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$.

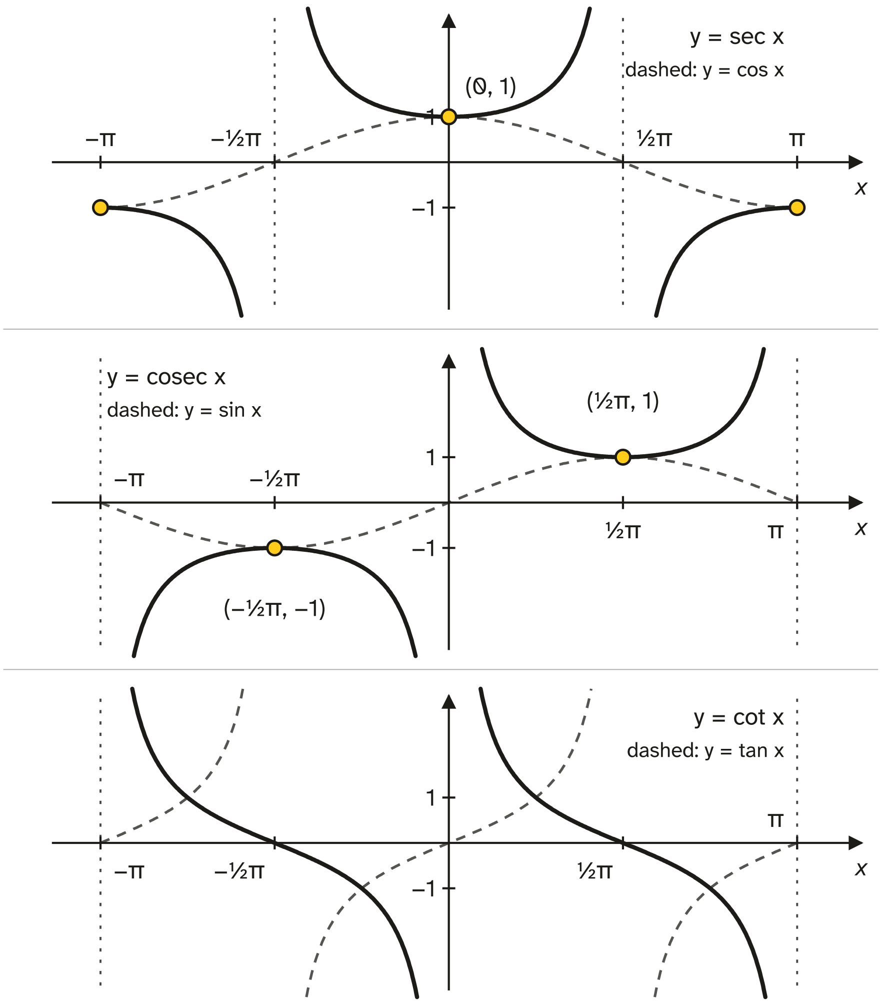
This is the same syllabus content as the Paper 2 Trigonometry note ("Read first"), which works through it slowly with many examples. This note covers every idea again, more briefly, and shows how Paper 3 asks it: more often in radians, with angles such as 3θ, $4x$ and $\frac{x}{2}$, longer identities with powers such as $\cos^4\theta$, and identities used to set up an integral. Factor formulas (such as $\sin P + \sin Q$) and the $t = \tan\frac{1}{2}\theta$ formulas are **not** in this topic; when you need something like them, you build it from the compound-angle formulas.

Understand it

Six functions

The three reciprocal functions are "one over" the three you know. The third letter tells you which: **sec** goes with **c**os, **cos**ec with **s**in, **cot** with **t**an. A calculator has no buttons for them, so turn the equation round first: $\sec\theta = -3$ means $\cos\theta = -\frac{1}{3}$.

Each reciprocal graph comes from its partner. Where the partner is ± 1, the reciprocal is the same; where the partner is small, the reciprocal is large; where the partner is 0, the reciprocal is undefined and its graph has a vertical asymptote.



What the graphs tell you:

- $\sec x$ and $\operatorname{cosec} x$ repeat every 2π (360°); $\cot x$, like $\tan x$, repeats every π (180°).
- $\sec x$ and $\operatorname{cosec} x$ are **never between** -1 **and** 1, so $\operatorname{cosec}\theta = \frac{1}{4}$ has no solution. $\cot x$ takes every value.
- $\sec x$ is undefined where $\cos x = 0$ ($x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$); $\operatorname{cosec} x$ and $\cot x$ are undefined where $\sin x = 0$ ($x = 0, \pm\pi, \ldots$).
- $\sec(-x) = \sec x$, like cos; $\operatorname{cosec}(-x) = -\operatorname{cosec} x$ and $\cot(-x) = -\cot x$, like $\sin$ and $\tan$.

"Angles of any size" works as in the Paper 1 note: the calculator gives one angle (the principal value), and the symmetry of the graph gives the others. $\sin$ repeats every 2π and $\sin(\pi - x) = \sin x$; $\cos$ repeats every 2π and $\cos(-x) = \cos x$; $\tan$ repeats every π.

Two more Pythagorean identities

Divide $\sin^2\theta + \cos^2\theta \equiv 1$ by $\cos^2\theta$ to get $\tan^2\theta + 1 \equiv \sec^2\theta$, or by $\sin^2\theta$ to get $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$.

Use them to turn an equation with two different functions into one with a single function. $\cot\theta$ with $\operatorname{cosec}^2\theta$: replace $\operatorname{cosec}^2\theta$ by $1 + \cot^2\theta$, and you have a quadratic in $\cot\theta$. $\tan^2\theta$ with $\sec\theta$: replace $\tan^2\theta$ by $\sec^2\theta - 1$.

Compound angles

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

In the $\cos$ formula the sign flips, and in the $\tan$ formula the bottom sign is the opposite of the top one. $\cos(x - 45^\circ)$ is **not** $\cos x - \cos 45^\circ$.

They give exact values of new angles, such as

$$\tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3},$$

and they break a bracket such as $\sin\left(\theta + \frac{\pi}{6}\right)$ into $\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$, so that an equation has only $\sin\theta$ and $\cos\theta$ in it.

