Cambridge AS & A Level · Mathematics (9709) · Notes

Vectors A2

Read online at pastopic.com/cambridge/mathematics-9709/notes/vectors

Last checked 28 September 2026

Read first: Quadratics

What you need to know

By the end of this topic you should be able to:

1. **Read and write vectors in every standard notation**: a column $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, the unit-vector form $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ (or $x\mathbf{i} + y\mathbf{j}$ in two dimensions), $\overrightarrow{AB}$ for the vector from A to B, and a bold letter $\mathbf{a}$ (underlined when you write it by hand).
2. **Add and subtract vectors and multiply them by a number**, and say what each of these means in a diagram. For example, "$OABC$ is a parallelogram" means the same as $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$, and the midpoint of AB has position vector $\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$.
3. **Find the length (magnitude) of a vector** in two or three dimensions, and use unit vectors, position vectors and displacement vectors.
4. **Understand every symbol in the vector equation of a line**, $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, and write the equation of a line from a point and a direction, or from two points.
5. **Decide whether two lines are parallel, intersect or are skew**, and find where they meet when they do.
6. **Use the scalar product**: find the angle between two vectors or two lines, test for perpendicular lines, and find the foot of the perpendicular from a point to a line. Questions may be set in 3D shapes such as cuboids and pyramids.

The shortest distance between two skew lines, the common perpendicular of two skew lines, the general ratio theorem and the vector (cross) product are **not** in the syllabus.

Understand it

What a vector is

A **vector** has a size and a direction; a number on its own (a **scalar**) has only a size. A move of 4 to the right and 2 down is the vector $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$, which is also written $4\mathbf{i} - 2\mathbf{j}$. In three dimensions a third number gives the move in the z-direction: $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$. Here $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are vectors of length 1 along the x-, y- and z-axes. Both forms mean the same thing; use whichever the question uses. Never mix them by putting $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ inside a column.

Two vectors are **equal** when they have the same length and the same direction, wherever they are drawn.

Position vectors and displacement vectors

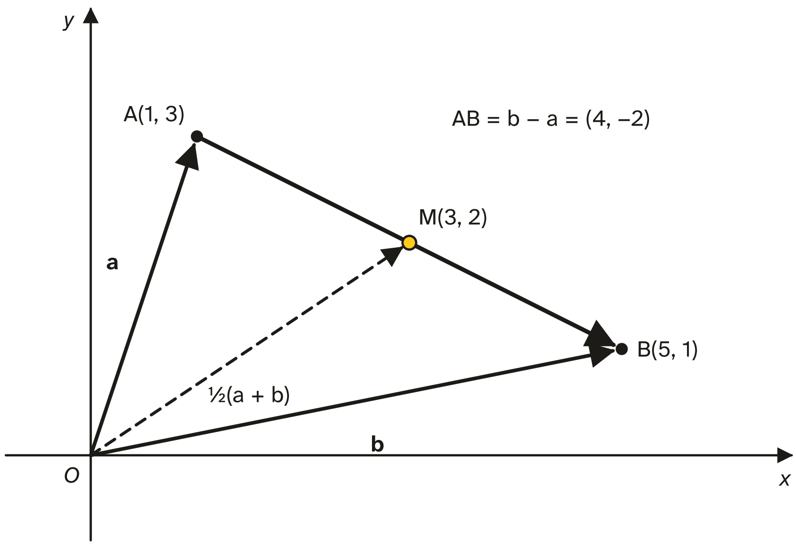
The **position vector** of a point A is the vector from the origin to A: $\overrightarrow{OA} = \mathbf{a}$. Its components are just A's coordinates.

To get from A to B, go back from A to O, then out to B:

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}.$$

So a **displacement vector** is "end minus start". For $A(1, 3)$ and $B(5, 1)$, $\overrightarrow{AB} = \begin{pmatrix} 5-1 \\ 1-3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, and $\overrightarrow{BA} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ points the other way.

The **midpoint** M of AB is halfway along: $\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b})$. Here $M = (3, 2)$.



Adding, subtracting and scaling

Adding vectors means doing one move after the other: $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$. Multiplying by a number k stretches the vector: $2\mathbf{v}$ is twice as long in the same direction, and $-\mathbf{v}$ points the opposite way. Work component by component: $2\begin{pmatrix}1\\-3\\2\end{pmatrix} - \begin{pmatrix}4\\0\\-1\end{pmatrix} = \begin{pmatrix}-2\\-6\\5\end{pmatrix}$.

This gives the two facts behind most geometry questions:

- **Parallel** vectors are multiples of each other. $\begin{pmatrix}2\\-4\\6\end{pmatrix} = 2\begin{pmatrix}1\\-2\\3\end{pmatrix}$, so they are parallel; $\begin{pmatrix}2\\3\\-1\end{pmatrix}$ and $\begin{pmatrix}-1\\-2\\1\end{pmatrix}$ are not, because no single number turns one into the other.
- **Shapes are sums of moves.** In a parallelogram $ABCD$ (vertices in that order round the shape) the opposite sides are equal vectors: $\overrightarrow{AD} = \overrightarrow{BC}$ and $\overrightarrow{DC} = \overrightarrow{AB}$. A point N on BC with $BN = 3NC$ is three-quarters of the way along, so $\overrightarrow{ON} = \mathbf{b} + \frac{3}{4}\overrightarrow{BC}$.