Double angles, and angles built from them

Put $B = A$ in the compound-angle formulas:

$$\sin 2A \equiv 2\sin A \cos A, \qquad \tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A},$$
$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A.$$

Pick the form of $\cos 2A$ that matches the rest of the equation: $1 - 2\sin^2 A$ when the other terms have sin, $2\cos^2 A - 1$ when they have cos. Rearranged, they remove a square: $\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A)$ and $\sin^2 A \equiv \frac{1}{2}(1 - \cos 2A)$.

"Double" means double **whatever angle is there**, and Paper 3 uses this all the time:

- **Half angles:** $\cos x = 2\cos^2\frac{x}{2} - 1$, because x is double $\frac{x}{2}$.
- **Quadruple angles:** $\cos 4x = 2\cos^2 2x - 1$, then use the formula again on $\cos 2x$. For example $\cos 4x = 2(2\cos^2 x - 1)^2 - 1 = 8\cos^4 x - 8\cos^2 x + 1$. In the same way $\tan 4\theta = \frac{2\tan 2\theta}{1 - \tan^2 2\theta}$, and then replace $\tan 2\theta$.
- **Triple angles:** write $3\theta = 2\theta + \theta$, expand with the compound-angle formula, then replace $\sin 2\theta$ and $\cos 2\theta$ (Example 2 does this).

Two patterns with squares turn up in identities with powers:

$$(\cos\theta \pm \sin\theta)^2 = \cos^2\theta + \sin^2\theta \pm 2\sin\theta\cos\theta = 1 \pm \sin 2\theta,$$

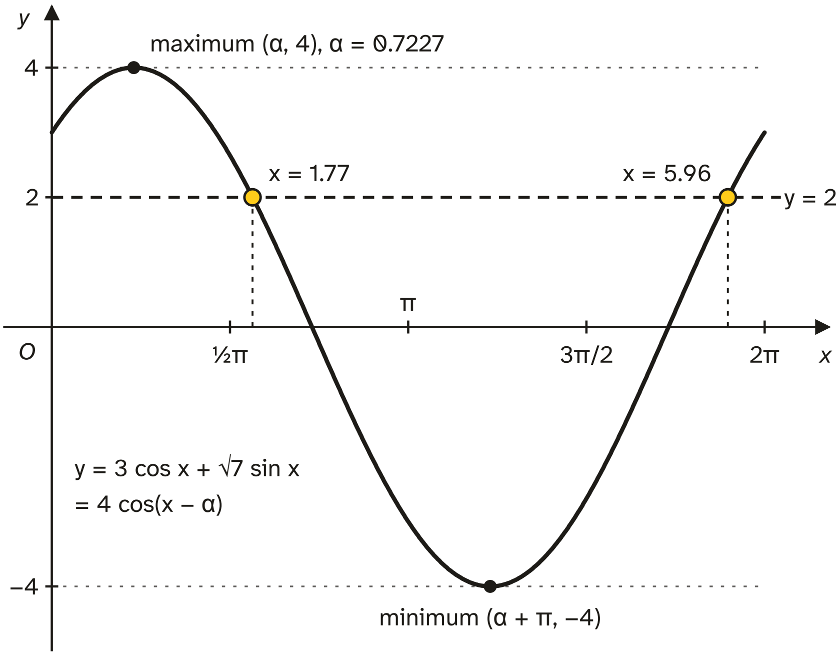
and a **difference of two squares**: $\cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta) = \cos 2\theta \times 1$. When you see fourth powers, look for one of these, or for $(\cos^2\theta + \sin^2\theta)^2 = 1$.

One wave: the R form

A sine wave plus a cosine wave of the same period is a single wave, shifted. For example $3\cos x + \sqrt{7}\sin x$ can be written $R\cos(x-\alpha)$. Expand the form and match coefficients:

$$R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha \quad\Rightarrow\quad R\cos\alpha = 3, \quad R\sin\alpha = \sqrt{7}.$$

Squaring and adding (Pythagoras) gives $R^2 = 3^2 + (\sqrt{7})^2 = 16$, so $R = 4$. Dividing gives $\tan\alpha = \frac{\sqrt{7}}{3}$, so $\alpha = 0.7227$ radians. So $3\cos x + \sqrt{7}\sin x \equiv 4\cos(x - 0.7227)$.



The single wave makes two jobs easy.

- **Solving.** $3\cos x + \sqrt{7}\sin x = 2$ becomes $\cos(x - 0.7227) = \frac{1}{2}$. For $0 \leqslant x \leqslant 2\pi$ the bracket runs from -0.7227 to $2\pi - 0.7227$, and in that range $x - 0.7227 = \frac{\pi}{3}$ or $-\frac{\pi}{3}$. So $x = 1.77$ or $x = -0.324$; the second is outside the interval, so add 2π: $x = 5.96$.
- **Greatest and least values.** cos is between -1 and 1, so $4\cos(x-\alpha)$ is between -4 and 4. The greatest value happens when the bracket is 0 ($x = \alpha$), the least when the bracket is π ($x = \alpha + \pi = 3.86$).

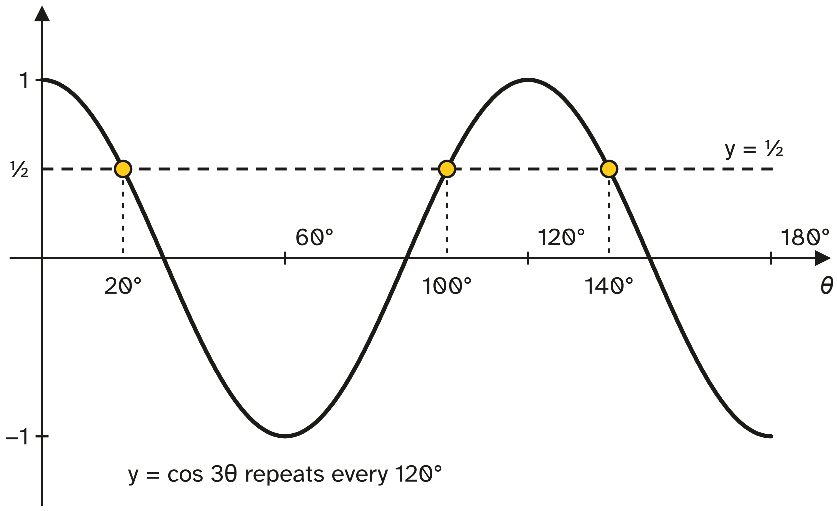
The four forms, for $a, b > 0$ (the question always says which to use):

Expression	Write as	Match
$a\sin\theta + b\cos\theta$	$R\sin(\theta+\alpha)$	$R\cos\alpha = a$, $R\sin\alpha = b$
$a\sin\theta - b\cos\theta$	$R\sin(\theta-\alpha)$	$R\cos\alpha = a$, $R\sin\alpha = b$
$a\cos\theta + b\sin\theta$	$R\cos(\theta-\alpha)$	$R\cos\alpha = a$, $R\sin\alpha = b$
$a\cos\theta - b\sin\theta$	$R\cos(\theta+\alpha)$	$R\cos\alpha = a$, $R\sin\alpha = b$

In each case $R = \sqrt{a^2+b^2}$ and $\tan\alpha = \frac{b}{a}$, with α acute.