Length and unit vectors

By Pythagoras (twice in 3D), the **magnitude** of $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is

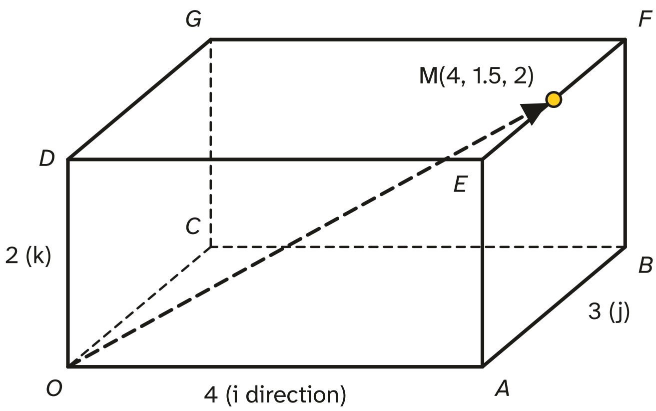
$$|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}.$$

The distance between two points is the length of the vector joining them: $AB = |\overrightarrow{AB}|$. For $\overrightarrow{AB} = \begin{pmatrix}4\\-2\end{pmatrix}$ it is $\sqrt{20} = 2\sqrt{5}$.

A **unit vector** has length 1. Divide any vector by its length to get the unit vector in its direction: $\begin{pmatrix}2\\-1\\2\end{pmatrix}$ has length 3, so its unit vector is $\frac{1}{3}\begin{pmatrix}2\\-1\\2\end{pmatrix}$.

3D shapes

Questions often describe a cuboid or pyramid with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ along three edges from O. Write down each vertex's coordinates by counting along those edges, then use them as position vectors. In the cuboid below, F is 4 along $\mathbf{i}$, 3 along $\mathbf{j}$ and 2 up, so $\overrightarrow{OF} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. M, the midpoint of EF, is at $(4, 1.5, 2)$, and $\overrightarrow{DM} = \overrightarrow{OM} - \overrightarrow{OD} = 4\mathbf{i} + 1.5\mathbf{j}$.

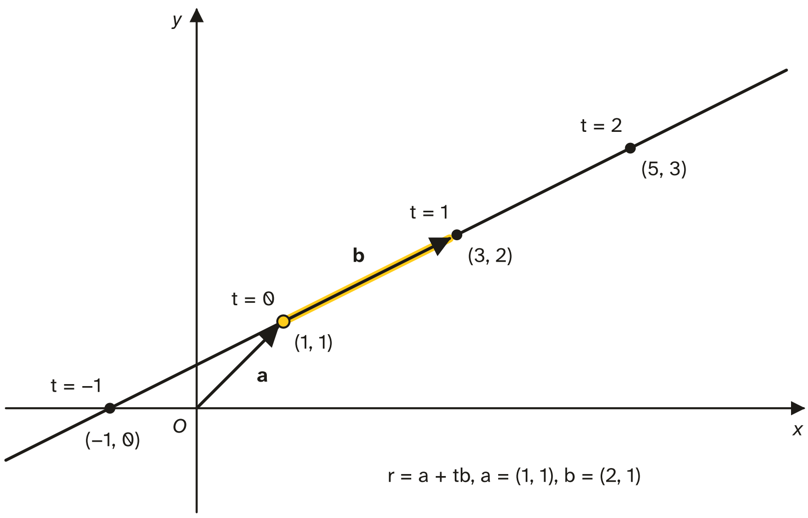


The vector equation of a line

A line is fixed by one point on it and its direction. If $\mathbf{a}$ is the position vector of a point on the line and $\mathbf{b}$ is a vector along it, every point on the line is reached by going to $\mathbf{a}$ and then some multiple of $\mathbf{b}$:

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}.$$

$\mathbf{r}$ is the position vector of a general point on the line, $\mathbf{a}$ is a **position vector** (a point), $\mathbf{b}$ is the **direction vector**, and the **parameter** t is a number: each value of t gives one point. For $\mathbf{r} = \begin{pmatrix}1\\1\end{pmatrix} + t\begin{pmatrix}2\\1\end{pmatrix}$, $t = 0$ gives $(1, 1)$, $t = 1$ gives $(3, 2)$ and $t = -1$ gives $(-1, 0)$.



For the line through two points A and B, use either point for $\mathbf{a}$ and $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ (or any multiple of it) for the direction: $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$. The same line has many correct equations, because you can start from any point on it and scale the direction.

Writing the line's general point in components, for example $(1+2t,\ 1+t)$, is the key step in almost every line question: it turns "a point on the line" into something you can calculate with.