Multiple angles have more solutions

$\cos 3\theta$ goes through a whole cycle every 120°, three times as fast as $\cos\theta$. So on $0^\circ \leqslant \theta \leqslant 180^\circ$, the equation $\cos 3\theta = \frac{1}{2}$ has three solutions, not one or two.



The safe method: **stretch the interval first.** If $0^\circ \leqslant \theta \leqslant 180^\circ$, then $0^\circ \leqslant 3\theta \leqslant 540^\circ$. Solve $\cos 3\theta = \frac{1}{2}$ for 3θ in that longer interval: $3\theta = 60^\circ, 300^\circ, 420^\circ$. Only then divide by 3: $\theta = 20^\circ, 100^\circ, 140^\circ$. For $\frac{x}{2}$ it works the other way: $0 \leqslant x \leqslant 2\pi$ means $0 \leqslant \frac{x}{2} \leqslant \pi$, a **shorter** interval with fewer solutions.

Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\sec\theta = \dfrac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$	Learn it
$\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ and $\cos^2\theta + \sin^2\theta \equiv 1$	Given
$1 + \tan^2\theta \equiv \sec^2\theta$ and $\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$	Given
$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$	Given
$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$	Given
$\tan(A \pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$	Given
$\sin 2A \equiv 2\sin A\cos A$	Given
$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$	Given
$\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$	Given
$\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A)$ and $\sin^2 A \equiv \frac{1}{2}(1 - \cos 2A)$	Learn it
$(\cos\theta \pm \sin\theta)^2 \equiv 1 \pm \sin 2\theta$	Learn it

Formula	Status
R form: $R = \sqrt{a^2 + b^2}$, $\tan\alpha = \dfrac{b}{a}$ (from matching $R\cos\alpha$ and $R\sin\alpha$)	Learn it
$R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$ lie between $-R$ and R	Learn it
sec and cosec: period 2π, never between -1 and 1; cot: period π	Learn it
Exact values of $\sin$, $\cos$, $\tan$ at $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ $(30^\circ, 45^\circ, 60^\circ)$	Learn it
Principal values: $-\frac{\pi}{2} \leqslant \sin^{-1} x \leqslant \frac{\pi}{2}$, $0 \leqslant \cos^{-1} x \leqslant \pi$, $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$	Given

Having a formula in MF19 does not mean you can skip it: write the identity with your angles in it before you simplify.

How to do it

In the Paper 3 questions we have tagged for this topic (44 questions, 2021–2025), the types came up this often. 19 of the 44 questions mix types, so the counts add up to more than 44.

Question type	Questions
Prove an identity, or show an equation can be rewritten	18
Solve an equation using double-angle formulas	16
Express in the form $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$, then use it	13
Solve an equation using compound-angle formulas	7
Use an identity to find an integral	6
Equations needing $\sec^2\theta = 1 + \tan^2\theta$ or $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$	3
A polynomial equation in $\cos\theta$	1

Every equation ends with lines like $\tan x = k$ or $\sin(2x + \alpha) = k$. Finish them as in "Multiple angles have more solutions": stretch the interval, find **every** value of the bracket in it, then rearrange. Work in radians when the interval has π in it, and in degrees when it has $^\circ$.

1. Prove an identity, or show an equation can be rewritten (18 questions)

1. **Start from one side**, usually the more complicated one, and work to the other. Never work on both sides at once.
2. **Write everything in $\sin$ and $\cos$ of single angles**: replace $\tan$, $\sec$, cosec, $\cot$ by their definitions, expand compound angles, and break multiple angles down step by step ($4x$ to $2x$ to x).
3. **Simplify**: common denominators, expanding brackets, $\sin^2\theta + \cos^2\theta = 1$, and the square patterns $(\cos\theta \pm \sin\theta)^2 = 1 \pm \sin 2\theta$ and $\cos^2\theta - \sin^2\theta = \cos 2\theta$. When the target has $\sin 2\theta$ or $\cos 2\theta$ in it, build them back up.
4. **Finish on exactly the printed expression.**

For example, $(\cos\theta+\sin\theta)^2-(\cos\theta-\sin\theta)^2=(1+\sin 2\theta)-(1-\sin 2\theta)=2\sin 2\theta$.

Variations you will meet:

- **Multiple angles in terms of single ones**: $\cos 4x$ in terms of $\sin x$ or $\cos x$; $\sin 4x$ in terms of $\sin x$ and $\cos x$ ($\sin 4x=2\sin 2x\cos 2x$, then expand each); $\cos 3\theta$ or $\sin 3\theta$ by writing $3\theta=2\theta+\theta$; $\tan 4\theta$ as $\tan(2\theta+2\theta)$. Use brackets when you square: $(2\cos^2\theta-1)^2=4\cos^4\theta-4\cos^2\theta+1$.
- **Fourth powers**: factorise as a difference of two squares ($\sec^4\theta-\tan^4\theta=(\sec^2\theta-\tan^2\theta)(\sec^2\theta+\tan^2\theta)$, and the first bracket is 1), or start from $(\cos^2\theta+\sin^2\theta)^2=1$ and expand it.
- **Reciprocal functions of 2θ**, such as $\operatorname{cosec} 2\theta$ or $\cot 2\theta$: write $\frac{1}{\sin 2\theta}$ and $\frac{\cos 2\theta}{\sin 2\theta}$, put them over the common denominator $\sin 2\theta$, then use the double-angle formulas. Fractions such as $\frac{1-\cos 2\theta}{1+\cos 2\theta}$ become $\frac{2\sin^2\theta}{2\cos^2\theta}$.
- **A sum from two expansions**: to show a result such as $\sin(2x+x)+\sin(2x-x)=2\sin 2x\cos x$, expand both brackets and add; the $\cos\ldots\sin$ terms cancel.
- **"Show that the equation can be written as ..."**: the target is usually a polynomial in one function, such as a quadratic in $\tan x$ or a quartic in $\sin\theta$. Replace every function by that one, **multiply through to clear every fraction**, and collect terms. A mixture like $\cos^2\theta+2\sin\theta\cos\theta-3\sin^2\theta=0$ counts as an answer in $\sin\theta$ and $\cos\theta$; divide it by $\cos^2\theta$ later to get a quadratic in $\tan\theta$.
- **"Hence solve"** comes next in most of these questions; see type 2 for the equations and type 5 for integrals.

2. Solve an equation using double-angle formulas (16 questions)

1. **Choose the formula that leaves one function of one angle.** $\cos 2\theta$ with $\sin\theta$: use $1-2\sin^2\theta$. $\tan 2x$ or $\cot 2x$ with $\tan x$ or $\cot x$: use $\tan 2x=\frac{2\tan x}{1-\tan^2 x}$, and $\cot x=\frac{1}{\tan x}$. $\cos x$ with $\cos\frac{x}{2}$: use $\cos x=2\cos^2\frac{x}{2}-1$.
2. **Clear the fractions**: multiply every term by the full denominator, such as $2\tan x(1-\tan^2 x)$.
3. **Rearrange to a polynomial = 0** and solve it. **Factorise out** a common $\sin\theta$ or $\tan x$ instead of dividing by it, or you lose solutions.
4. **Find every angle** in the interval, and reject impossible values ($\sin\theta=-1.44$, say).

For example, $\tan 2x=4\tan x$ for $0^\circ\leqslant x\leqslant 180^\circ$: with $t=\tan x$, $\frac{2t}{1-t^2}=4t$, so $2t=4t(1-t^2)$, which is $2t(2t^2-1)=0$. $t=0$ gives $x=0^\circ$ or 180°; $t^2=\frac{1}{2}$ gives $\tan x=\pm 0.7071$, so $x=35.3^\circ$ or 144.7°. Dividing by t at the start would have lost 0° and 180°.

Variations you will meet:

- $\cos 2\theta$ **with** $\sin\theta$ **or** $\cos\theta$: a quadratic. Both roots may give answers, and a negative $\cos\theta$ means the second and third quadrants.
- $\tan 2x$ **or** $\cot 2x$ **with** $\cot x$: after clearing fractions you get a quadratic in $\tan x$. A negative root of $\tan x$ gives an answer in the second quadrant (180° minus the acute angle), so it still counts.
- **Half angles.** $\cos x=3\cos\frac{x}{2}-2$ for $0\leqslant x\leqslant 2\pi$: $2\cos^2\frac{x}{2}-1=3\cos\frac{x}{2}-2$, so $2c^2-3c+1=0$ with $c=\cos\frac{x}{2}$, and $c=1$ or $\frac{1}{2}$. Now $0\leqslant\frac{x}{2}\leqslant\pi$: $c=1$ gives $\frac{x}{2}=0$, and $c=\frac{1}{2}$ gives $\frac{x}{2}=\frac{\pi}{3}$. So $x=0$ or $\frac{2\pi}{3}$. Doubling at the end is essential; $x=2\pi$ is **not** a solution.

- **Fourth powers**, such as $6\sin^4 x - 5\sin^2 x = 0$ or $\tan^4 x - 4\tan^2 x + 3 = 0$: treat them as quadratics in $\sin^2 x$ or $\tan^2 x$ (take out a factor $\sin^2 x$ if there is one). Reject a negative value of a square, then take **both** square roots, $\pm$.
- **A cubic with a common factor**, such as $3\cos^3\theta + 2\cos^2\theta - \cos\theta = 0$: take out $\cos\theta$, then solve $\cos\theta = 0$ and the quadratic $3\cos^2\theta + 2\cos\theta - 1 = 0$ separately.
- **Only $\sin^2\theta$, $\sin\theta\cos\theta$ and $\cos^2\theta$** (every term of degree 2): divide by $\cos^2\theta$ to get a quadratic in $\tan\theta$. For example $2\sin^2\theta - 3\sin\theta\cos\theta - 2\cos^2\theta = 0$ gives $2t^2 - 3t - 2 = 0$, so $\tan\theta = 2$ or $-\frac{1}{2}$: $\theta = 63.4^\circ$ or 153.4° for $0^\circ < \theta < 180^\circ$.
- **"Hence solve" with a changed angle.** If part (a) proved an identity in θ and part (b) has the same expression in 2α (or 2θ), the identity holds with θ replaced by 2α. Solve for 2α over the doubled interval, then halve.
- **A squared function of a double angle**, such as $\sin^2 2\theta = \frac{3}{4}$: $\sin 2\theta = \pm\frac{\sqrt{3}}{2}$. With $0^\circ < \theta < 180^\circ$, 2θ runs over $(0^\circ, 360^\circ)$: the positive root gives $2\theta = 60^\circ, 120^\circ$ and the negative root $2\theta = 240^\circ, 300^\circ$, so $\theta = 30^\circ, 60^\circ, 120^\circ, 150^\circ$.
- **A single fourth power**, such as $2\cos^4\theta = 1$: $\cos\theta = \pm\sqrt[4]{\frac{1}{2}} = \pm 0.8409$, which gives one acute and one obtuse answer in $0^\circ \leqslant \theta \leqslant 180^\circ$: 32.8° and 147.2°.

3. Express in R form, then use it (13 questions)

1. **Write out the expansion** of the form you are given, for example $R\sin(\theta + \alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$.
2. **Match coefficients** to get $R\cos\alpha$ and $R\sin\alpha$.
3. $R = \sqrt{(R\cos\alpha)^2 + (R\sin\alpha)^2}$, as an exact surd if asked, such as $\sqrt{13}$.
4. $\tan\alpha = \dfrac{R\sin\alpha}{R\cos\alpha}$; give α in the units of the question ($0 < \alpha < \frac{\pi}{2}$ means radians) and to the accuracy asked.