The scalar product

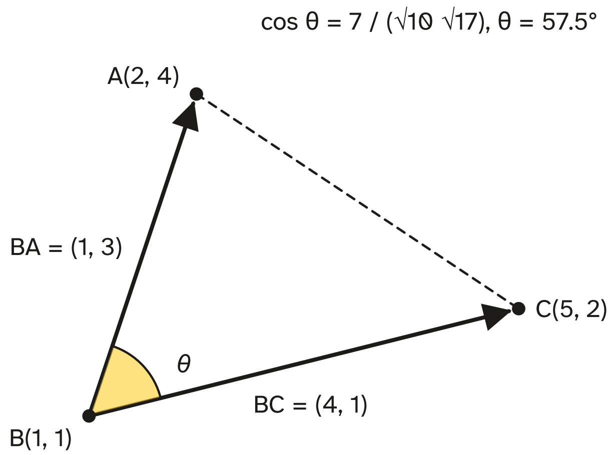
The **scalar product** (or dot product) of two vectors multiplies matching components and adds:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}|\,|\mathbf{b}|\cos\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ when they are drawn **from the same point**. The answer is a number, not a vector. Rearranged, it finds angles:

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}.$$

For the angle ABC in a triangle, both vectors must start at B: use $\overrightarrow{BA}$ and $\overrightarrow{BC}$. With $B(1,1)$, $A(2,4)$ and $C(5,2)$: $\overrightarrow{BA} = \begin{pmatrix}1\\3\end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix}4\\1\end{pmatrix}$, the scalar product is $4+3=7$, and $\cos\theta = \frac{7}{\sqrt{10}\sqrt{17}}$, so $\theta = 57.5^\circ$. Using $\overrightarrow{AB}$ with $\overrightarrow{BC}$ instead would give the outside angle, $180^\circ - 57.5^\circ$.



Three things follow:

- **Perpendicular** vectors have $\cos 90^\circ = 0$, so $\mathbf{a}\cdot\mathbf{b} = 0$. This is the test for perpendicular lines, and it is how you find a foot of a perpendicular.
- A **negative** scalar product means the angle is obtuse; a positive one means acute.
- The **angle between two lines** uses their **direction vectors**, never the position vectors of points on them. Lines make two angles that add to 180°; for the **acute** angle, take the modulus: $\cos\theta = \dfrac{|\mathbf{b}_1\cdot\mathbf{b}_2|}{|\mathbf{b}_1|\,|\mathbf{b}_2|}$.

Parallel, intersecting or skew

In a plane, two lines either are parallel or cross. In three dimensions there is a third possibility: lines that are not parallel and still never meet, like a road and a bridge crossing over it. These are **skew** lines.

- **Parallel**: the direction vectors are multiples of each other.
- **Intersect**: there are values of the two parameters that give the same point. Write both general points, set the x, y and z components equal, solve two of the equations for the two parameters, and **check the third**. If it works, the lines meet; substitute to find the point.
- **Skew**: not parallel, and the third equation fails.

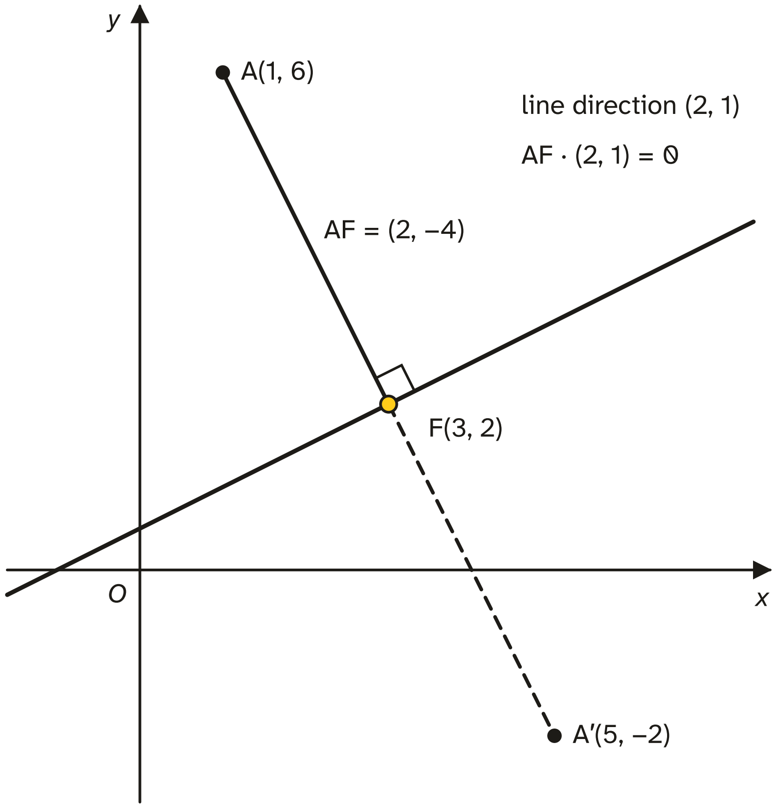
Use **different letters** for the two parameters (λ and μ): the two lines reach the meeting point at different parameter values.

The foot of the perpendicular

The shortest route from a point A to a line is at right angles to the line. It meets the line at the **foot of the perpendicular**, F. To find it, take the general point P of the line, form $\overrightarrow{AP}$, and make it perpendicular to the direction: $\overrightarrow{AP} \cdot \mathbf{b} = 0$. That gives one equation for the parameter.