Then use it:

- **Solve $R\sin(\theta + \alpha) = k$**: $\sin(\theta + \alpha) = \frac{k}{R}$. Change the interval to one for the bracket, find every value of the bracket in it, then subtract α.
- **Greatest and least values**: R and $-R$. For $R\cos(x - \alpha)$ the least value comes when the bracket is π (or 180°), not 0 or $\frac{\pi}{2}$; sketch the cosine curve if unsure. A **square** such as $(a\sin x - b\cos x)^2$ is greatest (R^2) when the bracket is $\pm R$, and least (0) when it is 0.

Variations you will meet:

- **Expand a compound angle first.** Expressions such as $k\sin\left(\theta + \frac{\pi}{6}\right) - m\sin\theta$ or $\cos x + k\cos(x - 60^\circ)$ must first become $a\sin\theta + b\cos\theta$: expand, put in the exact values, and collect.
- **Rearrange an equation first.** An equation with $\sec x$ and $\tan x$, such as $p\sec x + \tan x = q$, becomes $p + \sin x = q\cos x$ when you multiply by $\cos x$; then collect as $q\cos x - \sin x = p$.
- **A multiple or fraction of the angle** in the equation: 2θ, 3θ, $\frac{\theta}{2}$ or $\frac{x}{3}$. The bracket is then, say, $2x + \alpha$: find its range from the interval ($0 < x < \pi$ gives $\alpha < 2x + \alpha < 2\pi + \alpha$), list the values, subtract α and **then** divide by 2.

- **Wide or negative intervals**, such as $-4\pi < \theta < 4\pi$ with $\frac{1}{2}\theta$ in the bracket: the bracket's interval is $(-2\pi + \alpha, 2\pi + \alpha)$, and you may need four values. Adding or subtracting 2π from the calculator value finds them.
- **A multiple of the part (a) expression**: the equation may have every coefficient doubled (or halved) as well as a new angle, such as $2\left(a \sin \frac{\theta}{2} + b \cos \frac{\theta}{2}\right)$. Spot the factor and divide the equation by it first.
- **Exact α**: if $\tan \alpha = \frac{1}{\sqrt{3}}$, $\alpha = \frac{\pi}{6}$.
- **Then integrate** $\dfrac{1}{(a \cos 2x - b \sin 2x)^2} = \dfrac{1}{R^2} \sec^2(2x + \alpha)$: see type 5.

4. Solve an equation using compound-angle formulas (7 questions)

1. **Expand every bracket** with the formula and put in exact values ($\tan 45^\circ = 1$, $\tan 60^\circ = \sqrt{3}$, $\sin \frac{\pi}{6} = \frac{1}{2}$, and so on). Write $\cot x = \frac{1}{\tan x}$.
2. **Collect.** If you have only $\sin x$ and $\cos x$ to the first power, get $\tan x = k$. If the equation has $\tan(x \pm A)$ and $\cot x$, multiply out to get a **quadratic in** $\tan x$.
3. **Solve** and find every angle; $\tan$ repeats every 180°, so each root gives one answer in an interval of length 180°.

For example, $\cos(\theta + 30^\circ) = 2 \sin \theta$: $\frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta = 2 \sin \theta$, so $\frac{\sqrt{3}}{2} \cos \theta = \frac{5}{2} \sin \theta$ and $\tan \theta = \frac{\sqrt{3}}{5}$. For $0^\circ < \theta < 360^\circ$, $\theta = 19.1^\circ$ or 199.1°.

Variations you will meet:

- $\tan(x \pm 45^\circ)$ **or** $\tan(x \pm 60^\circ)$ **with** $\cot x$ (Example 1): a quadratic in $\tan x$. Replace $2 \cot x$ by $\frac{2}{\tan x}$, not $\frac{1}{2 \tan x}$. A negative root gives the obtuse answer: add 180° to the calculator's negative angle, don't reject it.
- **Brackets with $\frac{\pi}{6}$ or $\frac{\pi}{3}$ on both sides**: expand all four, collect, and find the **exact** value of $\tan x$ (it may be a surd fraction). Then solve in radians.
- **Show that $\tan x$ equals a given surd expression**, then solve: collect $\sin x$ and $\cos x$ terms on opposite sides, divide by $\cos x$, and tidy the fraction.
- **Two unknown angles**, such as $\tan(A + B) = k$ with $\tan A = m \tan B$: substitute $\tan A$ into the $\tan(A + B)$ formula to get a quadratic in $\tan B$, then find A from $\tan A$. Use the stated range (such as $0^\circ < A, B < 180^\circ$) to decide which roots and angles are allowed; both roots may give a pair.

5. Use an identity to find an integral (6 questions)

These are "prove, then hence integrate" questions: part (a) is type 1, and part (b) uses it to turn the integrand into something you can integrate. The integrating itself is in the Integration note.

1. **Replace the integrand** using part (a), with the right angle ($2x$ for x if needed).
2. **Aim for standard forms**: $\sin kx$, $\cos kx$, $\sec^2 kx$, $\tan x$, or a power of $\cos x$ times $\sin x$.
3. **Integrate and use exact values** at the limits, subtracting the right way round.

For example, $\tan^2 x = \sec^2 x - 1$, so $\displaystyle\int_0^{\pi/4} \tan^2 x \, dx = \Big[\tan x - x\Big]_0^{\pi/4} = 1 - \frac{\pi}{4}$.

Variations you will meet:

- A **product** such as $\sin 4x \cos x$ becomes $\frac{1}{2}(\sin 5x + \sin 3x)$, from adding the expansions of $\sin(4x + x)$ and $\sin(4x - x)$.
- **Squares** become double angles: $\sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta)$.
- **Powers times a sine**: $\sin 4x$ written with $\cos^3 x$ and $\cos x$, then multiplied by $\sin x$ or $\cos x$, gives terms like $\sin x \cos^3 x$, whose integral is $-\frac{1}{4}\cos^4 x$.
- **An R form in the denominator**: $\frac{1}{(R\cos(2x+\alpha))^2} = \frac{1}{R^2}\sec^2(2x + \alpha)$, which integrates to $\frac{1}{2R^2}\tan(2x + \alpha)$.
- An identity that simplifies to $\tan\theta$ leads to $\int \tan\theta \, d\theta = \ln\sec\theta$; one that gives $\tan^2\theta$ needs $\sec^2\theta - 1$ first.

6. Equations needing $\sec^2\theta = 1 + \tan^2\theta$ or $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ (3 questions)

1. **Replace the squared function** so that only one function is left: $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ when there is also $\cot\theta$; $\tan^2\theta = \sec^2\theta - 1$ when there is also $\sec\theta$. Keep the number in front: $4\operatorname{cosec}^2\theta = 4 + 4\cot^2\theta$.
2. **Solve the quadratic**, then turn each root round: $\cot\theta = k$ means $\tan\theta = \frac{1}{k}$; $\sec\theta = k$ means $\cos\theta = \frac{1}{k}$.
3. **Find every angle**, rejecting $|\sec\theta| < 1$ or $|\operatorname{cosec}\theta| < 1$.

For example, $\tan^2\theta = 5 - \sec\theta$ for $0 \leqslant \theta \leqslant 2\pi$: $\sec^2\theta - 1 = 5 - \sec\theta$, so $\sec^2\theta + \sec\theta - 6 = 0$ and $(\sec\theta - 2)(\sec\theta + 3) = 0$. $\cos\theta = \frac{1}{2}$ gives $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$; $\cos\theta = -\frac{1}{3}$ gives $\theta = 1.91, 4.37$.

Variations you will meet:

- **An interval such as $-\pi \leqslant \theta \leqslant \pi$**: from $\tan\theta = k$, the other answer is the calculator value $\pm\pi$ (whichever stays in the interval).
- **Fourth powers and a changed angle**: part (a) simplifies an expression in $\sec^4\theta$ and $\tan^4\theta$, and part (b) uses it with 2α in place of θ. Replace $\sec^2 2\alpha$ by $1 + \tan^2 2\alpha$, get a quadratic in $\tan^2 2\alpha$, take $\pm$ square roots and solve for 2α over the doubled interval.
- **Squares of $\cot$ with $\cos 2\theta$**: replace $\cot^2\theta$ by $\operatorname{cosec}^2\theta - 1 = \frac{1}{\sin^2\theta} - 1$ and $\cos 2\theta$ by $1 - 2\sin^2\theta$, then multiply by $\sin^2\theta$.

7. A polynomial equation in $\cos\theta$ (1 question)

The first parts use the factor theorem on a cubic $f(x)$ (see the Paper 3 Algebra note). The last part replaces x by $\cos\theta$.

1. **Use the factorised cubic**: the roots of $f(x) = 0$ are the possible values of $\cos\theta$.
2. **Reject impossible roots**: $\cos\theta$ must lie between -1 and 1.
3. **Solve** each remaining $\cos\theta = k$ in the interval.

For example, if $f(x) = 0$ has roots 2, $\frac{1}{2}$ and $-\frac{1}{3}$, then $f(\cos\theta) = 0$ for $0^\circ \leqslant \theta \leqslant 360^\circ$: $\cos\theta = 2$ is impossible; $\cos\theta = \frac{1}{2}$ gives $60^\circ, 300^\circ$; $\cos\theta = -\frac{1}{3}$ gives $109.5^\circ, 250.5^\circ$.

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

By first expressing the equation $\tan(x - 45^\circ) = 6\cot x$ as a quadratic equation in $\tan x$, solve the equation for $0^\circ < x < 180^\circ$. **[6]**

Let $t = \tan x$. Expand with $\tan 45^\circ = 1$: **B1** for $\tan(x - 45^\circ) = \dfrac{t-1}{1+t}$.

Write $6\cot x = \dfrac{6}{t}$. **B1**

$$\frac{t-1}{1+t} = \frac{6}{t} \;\Rightarrow\; t(t-1) = 6(1+t) \;\Rightarrow\; t^2 - 7t - 6 = 0.$$

B1 for the correct three-term quadratic.

$$t = \frac{7 \pm \sqrt{49+24}}{2} = \frac{7 \pm \sqrt{73}}{2}, \quad \text{so } \tan x = 7.772 \text{ or } -0.772.$$

M1 for solving the quadratic and finding an angle.

- $\tan x = 7.772$: $x = 82.7^\circ$. **A1**
- $\tan x = -0.772$: the calculator gives -37.7°, outside the interval; $\tan$ repeats every 180°, so $x = 180^\circ - 37.7^\circ = 142.3^\circ$. **A1**, with no others in the interval.

Example 2

(a) By expressing 3θ as $2\theta + \theta$, prove that $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$. **[3]**

(b) Hence solve the equation $\sin 3\theta + \sin\theta = 3\sin^2\theta$ for $0^\circ \leqslant \theta \leqslant 180^\circ$. **[5]**

(a)

$$\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta.$$

M1 for the correct compound-angle expansion.

$$= 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta = 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta.$$

M1 for correct double-angle formulas and $\cos^2\theta = 1 - \sin^2\theta$, giving an expression in $\sin\theta$ only.

$$= 3\sin\theta - 4\sin^3\theta.$$

A1, the given answer from full working.

(b) Use (a) with $s = \sin\theta$: **M1** for an equation in $\sin\theta$ only.

$$3s - 4s^3 + s = 3s^2 \Rightarrow 4s^3 + 3s^2 - 4s = 0 \Rightarrow s(4s^2 + 3s - 4) = 0.$$

$s = 0$ gives $\theta = 0^\circ$ or 180°. **B1**

$4s^2 + 3s - 4 = 0$. **A1**

$$s = \frac{-3 \pm \sqrt{9+64}}{8} = \frac{-3 \pm \sqrt{73}}{8}, \quad \text{so } \sin\theta = 0.693 \text{ or } -1.443.$$

$\sin\theta = -1.443$ is impossible. $\sin\theta = 0.693$ gives $\theta = 43.9^\circ$ or $180^\circ - 43.9^\circ = 136.1^\circ$. **M1 A1**

$\theta = 0^\circ, 43.9^\circ, 136.1^\circ, 180^\circ$. Dividing by $\sin\theta$ in the first line would have lost two of them.