For the line $\mathbf{r} = \begin{pmatrix}1\\1\end{pmatrix} + t\begin{pmatrix}2\\1\end{pmatrix}$ and $A(1,6)$: $\overrightarrow{AP} = \begin{pmatrix}2t\\t-5\end{pmatrix}$, and $\overrightarrow{AP} \cdot \begin{pmatrix}2\\1\end{pmatrix} = 4t + t - 5 = 0$ gives $t = 1$. So $F = (3,2)$, $\overrightarrow{AF} = \begin{pmatrix}2\\-4\end{pmatrix}$, and the **perpendicular distance** from A to the line is $|\overrightarrow{AF}| = \sqrt{20} = 2\sqrt{5}$.

The **reflection** of A in the line is as far past F as A is before it: $\overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AF}$, which is the same as $2\overrightarrow{OF} - \overrightarrow{OA}$. Here $A' = (5,-2)$.



Key facts and formulas

"Given" means it is in the MF19 list of formulae in the exam. "Learn it" means you must remember it.

Formula	Status
$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \lvert\mathbf{a}\rvert\,\lvert\mathbf{b}\rvert\cos\theta$	Given
$\lvert x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\rvert = \sqrt{x^2+y^2+z^2}$	Learn it
Unit vector in the direction of $\mathbf{v}$: $\dfrac{\mathbf{v}}{\lvert\mathbf{v}\rvert}$	Learn it
$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ ("end minus start")	Learn it
Midpoint of AB: $\frac{1}{2}(\mathbf{a}+\mathbf{b})$	Learn it
Line: $\mathbf{r} = \mathbf{a} + t\mathbf{b}$; through A and B: $\mathbf{r} = \mathbf{a} + t(\mathbf{b}-\mathbf{a})$	Learn it
Parallel: one direction vector is a multiple of the other	Learn it
Perpendicular: $\mathbf{a}\cdot\mathbf{b} = 0$	Learn it
Acute angle between lines: $\cos\theta = \dfrac{\lvert\mathbf{b}_1\cdot\mathbf{b}_2\rvert}{\lvert\mathbf{b}_1\rvert\,\lvert\mathbf{b}_2\rvert}$	Learn it
Foot of perpendicular from A: $\overrightarrow{AP}\cdot\mathbf{b} = 0$ for the general point P	Learn it

Formula	Status
Reflection of A in a line with foot F: $\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA}$	Learn it
Area of triangle ABC: $\frac{1}{2}\,AB \times AC \times \sin A$; parallelogram: twice that	Learn it
$\sin\theta = \sqrt{1-\cos^2\theta}$ for $0^\circ < \theta < 180^\circ$	Learn it

How to do it

In the Paper 3 questions we have tagged for this topic (37 questions, 2021–2025), the types came up this often. Every one of the 37 questions has several parts that mix two or more types, so the counts add up to more than 37.

Question type	Questions
Angles using the scalar product (including perpendicular lines)	27
Vector equation of a line	20
Parallel, intersecting or skew lines	18
Position vectors in shapes	15
Foot of the perpendicular, distance to a line and reflection	13
Area of a triangle or parallelogram	6
A point on a line at a given distance	4

1. Angles using the scalar product (27 questions)

1. **Choose the right two vectors.** For the angle between two lines, use their direction vectors. For an angle ABC in a shape, use $\overrightarrow{BA}$ and $\overrightarrow{BC}$ (both leaving B). For the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, use the position vectors themselves.
2. **Scalar product**: multiply matching components and add. Show the sum, such as $(-2)+6+(-6)=-2$.
3. **Magnitudes** of the **same two** vectors.
4. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$, then $\cos^{-1}$. Give degrees to 1 d.p. (or radians to 3 s.f.) unless an exact cosine is asked for.

Variations you will meet:

- **"Exact value of $\cos\theta$"**: stop at the fraction and simplify the surds, for example $\frac{4}{\sqrt{12}\sqrt{18}} = \frac{4}{\sqrt{216}} = \frac{4}{6\sqrt{6}} = \frac{\sqrt{6}}{9}$. Always write "$\cos\theta = \ldots$", not just the number.
- **Acute or obtuse.** If the question wants the acute angle and your cosine is negative, the answer is 180° minus your angle (or use the modulus from the start). For the obtuse angle, do the opposite.
- **Perpendicular lines**: set the scalar product of the direction vectors to 0. With an unknown, such as directions $3\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $\mathbf{i}+4\mathbf{j}+a\mathbf{k}$: $3-4+2a=0$, so $a=\frac{1}{2}$. The same test shows a corner of a shape is a right angle, for example in a rectangle.

- **The angle is given and a constant is unknown.** Write $\dfrac{|\mathbf{b}_1 \cdot \mathbf{b}_2|}{|\mathbf{b}_1|\,|\mathbf{b}_2|} = \cos\theta$, clear the fractions, **square both sides** to remove the roots, and solve the quadratic. Keep both roots when the question asks for the acute angle, because the modulus makes a negative scalar product acceptable. See Example 2.
- **Angle between a line and a vector $\overrightarrow{OA}$**: use the line's direction vector and $\overrightarrow{OA}$.

2. Vector equation of a line (20 questions)

1. **Find a direction vector**: given, or $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ for a line through A and B. You may divide it by a common factor.
2. **Pick a point** on the line for $\mathbf{a}$.
3. Write $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, **starting with "$\mathbf{r} =$"**. Writing "$l =$" or leaving the left side blank loses the mark.

Variations you will meet:

- **Does a point lie on the line?** Set the general point equal to it, solve one component for t, and check that the same t fits the other two.
- **A point with an unknown lies on the line**, such as $(2s, s, -s)$: equate components with the line's general point and solve the pair of equations.
- **The direction is described, not given**, for example "perpendicular to $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, with no x-component". Write the direction as $y\mathbf{j} + z\mathbf{k}$, set its scalar product with the given vector to 0 ($3y - z = 0$), and pick simple numbers, such as $\mathbf{j} + 3\mathbf{k}$.
- **Lines through points of a shape** (a diagonal, or the line through two midpoints): work out the points' position vectors first, then use step 1.

3. Parallel, intersecting or skew lines (18 questions)

1. **Parallel?** Compare the direction vectors. If one is a multiple of the other, the lines are parallel (and either the same line or never meeting). If not, **say so with the vectors written out**, for example "$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \neq k\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ for any k, so the lines are not parallel". "The directions are different" is not enough.
2. **Write both general points in components**, with different parameters λ and μ.
3. **Equate two pairs of components** and solve the simultaneous equations for λ and μ.
4. **Check the third component** with those values.
   - Both sides equal: the lines **intersect**. Substitute λ (or μ) back into its line to get the point, and give it as the question asks (a position vector or coordinates).
   - Not equal: the lines do **not** intersect. Show the two different numbers, such as "$4 \neq -1$". If they are also not parallel, they are **skew**.

Variations you will meet:

- **"Show that the lines do not intersect"**: steps 2–4 and a clear conclusion.

- **"Given that the lines intersect, find the constant a"**: solve the two components that do not contain a for λ and μ, then put them into the third equation to find a. See Example 3.
- **Where a line meets another line through two given points**, or where the diagonals of a shape cross: form the second line first, then intersect as above.
- **The lines share a point in their equations** (both start at the same position vector): that point is where they meet, and you can simply state it.
- **Two unknown constants a and b** ("show that a and b satisfy ..."): eliminate the two parameters from the three component equations.

4. Position vectors in shapes (15 questions)

1. **Write every point you know as a position vector.** In a 3D solid, count along the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ edges.
2. **Turn each fact into vectors**:
   - midpoint: $\frac{1}{2}(\mathbf{a}+\mathbf{b})$;
   - a point dividing BC with $BN = 3NC$: $\mathbf{b} + \frac{3}{4}(\mathbf{c}-\mathbf{b})$;
   - parallelogram $ABCD$: $\overrightarrow{AD} = \overrightarrow{BC}$, so $\mathbf{d} = \mathbf{a}+\mathbf{c}-\mathbf{b}$;
   - trapezium with $\overrightarrow{DC} = 2\overrightarrow{AB}$: $\mathbf{d} = \mathbf{c} - 2(\mathbf{b}-\mathbf{a})$;
   - equal lengths such as $AB = BC$: set $|\overrightarrow{AB}|^2 = |\overrightarrow{BC}|^2$ and solve.
3. **Keep the letters in order.** "$ABCD$" means the vertices go round the shape in that order, so AB and DC are opposite sides, not AB and CD in the same direction.

Variations you will meet:

- **Unit vector** in the direction of a line: divide its direction vector by its length.
- **"Show that $OABC$ is a rectangle"**: one pair of opposite sides are equal vectors (so it is a parallelogram) and two adjacent sides have scalar product 0.
- **A rhombus** $ABCD$ has $AB = BC$ as well as being a parallelogram, so first use the equal lengths to find the unknown, then $\mathbf{d} = \mathbf{a}+\mathbf{c}-\mathbf{b}$.
- **"Find the vectors $\overrightarrow{MB}$ and $\overrightarrow{MC}$"**: end minus start, with the midpoint worked out first. Don't confuse $\overrightarrow{OM}$ (a position vector) with $\overrightarrow{AM}$.

5. Foot of the perpendicular, distance to a line and reflection (13 questions)

1. **General point** P of the line in components: $\mathbf{p} = \mathbf{a} + t\mathbf{b}$.
2. $\overrightarrow{AP} = \mathbf{p} - \overrightarrow{OA}$, still in terms of t. It must start at the given point A, not at O.
3. **Set** $\overrightarrow{AP} \cdot \mathbf{b} = 0$ and solve the linear equation for t.
4. **Substitute** t to get the foot F (the question may call it P, N or Q).