Example 3

(a) Prove that $\dfrac{1+\sin 2\theta}{\cos 2\theta} \equiv \dfrac{1+\tan\theta}{1-\tan\theta}$. **[4]**

(b) Hence solve the equation $\dfrac{1+\sin 4x}{\cos 4x} = 3$ for $0^\circ < x < 180^\circ$. **[4]**

(a) Start from the left. The top is a perfect square and the bottom a difference of two squares:

$$1 + \sin 2\theta = \cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta = (\cos\theta + \sin\theta)^2.$$

B1 for using $\sin 2\theta = 2\sin\theta\cos\theta$ with $1 = \cos^2\theta + \sin^2\theta$.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = (\cos\theta - \sin\theta)(\cos\theta + \sin\theta).$$

B1 for the correct form of $\cos 2\theta$.

$$\frac{(\cos\theta + \sin\theta)^2}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}.$$

M1 for cancelling the common factor.

Divide the top and bottom by $\cos\theta$: $\dfrac{1+\tan\theta}{1-\tan\theta}$. **A1**

(b) The left-hand side is part (a) with θ replaced by $2x$, so it equals $\dfrac{1+\tan 2x}{1-\tan 2x}$. **M1** for using the identity with $2x$.

$$1 + \tan 2x = 3 - 3\tan 2x \Rightarrow \tan 2x = \frac{1}{2}.$$

A1

For $0^\circ < x < 180^\circ$, $0^\circ < 2x < 360^\circ$: $2x = 26.57^\circ$ or $180^\circ + 26.57^\circ = 206.57^\circ$. **M1** for using the doubled interval.

$x = 13.3^\circ$ or 103.3°. **A1**, with no others.

Example 4

(a) Express $4\sin\left(\theta + \frac{\pi}{3}\right) - \sqrt{3}\cos\theta$ in the form $R\sin(\theta+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and the value of α correct to 3 decimal places. **[4]**

(b) Hence solve the equation $4\sin\left(2x + \frac{\pi}{3}\right) - \sqrt{3}\cos 2x = 1$ for $0 < x < \pi$. **[4]**

(c) Find the greatest value of $\dfrac{1}{4\sin\left(\theta + \frac{\pi}{3}\right) - \sqrt{3}\cos\theta + 3}$, giving your answer in an exact form. **[2]**

(a) Expand the bracket and collect:

$$4\left(\sin\theta \cdot \tfrac{1}{2} + \cos\theta \cdot \tfrac{\sqrt{3}}{2}\right) - \sqrt{3}\cos\theta = 2\sin\theta + 2\sqrt{3}\cos\theta - \sqrt{3}\cos\theta = 2\sin\theta + \sqrt{3}\cos\theta.$$

B1

$R\sin(\theta+\alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$, so $R\cos\alpha = 2$ and $R\sin\alpha = \sqrt{3}$.

$R = \sqrt{2^2 + (\sqrt{3})^2} = \sqrt{7}$. **B1**

$\tan\alpha = \dfrac{\sqrt{3}}{2}$, **M1**, so $\alpha = 0.714$. **A1**

(b) The left-hand side is part (a) with $\theta = 2x$: $\sqrt{7}\sin(2x+\alpha) = 1$, so $\sin(2x+\alpha) = \dfrac{1}{\sqrt{7}}$ and the calculator gives 0.3876. **B1**

For $0 < x < \pi$, the bracket runs over $\alpha < 2x + \alpha < 2\pi + \alpha$, that is from 0.7137 to 6.9969. 0.3876 itself is too small, so the values are

$$2x + \alpha = \pi - 0.3876 = 2.7540 \quad \text{or} \quad 2x + \alpha = 2\pi + 0.3876 = 6.6708.$$

M1 for a correct method for a value of x in the interval.

$x = \frac{1}{2}(2.7540 - 0.7137) = 1.02$ **A1**, and $x = \frac{1}{2}(6.6708 - 0.7137) = 2.98$ **A1**, with no others.

(c) The fraction is greatest when its denominator $\sqrt{7}\sin(\theta+\alpha) + 3$ is least, which is $3 - \sqrt{7}$ (still positive). **B1**

Greatest value $= \dfrac{1}{3-\sqrt{7}} = \dfrac{3+\sqrt{7}}{(3-\sqrt{7})(3+\sqrt{7})} = \dfrac{3+\sqrt{7}}{2}$. **B1**

Common mistakes

- **Writing $2\cot x$ as $\frac{1}{2\tan x}$.** It is $\frac{2}{\tan x}$. Examiners report this in several years of the $\tan(x \pm 45^\circ) = 2\cot x$ type, along with forgetting to multiply **every** term when clearing the fractions.

- **Losing the second root of $\tan x$.** A negative value of $\tan x$ still gives an answer: in $0^\circ < x < 180^\circ$ it is 180° plus the calculator's negative angle. Examiners see many candidates reject it, or use 180° minus the positive root instead.
- **Working with θ instead of 2θ.** In "hence solve $\ldots 2\theta \ldots$", candidates often carry on with θ, or write $2(\theta - \alpha)$ when they mean $2\theta - \alpha$. Find the bracket's values first, then subtract α, then halve.
- **Finding the second angle the wrong way**, such as taking 180° minus the first answer for θ when that answer came from a bracket like $2\theta - \alpha$. Work out the second value of the **whole bracket** first.
- **An inexact R, or α from the wrong ratio.** $R = \sqrt{a^2 + b^2}$ exactly; don't find α first and then $R = \frac{a}{\cos \alpha}$ as a decimal. Writing $\cos \alpha = a$ and $\sin \alpha = b$ (instead of $R \cos \alpha = a$, $R \sin \alpha = b$) loses the method mark, and a sign slip such as $R \sin \alpha = -b$ gives the wrong α.
- **The least value in the wrong place.** $R \cos(x - \alpha)$ is least when $x - \alpha = 180^\circ$, not 0°, 90° or 270°. A quick sketch of the cosine curve settles it.
- **Wrong half-angle formula.** $\cos x = 2\cos^2 \frac{x}{2} - 1$; examiners have seen "$\cos x = \cos^2 \frac{x}{2} - 1$" and answers in x that were never doubled.
- **Dropped brackets in identities.** $1 - \cos 2\theta = 1 - (1 - 2\sin^2 \theta)$; without the bracket the sign goes wrong and the method mark is lost with it.
- **Using an identity that isn't one**, such as $\cos(x - 60^\circ) = \cos x \cos 60^\circ - \sin x \sin 60^\circ$ (the sign should be $+$), or $\operatorname{cosec} 2\theta = \frac{1}{\cos 2\theta}$.
- **Ignoring "hence".** Candidates who start again instead of using part (a) usually get stuck; the identity is there to make part (b) short.
- **Treating $\sin^2 2\theta$ as $\sin(2\theta)^2$ or $\sin 2\theta^2$.** $\sin^2 2\theta = (\sin 2\theta)^2$: find $\sin 2\theta = \pm \ldots$ first.
- **Rounding a root of the quadratic too early.** Keep full calculator values of $\cos \theta$ or $\tan x$ until the angle is found.
- **Mixing degrees and radians** between parts (a) and (b), or giving radians when the interval is in degrees.