Variations you will meet:

- **Perpendicular distance** ("show that the length of the perpendicular from A to the line is $\sqrt{26}$"): find F as above, then $|\overrightarrow{AF}|$. Show the components and the square root; the answer is given, so every step must be visible.
- **Perpendicular from the origin**: the same method with $A = O$, so $\overrightarrow{OP} \cdot \mathbf{b} = 0$.
- **Reflection of A in the line**: $\overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AF}$.
- **Extending the perpendicular** in a ratio, such as R on PQ extended with $PQ : QR = 1 : 2$: $\overrightarrow{OR} = \overrightarrow{OQ} + 2\overrightarrow{PQ}$.
- **Distance between two parallel lines** (for example the parallel sides of a trapezium): take a point on one line and find its perpendicular distance to the other. The area of a trapezium is then $\frac{1}{2}(\text{sum of the parallel sides}) \times \text{distance}$.
- **Minimising the length** also works: $|\overrightarrow{AP}|^2$ is a quadratic in t; complete the square or differentiate to find its minimum. The scalar-product method is usually shorter.

6. Area of a triangle or parallelogram (6 questions)

1. **Find the cosine of an angle** of the triangle with the scalar product (often the part before, marked "hence").
2. **Exact sine**: $\sin\theta = \sqrt{1-\cos^2\theta}$ (positive, because the angle is between 0° and 180°). For $\cos\theta = \frac{\sqrt{6}}{9}$, $\cos^2\theta = \frac{6}{81}$ and $\sin\theta = \sqrt{\frac{75}{81}} = \frac{5\sqrt{3}}{9}$.
3. **Area** $= \frac{1}{2} \times$ (the two sides next to that angle) $\times \sin\theta$. A parallelogram is twice the triangle.
4. Simplify the surd. With sides $\sqrt{12}$ and $\sqrt{18}$ and that sine, the triangle's area is $\frac{1}{2}\sqrt{12}\sqrt{18} \times \frac{5\sqrt{3}}{9} = \frac{1}{2} \times 6\sqrt{6} \times \frac{5\sqrt{3}}{9} = 5\sqrt{2}$.

Variations you will meet:

- **A triangle made of two halves.** If M is the midpoint of AB, triangles AMC and BMC have equal areas, so area $ABC = 2\times$ area BMC.
- **The sides must match the angle.** With angle BAC use AB and AC; with angle CMB use MB and MC.

7. A point on a line at a given distance (4 questions)

1. **General point** P of the line in components.
2. **Write the length condition** with the vector it describes: $|\overrightarrow{OP}| = 5$, or $AB = BD$, or $AP = BP$, or $PA = 3\,OA$.
3. **Square both sides** so there are no roots, expand, and solve for the parameter. A length condition usually gives a quadratic with **two** points; a condition like $AP = BP$ gives a linear equation because the t^2 terms cancel.
4. Substitute each value back to get the position vectors. Throw out $t = 0$ only if it gives a point the question has ruled out (for example D can't be the same point as A).

Worked examples

The bold codes show where marks are earned, as in Cambridge mark schemes. The number is how many marks.

- **M1** method mark: for a correct method, even if the numbers slip.
- **DM1** method mark that needs the M mark before it.
- **A1** accuracy mark: for a correct answer; it needs the M mark before it.
- **B1** mark for a correct result on its own, with no method needed.

Example 1

The points A, B and C have position vectors $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j}$, $\overrightarrow{OB} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OC} = 4\mathbf{i} + \mathbf{j} + 5\mathbf{k}$.

(a) Find the unit vector in the direction of $\overrightarrow{AB}$. **[2]**

(b) $ABCD$ is a parallelogram. Find the position vector of D. **[2]**

(c) Using a scalar product, find angle ABC. **[4]**

(d) Hence find the exact area of the parallelogram $ABCD$. **[3]**

(a) $\overrightarrow{AB} = (3-1)\mathbf{i} + (-1-2)\mathbf{j} + (2-0)\mathbf{k} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, and $|\overrightarrow{AB}| = \sqrt{4+9+4} = \sqrt{17}$. **M1**

Unit vector $= \dfrac{1}{\sqrt{17}}(2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$. **A1**

(b) In $ABCD$, $\overrightarrow{AD} = \overrightarrow{BC} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. **M1**

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$. **A1**

(c) Both vectors leave B: $\overrightarrow{BA} = -2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{BC} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. **B1**

$\overrightarrow{BA} \cdot \overrightarrow{BC} = -2 + 6 - 6 = -2$. **M1**

$$\cos ABC = \frac{-2}{\sqrt{17}\sqrt{14}} = \frac{-2}{\sqrt{238}}.$$

M1 for dividing by the product of the magnitudes.

Angle $ABC = 97.4^\circ$. **A1** (The negative cosine is right: this angle is obtuse.)

(d) $\sin ABC = \sqrt{1 - \dfrac{4}{238}} = \sqrt{\dfrac{234}{238}}$. **M1** for an exact method for the sine.

The parallelogram is two copies of triangle ABC:

$$\text{area} = 2 \times \frac{1}{2} \times \sqrt{17}\sqrt{14} \times \sqrt{\frac{234}{238}} = \sqrt{238} \times \sqrt{\frac{234}{238}} = \sqrt{234} = 3\sqrt{26}.$$

M1 for $2 \times \frac{1}{2} \times BA \times BC \times \sin B$, **A1**.

Example 2

The lines l and m have equations

$$l: \ \mathbf{r} = \mathbf{i} + 3\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}), \qquad m: \ \mathbf{r} = \mathbf{j} + 4\mathbf{k} + \mu(a\mathbf{i} + 2\mathbf{j} + \mathbf{k}),$$

where a is a constant.

(a) Given that the acute angle between l and m is $\cos^{-1}\frac{4}{9}$, find the possible values of a. **[5]**

(b) Given instead that l and m are perpendicular, find a. **[1]**

(a) Direction vectors: $(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (a\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 2a - 2 + 2 = 2a$. **M1**

Magnitudes 3 and $\sqrt{a^2 + 5}$. For the acute angle use the modulus:

$$\frac{|2a|}{3\sqrt{a^2+5}} = \frac{4}{9}.$$

M1 for dividing by the magnitudes and equating to $\frac{4}{9}$, **A1** for a correct equation.

Multiply up: $18|a| = 12\sqrt{a^2+5}$, so $3|a| = 2\sqrt{a^2+5}$. Square: $9a^2 = 4a^2 + 20$, so $5a^2 = 20$ and $a^2 = 4$. **DM1** for squaring and solving.

$a = 2$ or $a = -2$. **A1**. Both are valid: with $a = -2$ the scalar product is negative, so the lines' directions make an obtuse angle, but the acute angle between the lines is still $\cos^{-1}\frac{4}{9}$.

(b) Perpendicular: $2a = 0$, so $a = 0$. **B1**

Example 3

The lines l and m have equations

$$l: \ \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \qquad m: \ \mathbf{r} = \begin{pmatrix} 5 \\ -1 \\ a \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

(a) Given that l and m intersect, find the value of a and the position vector of the point of intersection. **[5]**

(b) When $a = 2$, show that the lines are skew. **[3]**

(a) General points: $(1 + 2\lambda, \ \lambda, \ 3 - 2\lambda)$ on l and $(5 + \mu, \ -1 - \mu, \ a + 2\mu)$ on m. **B1**

Equate the x and y components, which don't involve a: $1 + 2\lambda = 5 + \mu$ and $\lambda = -1 - \mu$. **M1**

Substituting: $1 + 2(-1 - \mu) = 5 + \mu$, so $-1 - 2\mu = 5 + \mu$, giving $\mu = -2$ and $\lambda = 1$. **A1**

z components: $3 - 2(1) = a + 2(-2)$, so $1 = a - 4$ and $a = 5$. **A1**

Point of intersection: $\lambda = 1$ in l gives $3\mathbf{i} + \mathbf{j} + \mathbf{k}$. **A1**

(b) The x and y components still give $\lambda = 1$, $\mu = -2$. The z components are then $3 - 2 = 1$ on l and $2 - 4 = -2$ on m, and $1 \neq -2$, so the lines do not meet. **M1 A1**

$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ for any k (the first components need $k = 2$, the second $k = -1$), so the lines are not parallel. Not parallel and not meeting: they are skew. **B1**

Example 4

The points A and B have position vectors $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} + 5\mathbf{k}$. The point P has position vector $9\mathbf{i} + 3\mathbf{k}$.

(a) Find a vector equation for the line l through A and B. **[2]**

(b) Find the position vector of the foot of the perpendicular from P to l. **[4]**

(c) Find the exact perpendicular distance from P to l. **[2]**

(d) Find the position vector of the reflection of P in l. **[2]**

(a) $\overrightarrow{AB} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. **M1**

$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$. **A1**

(b) A general point F on l is $(1 + 2t,\ 2 - t,\ 3 + 2t)$, so

$$\overrightarrow{PF} = \begin{pmatrix} 1 + 2t - 9 \\ 2 - t - 0 \\ 3 + 2t - 3 \end{pmatrix} = \begin{pmatrix} 2t - 8 \\ 2 - t \\ 2t \end{pmatrix}.$$

B1

Perpendicular to l: $\overrightarrow{PF} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 2(2t - 8) - (2 - t) + 2(2t) = 9t - 18 = 0$. **M1**

$t = 2$. **A1**. The foot is $5\mathbf{i} + 7\mathbf{k}$. **A1**

(c) $\overrightarrow{PF} = -4\mathbf{i} + 4\mathbf{k}$ (check: $-8 + 0 + 8 = 0$), so the distance is $\sqrt{16 + 0 + 16} = \sqrt{32} = 4\sqrt{2}$. **M1 A1**

(d) $\overrightarrow{OP'} = \overrightarrow{OP} + 2\overrightarrow{PF} = (9\mathbf{i} + 3\mathbf{k}) + (-8\mathbf{i} + 8\mathbf{k}) = \mathbf{i} + 11\mathbf{k}$. **M1 A1**

Common mistakes

- **Using position vectors instead of direction vectors** for the angle between two lines. The angle depends only on the directions, the vectors multiplied by the parameter.