How to get the marks

- **Show the identity step.** Method marks are for "use of a correct formula": write the expansion with your angles in it, then the simplified form.
- **"Prove" or "show that"**: the answer is printed, so every step must be there and your last line must match it exactly. Jumping from the first substitution straight to the printed result can lose the final mark: show the lines in between. If a question asks you to show that an equation can be written in a form, end with that equation, "$= 0$" and all.
- **"Hence"** means use the previous part, and say how (for example "using (a) with $\theta = 2x$").
- **"Exact"** means surds, fractions and π: $R = \sqrt{13}$, $x = \frac{5\pi}{6}$. Get R from $\sqrt{a^2 + b^2}$, not from a decimal α.
- **Accuracy.** Unless the question says otherwise, give angles in degrees to 1 decimal place and non-exact radians to 3 significant figures; when it asks for α to 2 or 3 decimal places, give exactly that. Mark schemes accept a more accurate answer if it is correct, so keep full values in your working and round only at the end.

- **"Solve … for $0^\circ < \theta < 180^\circ$"**: the last mark needs **every** solution in the interval **and no others**. Extra answers inside the interval lose it; answers outside it are ignored. Answers in the wrong unit are treated as a misread and cost marks.
- **Quadratics in a trigonometric function**: the method mark needs a correct method to solve a three-term quadratic; show the factors or the formula, then the values of the function, then the angles.
- **Follow-through.** Part (b) marks usually follow your R and α from part (a), so use them even if you doubt them.

Quick check

1. The acute angles A and B satisfy $\tan(A+B) = 3$ and $\tan A = 2\tan B$. Find A and B.
2. Solve $2\tan^2\theta + \sec\theta = 1$ for $0 \leqslant \theta \leqslant 2\pi$.
3. Express $\sqrt{3}\sin x - \cos x$ in the form $R\sin(x-\alpha)$, with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving exact values. Hence solve $\sqrt{3}\sin x - \cos x = \sqrt{2}$ for $0 < x < 2\pi$, giving your answers in terms of π.
4. Prove that $\cot\theta - \cot 2\theta \equiv \operatorname{cosec} 2\theta$. Hence find the exact value of $\cot\frac{\pi}{12}$.
5. Use $(\cos x + \sin x)^2 \equiv 1 + \sin 2x$ to find the exact value of $\displaystyle\int_0^{\pi/4} (\cos x + \sin x)^2\,dx$.

Answers

1. Let $t = \tan B$. $\dfrac{2t + t}{1 - 2t^2} = 3$, so $3t = 3 - 6t^2$, that is $2t^2 + t - 1 = 0$ and $(2t - 1)(t + 1) = 0$. B is acute, so $\tan B = \frac{1}{2}$: $B = 26.6^\circ$. Then $\tan A = 1$: $A = 45^\circ$.
2. $2(\sec^2\theta - 1) + \sec\theta = 1$, so $2\sec^2\theta + \sec\theta - 3 = 0$ and $(\sec\theta - 1)(2\sec\theta + 3) = 0$. $\sec\theta = 1$ gives $\cos\theta = 1$: $\theta = 0$ or 2π. $\sec\theta = -\frac{3}{2}$ gives $\cos\theta = -\frac{2}{3}$: $\theta = 2.30$ or $2\pi - 2.30 = 3.98$. So $\theta = 0, 2.30, 3.98, 2\pi$.
3. $R\cos\alpha = \sqrt{3}$, $R\sin\alpha = 1$: $R = 2$, $\tan\alpha = \frac{1}{\sqrt{3}}$, $\alpha = \frac{\pi}{6}$. So $2\sin\left(x - \frac{\pi}{6}\right) = \sqrt{2}$, $\sin\left(x - \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}$. The bracket is between $-\frac{\pi}{6}$ and $\frac{11\pi}{6}$: $x - \frac{\pi}{6} = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, so $x = \frac{5\pi}{12}$ or $\frac{11\pi}{12}$.
4. $\cot\theta - \cot 2\theta = \dfrac{\cos\theta}{\sin\theta} - \dfrac{\cos 2\theta}{2\sin\theta\cos\theta} = \dfrac{2\cos^2\theta - \cos 2\theta}{2\sin\theta\cos\theta} = \dfrac{2\cos^2\theta - (2\cos^2\theta - 1)}{\sin 2\theta} = \dfrac{1}{\sin 2\theta} = \operatorname{cosec} 2\theta$. With $\theta = \frac{\pi}{12}$: $\cot\frac{\pi}{12} - \cot\frac{\pi}{6} = \operatorname{cosec}\frac{\pi}{6}$, so $\cot\frac{\pi}{12} = \sqrt{3} + 2$.
5. $\displaystyle\int_0^{\pi/4} (1 + \sin 2x)\,dx = \left[x - \tfrac{1}{2}\cos 2x\right]_0^{\pi/4} = \left(\frac{\pi}{4} - 0\right) - \left(0 - \frac{1}{2}\right) = \frac{\pi}{4} + \frac{1}{2} = \frac{\pi + 2}{4}$.

Past questions to try

- **9709/32/M/J/25 Q4**: $\cot x$ and $\cot 2x$ together, leading to a quadratic in $\tan x$.
- **9709/33/O/N/23 Q6**: show an equation with $\cot^2\theta$ and $\cos 2\theta$ becomes a quartic in $\sin\theta$, then find four angles.
- **9709/31/M/J/25 Q7**: $R\sin(x - \alpha)$ after expanding a compound angle, then solving with 2θ in radians.
- **9709/32/O/N/21 Q8**: prove a fourth-power identity, then "hence solve" with four answers.