- **Using the wrong pair of vectors for an angle in a shape.** For angle BAC you need $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (or $\overrightarrow{BA}$ and $\overrightarrow{CA}$), not one of each, not $\overrightarrow{BC}$ and not the position vectors. Draw a sketch and put both arrows at the vertex.
- **Mixing vectors between the scalar product and the magnitudes.** The bottom of the fraction must use the lengths of the same two vectors as the top.
- **Getting $\overrightarrow{AB}$ the wrong way round.** It is $\mathbf{b} - \mathbf{a}$, "end minus start". Writing $\mathbf{a} - \mathbf{b}$ reverses every later answer that depends on direction.
- **Leaving out "$\mathbf{r} =$"** in the equation of a line, or writing "$l =$" or "line $=$". Examiners remove a mark for this every year.
- **Forgetting to check the third component** when two lines seem to meet. Two equations can always be solved; only the third decides whether the lines really intersect.
- **Using the same parameter for both lines.** Call them λ and μ; the meeting point is at different parameter values on each line.
- **Saying the lines are not parallel without showing it.** "The directions are not equal" is not enough; write the two direction vectors and say that neither is a multiple of the other.
- **Using $\overrightarrow{OP}$ instead of $\overrightarrow{AP}$** for a foot of a perpendicular. The vector that is perpendicular to the line starts at the given point A.
- **Finding the foot but giving the wrong length**, for example $|\overrightarrow{OF}|$ instead of $|\overrightarrow{AF}|$.
- **Giving an angle when the question asks for the exact cosine**, or a decimal area when it says "exact".
- **Arithmetic slips and miscopying.** Vector questions have many small calculations. Write components in columns and check each line as you go.

How to get the marks

- **Show the scalar product as a sum** (for example $-2 + 6 - 6$), then the division by the magnitudes. The method marks are for each of these steps, and they are given even if an earlier vector was wrong.
- **Lines:** start with "$\mathbf{r} =$". Any correct point and any multiple of the direction is accepted. Don't put $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ inside a column vector.
- **Intersections:** show the general points, the two equations you solved, the values of λ and μ, and the third component checked. Give the point in the form asked: a "position vector" needs a vector, not coordinates.
- **"Show that the lines are skew"** needs two things: not parallel (with the direction vectors named) **and** not intersecting (the third component fails, with both numbers shown). Missing either loses marks.
- **"Show that"** answers are given: every step must be visible, and the last line must match the printed answer.
- **"Hence"** means use the result of the part before, for example the cosine you just found to get the sine and then the area.
- **"Exact"** means surds and fractions, no decimals, simplified. Otherwise give angles to 1 d.p. in degrees (or 3 s.f. in radians) and lengths to 3 s.f.
- **"Acute" or "obtuse"**: check which one the question wants, and adjust with $180^\circ - \theta$ if needed.

- **"State"** (for example "state the coordinates of P") needs only the answer, with no working.

Quick check

1. $\overrightarrow{OA} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$. Find $|\overrightarrow{AB}|$ and the position vector of the midpoint M of AB.
2. Find the value of p for which the directions $2\mathbf{i} + p\mathbf{j} - \mathbf{k}$ and $p\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ are perpendicular.
3. Decide whether the lines $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} + 4\mathbf{k} + t(-2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$ are the same line, parallel, intersecting or skew.
4. The point P lies on the line $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + t(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ and $OP = \sqrt{11}$. Find the possible position vectors of P.
5. $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{k}$. Find the exact value of $\cos AOB$ and the exact area of triangle OAB.

Answers

1. $\overrightarrow{AB} = -2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$, so $|\overrightarrow{AB}| = \sqrt{4+16+16} = 6$. $\overrightarrow{OM} = \frac{1}{2}(4\mathbf{i} + 2\mathbf{j}) = 2\mathbf{i} + \mathbf{j}$.
2. $2p + 3p - 4 = 0$, so $p = \frac{4}{5}$.
3. $-2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} = -2(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$, so the lines are parallel. The vector between the two given points is $2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, which is not a multiple of $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, so $(3, 0, 4)$ is not on the first line: they are parallel and distinct.
4. $\overrightarrow{OP} = (1+2t)\mathbf{i} + (t-2)\mathbf{j} + (3-2t)\mathbf{k}$. $(1+2t)^2 + (t-2)^2 + (3-2t)^2 = 11$ gives $9t^2 - 12t + 14 = 11$, so $3t^2 - 4t + 1 = 0$ and $t = 1$ or $t = \frac{1}{3}$. $\overrightarrow{OP} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ or $\frac{5}{3}\mathbf{i} - \frac{5}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}$.
5. $\overrightarrow{OA} \cdot \overrightarrow{OB} = 3 + 0 + 8 = 11$, $|\overrightarrow{OA}| = 3$, $|\overrightarrow{OB}| = 5$, so $\cos AOB = \frac{11}{15}$. $\sin AOB = \sqrt{1 - \frac{121}{225}} = \frac{\sqrt{104}}{15} = \frac{2\sqrt{26}}{15}$. Area $= \frac{1}{2} \times 3 \times 5 \times \frac{2\sqrt{26}}{15} = \sqrt{26}$.

Past questions to try

- **9709/33/O/N/23 Q11**: a unit vector, a line through two points, then parallel, intersecting or skew.
- **9709/33/M/J/22 Q9**: an angle, the foot of a perpendicular and a reflection.
- **9709/33/M/J/24 Q10**: a constant from a given angle, then from intersecting lines.
- **9709/32/O/N/24 Q9**: a trapezium, where its diagonals cross, and an angle.